

Polarization

QuadriviumPress

Abstract

This comprehensive textbook covers the fundamental principles of optics, from geometrical optics and wave phenomena to advanced topics including lasers, fiber optics, and optical instrumentation. Designed for undergraduate students in physics and electrical engineering, it provides a rigorous yet accessible treatment of topics including ray optics, interference, diffraction, polarization, lasers, and fiber optics.

Keywords optics, physics, education, textbook, light, lenses, lasers, fiber optics, polarization, interference, diffraction

What you should know and be able to do after studying this chapter

- Understand how different states of polarization are related to the ratio of the amplitudes and the difference in phase between two orthogonal components of the electric field.
- Know that elliptical polarization is the most general state of polarization.
- Know that linear polarization and circular polarization are special cases.
- Know how to compute the degree of polarization.
- Be able to work with Jones vectors and Jones matrices.
- Know how birefringence is exploited to create wave plates and know the types of wave plates.
- Know how to rotate a state of linear polarization through a given angle.
- Know how to change linear polarization into circular polarization and conversely.
- Be able to show that elliptical polarization can be written as the sum of two orthogonal linear polarizations and as the sum of two circular polarizations.

1. POLARIZATION STATES AND JONES VECTORS

We have seen in [Nature of Light](#) that light is an electromagnetic wave which satisfies Maxwell's equations and the wave equation derived therefrom. Since the electric field is a vector which oscillates as a function of time in a certain direction, we say that the wave has a certain polarization. In this chapter we look at the different types of polarization and how the polarization of a light beam can be manipulated.

From Maxwell's equations and the wave equation, we know that the (real) electric field $\mathcal{E}(\mathbf{r}, t)$ of a time-harmonic plane wave is always perpendicular to the direction of propagation, which is the direction of the wave vector $\mathbf{k}$ as well as the direction of the Poynting vector (the direction of the power flow). Let the wave propagate in the z -direction:

$$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}. \quad (1)$$

Then the electric field vector does not have a z -component and hence the real electric field at z and at time t can be written as

$$\mathcal{E}(z, t) = \begin{pmatrix} \text{cal } A_x \cos(kz - \omega t + \phi_x) \\ \text{cal } A_y \cos(kz - \omega t + \phi_y) \\ 0 \end{pmatrix}. \quad (2)$$

where $\text{cal } A_x$ and $\text{cal } A_y$ are positive amplitudes and ϕ_x, ϕ_y are the phases of the electric field components. While k and ω are fixed, we can vary $\text{cal } A_x$, $\text{cal } A_y$, ϕ_x and ϕ_y . This degree of freedom is why different states of polarization exist: **the state of polarization is determined by the ratio of the amplitudes and by the phase difference $\phi_y - \phi_x$ between the two orthogonal components of the light wave** (Fowles, 1989a). Consider the electric field in a fixed plane $z = 0$:

$$\begin{pmatrix} \text{cal } E_x(0, t) \\ \text{cal } E_y(0, t) \end{pmatrix} = \begin{pmatrix} \text{cal } A_x \cos(-\omega t + \phi_x) \\ \text{cal } A_y \cos(-\omega t + \phi_y) \end{pmatrix} = \text{Re} \left\{ \begin{pmatrix} \text{cal } A_x e^{i\phi_x} \\ \text{cal } A_y e^{i\phi_y} \end{pmatrix} e^{-i\omega t} \right\} = \text{Re} \left\{ \begin{pmatrix} E_x(0) \\ E_y(0) \end{pmatrix} e^{-i\omega t} \right\}$$

The complex vector

$$\mathbf{J} = \begin{pmatrix} E_x(0) \\ E_y(0) \end{pmatrix} = \begin{pmatrix} \text{cal } A_x e^{i\phi_x} \\ \text{cal } A_y e^{i\phi_y} \end{pmatrix}, \quad (4)$$

is called the **Jones vector**. It is used to characterize the polarization state. Let us see how, at a fixed position in space, the electric field vector behaves as a function of time for different choices of $\text{cal } A_x$, $\text{cal } A_y$ and $\phi_y - \phi_x$.

a) Linear polarization: $\phi_y - \phi_x = 0$ or $\phi_y - \phi_x = \pi$. When $\phi_y - \phi_x = 0$ we have

$$\mathbf{J} = \begin{pmatrix} \text{cal } A_x \\ \text{cal } A_y \end{pmatrix} e^{i\phi_x}. \quad (5)$$

Equality of the phases, $\phi_y = \phi_x$, means that the field components $\text{cal } E_x(z, t)$ and $\text{cal } E_y(z, t)$ are in phase: when $\text{cal } E_x(z, t)$ is large, $\text{cal } E_y(z, t)$ is large, and when $\text{cal } E_x(z, t)$ is small, $\text{cal } E_y(z, t)$ is small. We can write

$$\begin{pmatrix} \text{cal } E_x(0, t) \\ \text{cal } E_y(0, t) \end{pmatrix} = \begin{pmatrix} \text{cal } A_x \\ \text{cal } A_y \end{pmatrix} \cos(\omega t - \phi_x), \quad (6)$$

which shows that for $\phi_y - \phi_x = 0$ the electric field simply oscillates in one direction given by the real vector $\text{cal } A_x \hat{x} + \text{cal } A_y \hat{y}$. See [Figure 1](#).

If $\phi_y - \phi_x = \pi$ we have

$$\mathbf{J} = \begin{pmatrix} \text{cal } A_x \\ -\text{cal } A_y \end{pmatrix} e^{i\phi_x}. \quad (7)$$

In this case $\text{cal } E_x(z, t)$ and $\text{cal } E_y(z, t)$ are out of phase and the electric field oscillates in the direction given by the real vector $\text{cal } A_x \hat{x} - \text{cal } A_y \hat{y}$.

b) Circular polarization: $\phi_y - \phi_x = \pm\pi/2$, $\text{cal } A_x = \text{cal } A_y$. In this case the Jones vector is:

$$\mathbf{J} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \text{cal } A_x e^{i\phi_x}. \quad (8)$$

The field components $\text{cal } E_x(z, t)$ and $\text{cal } E_y(z, t)$ are $\pi/2$ radians (90 degrees) out of phase: when $\text{cal } E_x(z, t)$ is large, $\text{cal } E_y(z, t)$ is small, and when $\text{cal } E_x(z, t)$ is small, $\text{cal } E_y(z, t)$ is large. We can write for $z = 0$ and with $\phi_x = 0$:

$$\begin{pmatrix} \text{cal } E_x(0, t) \\ \text{cal } E_y(0, t) \end{pmatrix} = \begin{pmatrix} \text{cal } A_x \cos(-\omega t) \\ \text{cal } A_x \cos(-\omega t \pm \pi/2) \end{pmatrix} = \text{cal } A_x \begin{pmatrix} \cos(\omega t) \\ \pm \sin(\omega t) \end{pmatrix} \quad (9)$$

At a given position, the electric field vector moves along a circle as time proceeds. For an observer looking towards the source, when the electric field rotates counterclockwise, we call it **left-circularly polarized** (+ sign in Eq. (9)); clockwise rotation is **right-circularly polarized** (- sign). This states the viewing direction explicitly because handedness names depend on convention.

c) Elliptical polarization: $\phi_y - \phi_x = \pm\pi/2$, $\text{cal } A_x$ and $\text{cal } A_y$ arbitrary. The Jones vector is:

$$\mathbf{J} = \begin{pmatrix} \text{cal } A_x \\ \pm i \text{cal } A_y \end{pmatrix} e^{i\phi_x}. \quad (10)$$

In this case we get instead of Eq. (9) (again taking $\phi_x = 0$):

$$\begin{pmatrix} \text{cal } E_x(0, t) \\ \text{cal } E_y(0, t) \end{pmatrix} = \begin{pmatrix} \text{cal } A_x \cos(\omega t) \\ \pm \text{cal } A_y \sin(\omega t) \end{pmatrix}. \quad (11)$$

This shows that the electric vector moves along an ellipse with major and minor axes parallel to the x - and y -axes. When the + sign applies, the field is called left-elliptically polarized, otherwise it is called right-elliptically polarized.

d) Elliptical polarization: $\phi_y - \phi_x = \text{anything else}$, $\text{cal } A_x$ and $\text{cal } A_y$ arbitrary. The Jones vector is now the most general one:

$$\mathbf{J} = \begin{pmatrix} \text{cal } A_x e^{i\phi_x} \\ \text{cal } A_y e^{i\phi_y} \end{pmatrix}. \quad (12)$$

It can be shown that the electric field vector always moves along an ellipse. The exact shape and orientation of this ellipse vary with the difference in phase $\phi_y - \phi_x$ and the ratio of the amplitudes $\text{cal } A_x$ and $\text{cal } A_y$ and, except when $\phi_y - \phi_x = \pm\pi/2$, the major and minor axes of the ellipse are not parallel to the x - and y -axis. See [Figure 3](#).

Remarks.

- Frequently the Jones vector is normalized such that

$$|J_x|^2 + |J_y|^2 = 1. \quad (13)$$

The normalized vector represents of course the same polarization state as the unnormalized one. In general, multiplying the Jones vector by a complex number does not change the polarization state. If we multiply for example by $e^{i\theta}$, this has the same result as changing the instant at which $t = 0$, hence it does not change the polarization state. In fact:

$$\mathcal{E}(0, t) = \text{Re}[e^{i\theta} J e^{-i\omega t}] = \text{Re}[J e^{-i\omega(t-\theta/\omega)}] \quad (14)$$

- A general time-harmonic field may superpose plane waves traveling in different directions, as near the focus of a high-numerical-aperture lens. There need not be one propagation direction perpendicular to the total electric field. At any fixed point, however, a harmonic real field has the form $\mathbf{a} \cos \omega t + \mathbf{b} \sin \omega t$ and traces an ellipse in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$; a line and a circle are special cases. The plane may vary with position (Born & Wolf, 1999). The Jones-vector treatment below applies to a single plane wave with a fixed transverse basis.

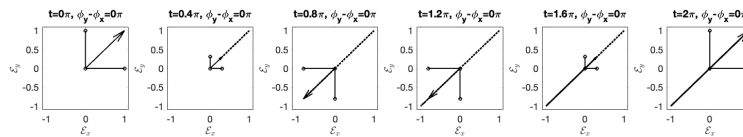


Figure 1: Linear polarization state of electromagnetic waves. The electric field vector oscillates along a single fixed direction perpendicular to the direction of propagation, tracing out a straight line in the plane perpendicular to the wave vector.

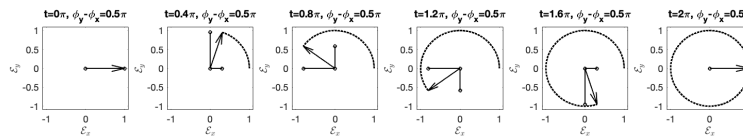


Figure 2: Circular polarization

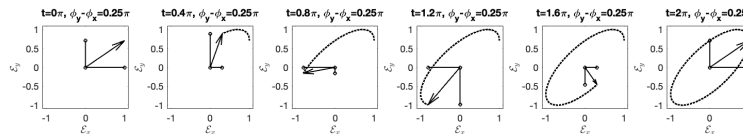


Figure 3: Elliptical polarization

Illustration of different types of polarization. The horizontal and vertical arrows indicate the momentary field components $\text{cal } E_x$, $\text{cal } E_y$. The thick arrow indicates the vector $\mathcal{E}$. The black curve indicates the trajectory of $\mathcal{E}(t)$.

External source

- [KhanAcademy - Polarization of light, linear and circular](#): Explanation of different polarization states and their applications.

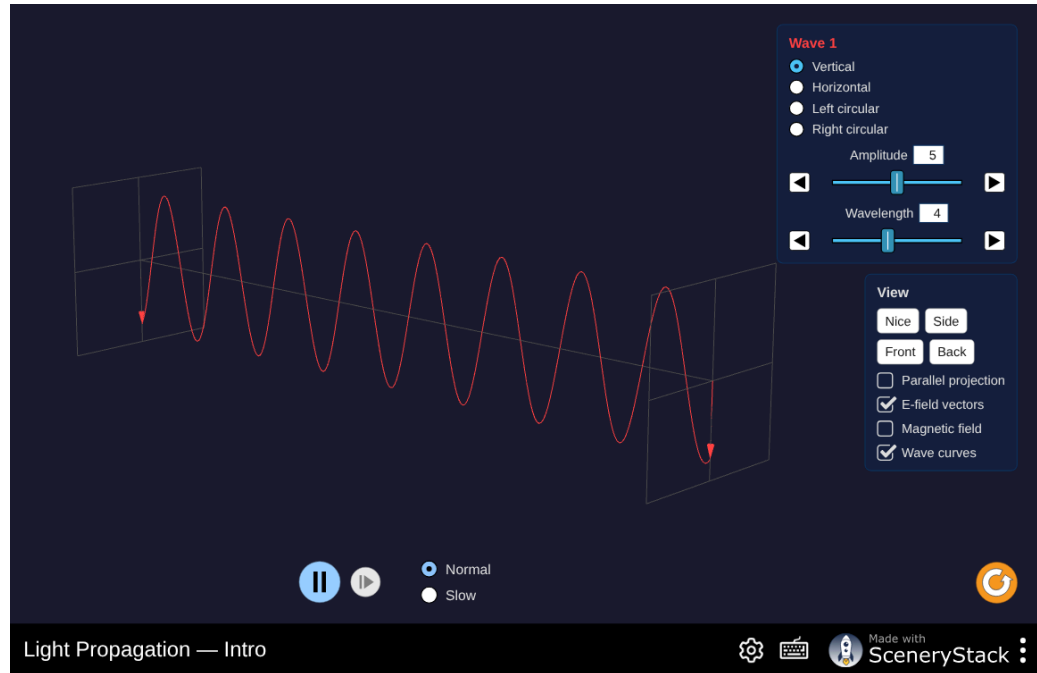


Figure 4: A monochromatic electromagnetic wave with its electric field drawn perpendicular to the propagation direction. Later screens of this simulation send the same wave through polarizers and birefringent plates — the interactive counterpart of the Jones-matrix section that follows.

Interactive simulation: [Light Propagation](#)

2. CREATING AND MANIPULATING POLARIZATION STATES

We have seen how Maxwell’s equations allow the existence of plane waves with many different states of polarization. But how can we create these states, and how do these states manifest themselves?

Natural light often does not have a definite polarization. Instead, the polarization fluctuates rapidly with time. To turn such randomly polarized light into linearly polarized light in a certain direction, we must extinguish the light polarized in the perpendicular direction. The remaining light is then linearly polarized along the desired direction. One could do this by using light reflected at the Brewster angle (which extinguishes p-polarized light), or one could let light pass through a dichroic crystal, which is a material which absorbs light polarized perpendicular to its so-called optic axis. A third method is sending the light through a wire grid polarizer, which consists of a metallic grating with sub-wavelength slits. Such a grating only transmits the electric field component that is perpendicular to the slits.

So suppose that with one of these methods we have obtained linearly polarized light. Then the question arises: how can linearly polarized light be changed into circularly or elliptically polarized light? Or how the state of linear polarization can be rotated over a certain angle? We have seen that the polarization state depends on the ratio of the amplitudes and on the phase difference $\phi_y - \phi_x$ of the orthogonal components

cal E_y and cal E_x of the electric field. Thus, to change linearly polarized light to some other state of polarization, a certain phase shift (say $\Delta\phi_x$) must be introduced to one component (say cal E_x), and another phase shift $\Delta\phi_y$ to the orthogonal component cal E_y . We can achieve this with a **birefringent crystal**, such as calcite. What is special about such a crystal is that it has two refractive indices: light polarized in a certain direction experiences a refractive index n_o , while light polarized perpendicular to it feels another refractive index n_e (the subscripts o and e stand for “ordinary” and “extraordinary”, but for our purpose we do not need to understand this terminology). The direction for which the refractive index is *smallest* (which can be either n_o or n_e) is called the **fast axis** because its phase velocity is largest, and the other direction is the **slow axis**. Because there are two different refractive indices, one can see double images through a birefringent crystal (Horvath et al., 2011). The difference between the two refractive indices $\Delta n = n_e - n_o$ is called the **birefringence**.

Suppose $n_e > n_o$ and that the fast axis, which corresponds to n_o , is aligned with cal E_x , while the slow axis (which then has refractive index n_e) is aligned with cal E_y . If the wave travels a distance d through the crystal, cal E_y will accumulate a phase $\Delta\phi_y = 2\pi n_e d / \lambda_0$, and cal E_x will accumulate $\Delta\phi_x = 2\pi n_o d / \lambda_0$, where λ_0 is the vacuum wavelength. Thus, the phase difference $\phi_y - \phi_x$ increases by

$$\Delta\phi_y - \Delta\phi_x = \frac{2\pi d}{\lambda_0} (n_e - n_o). \quad (15)$$

2.a. Jones Matrices

By letting light pass through crystals of different thicknesses d we can create different phase differences between the orthogonal field components and in this way we can create different states of polarization. To be specific, let $\mathbf{J}$, as given by Eq. (4), be the Jones vector of the plane wave before the crystal. Then we have, for the Jones vector after the passage through the crystal:

$$\tilde{\mathbf{J}} = \text{cal } M \mathbf{J}, \quad (16)$$

where

$$\text{cal } M = \begin{pmatrix} e^{\frac{2\pi i}{\lambda_0} d n_o} & 0 \\ 0 & e^{\frac{2\pi i}{\lambda_0} d n_e} \end{pmatrix} = e^{\frac{2\pi i}{\lambda_0} d n_o} \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{\lambda_0} d (n_e - n_o)} \end{pmatrix}. \quad (17)$$

A matrix such as cal M , which transfers one state of polarization of a plane wave into another, is called a **Jones matrix**. Depending on the phase difference which a wave accumulates by traveling through the crystal, these devices are called **quarter-wave plates** (phase difference $\pi/2$), **half-wave plates** (phase difference π), or **full-wave plates** (phase difference 2π). The applications of these wave plates will be discussed in later sections.

Consider as an example the Jones matrix which describes the change of linearly polarized light into circular polarization. Assume that we have diagonally (linearly) polarized light, so that

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (18)$$

We want to change it to circularly polarized light, for which

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad (19)$$

where one can check that indeed $\phi_y - \phi_x = \pi/2$. This can be done by passing the light through a crystal such that $\text{cal } E_y$ accumulates a phase difference of $\pi/2$ with respect to $\text{cal } E_x$. The transformation by which this is accomplished can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}. \quad (20)$$

The matrix on the left is the Jones matrix describing the operation of a quarter-wave plate.

Another important Jones matrix is the **rotation matrix**. In the preceding discussion it was assumed that the fast and slow axes were aligned with the x - and y -direction (i.e. they were parallel to $\text{cal } E_x$ and $\text{cal } E_y$). Suppose now that the slow and fast axes of the wave plate no longer coincide with $\hat{x}$ and $\hat{y}$, but rather with some other $\hat{x}'$ and $\hat{y}'$ as in [Figure 5](#). In that case we apply a basis transformation: the electric field vector which is expressed in the $\hat{x}, \hat{y}$ basis should first be expressed in the $\hat{x}', \hat{y}'$ basis before applying the Jones matrix of the wave plate to it. After applying the Jones matrix, the electric field has to be transformed back from the $\hat{x}', \hat{y}'$ basis to the $\hat{x}, \hat{y}$ basis.

Let $\mathbf{E}$ be given in terms of its components on the $\hat{x}, \hat{y}$ basis:

$$\mathbf{E} = E_x \hat{x} + E_y \hat{y}. \quad (21)$$

To find the components $E_{x'}, E_{y'}$ on the $\hat{x}', \hat{y}'$ basis:

$$\mathbf{E} = E_{x'} \hat{x}' + E_{y'} \hat{y}'. \quad (22)$$

We first write the unit vectors $\hat{x}'$ and $\hat{y}'$ in terms of the basis $\hat{x}, \hat{y}$ (see [Figure 5](#))

$$\hat{x}' = \cos \theta \hat{x} + \sin \theta \hat{y}. \quad (23)$$

$$\hat{y}' = -\sin \theta \hat{x} + \cos \theta \hat{y}. \quad (24)$$

By substituting [Eq. \(23\)](#) and [Eq. \(24\)](#) into [Eq. \(22\)](#) we find

$$\begin{aligned} \mathbf{E} &= E_{x'} \hat{x}' + E_{y'} \hat{y}' \\ &= E_{x'} (\cos \theta \hat{x} + \sin \theta \hat{y}) + E_{y'} (-\sin \theta \hat{x} + \cos \theta \hat{y}), \\ &= (\cos \theta E_{x'} - \sin \theta E_{y'}) \hat{x} + (\sin \theta E_{x'} + \cos \theta E_{y'}) \hat{y}. \end{aligned} \quad (25)$$

Comparing with [Eq. \(21\)](#) implies

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{x'} \cos \theta - E_{y'} \sin \theta \\ E_{x'} \sin \theta + E_{y'} \cos \theta \end{pmatrix} = \text{cal } R_\theta \begin{pmatrix} E_{x'} \\ E_{y'} \end{pmatrix}, \quad (26)$$

where $\text{cal } R_\theta$ is the rotation matrix over an angle θ in the counterclockwise direction: (Hecht, 2002).

That $\text{cal } R(\theta)$ indeed is a rotation through an angle θ in the counterclockwise direction is easy to see by considering what happens when $\text{cal } R_\theta$ is applied to the vector $(1, 0)^T$ (Hecht, 2002).

This relationship converts Jones-vector components from the crystal's x', y' axes to the original x, y axes. The reverse conversion is $\text{cal } R_{-\theta}$. If $\text{cal } M$ is the wave-plate matrix in Eq. (17), with x' as the fast axis and y' as the slow axis, then the matrix $\text{cal } M_\theta$ of the rotated plate in the $\hat{x}, \hat{y}$ basis is

$$\text{cal } M_\theta = \text{cal } R_\theta \text{cal } M \text{cal } R_{-\theta}. \quad (27)$$

This is a standard result from linear algebra involving basis transformations.

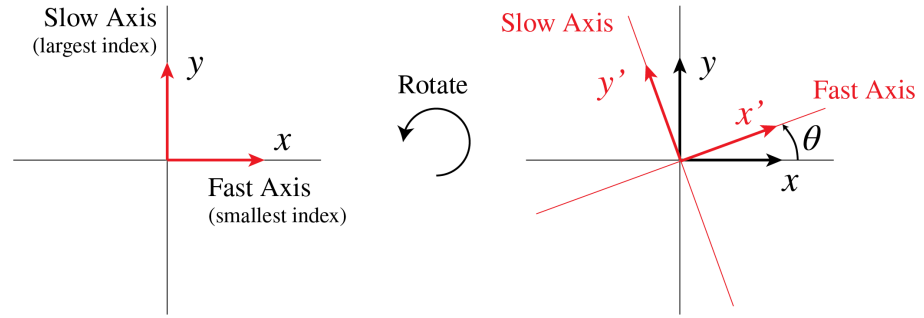


Figure 5: If the wave plate is rotated, the fast and slow axes no longer correspond to x and y . Instead, we have to introduce a new coordinate system (x', y') .

2.b. Linear polarizers

A polarizer that only transmits horizontally polarized light is described by the Jones matrix:

$$\text{cal } M_{LP} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (28)$$

Clearly, horizontally polarized light is completely transmitted, while vertically polarized light is not transmitted at all. More generally, for light that is polarized at an angle α , we get

$$\text{cal } M_{LP} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \begin{pmatrix} \cos \alpha \\ 0 \end{pmatrix}. \quad (29)$$

The amplitude of the transmitted field is reduced by the factor $\cos \alpha$, which implies that the intensity of the transmitted light is reduced by the factor $\cos^2 \alpha$. This relation is known as **Malus' law**.

2.c. Degree of Polarization

Natural light such as sunlight is unpolarized. The instantaneous polarization of unpolarized light fluctuates rapidly in a random manner. A linear polarizer produces linearly polarized light from unpolarized light. It follows from Eq. (29) that the intensity transmitted by a linear polarizer when unpolarized light is incident is the average value of $\cos^2 \alpha$ namely $\frac{1}{2}$, times the incident intensity.

Light that is a mixture of polarized and unpolarized light is called partially polarized. The **degree of polarization** is defined as the fraction of the total intensity that is polarized:

$$\text{degree of polarization} = \frac{I_{pol}}{I_{pol} + I_{unpol}}. \quad (30)$$

2.d. Quarter-Wave Plates

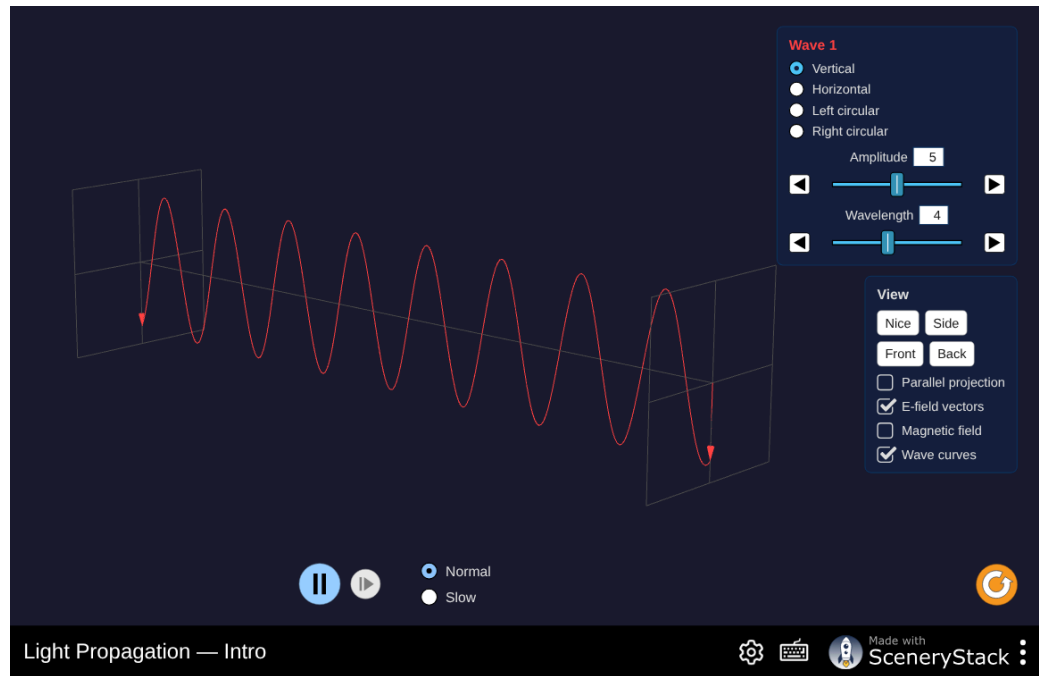


Figure 6: A birefringent slab with adjustable retardation. Set a quarter-wave or half-wave plate, rotate the fast axis relative to a linear input, and watch linear light become circular (or rotate) — the same phase shift treated by the Jones matrix below.

Interactive simulation: [Light Propagation](#)

A quarter-wave plate has already been introduced above. It introduces a phase shift of $\pi/2$, so its Jones matrix is

$$\text{cal } M_{QWP} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad (31)$$

because $\exp(i\pi/2) = i$. To describe the actual transmission through the quarter-wave plate, the matrix should be multiplied by some global phase factor, but because we only care about the **phase difference** between the field components, this global

phase factor can be omitted without problem. The quarter-wave plate is typically used to **convert linearly polarized light to elliptically polarized light and vice-versa**(Saleh & Teich, 2007). If the incident light is linearly polarized at angle α , the state of polarization after the quarter-wave plate is

$$\begin{pmatrix} \cos \alpha \\ i \sin \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}. \quad (32)$$

In particular, if incident linear polarization is at 45° to either plate axis, the output is circularly polarized. Conversely, circular input becomes linearly polarized after an ideal quarter-wave plate.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (33)$$

A demonstration is shown in(Bartholinus, 1669).

2.e. Half-Wave Plates

A half-wave plate introduces a phase shift of π , so its Jones matrix is

$$\text{cal } M_{HWP} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (34)$$

because $\exp(i\pi) = -1$. An important application of the half-wave plate is to **change the orientation of linearly polarized light**(Fowles, 1989b). After all, what this matrix does is mirror the polarization state in the x -axis. Thus, if we choose our mirroring axis correctly (i.e. if we choose the orientation of the wave plate correctly), we can change the direction in which the light is linearly polarized arbitrarily(Goldstein, 2011). To give an example: a wave with linear polarization parallel to the x -direction can be rotated by an angle α by rotating the crystal such that the fast axis makes an angle $\alpha/2$ with the x -axis. Upon propagation through the crystal, the slow axis gets an additional phase of π , due to which the electric vector makes angle α with the x -axis (see [Figure 5](#)). It is not difficult to verify that when the fast and slow axes are interchanged, the same linear state of polarization results.

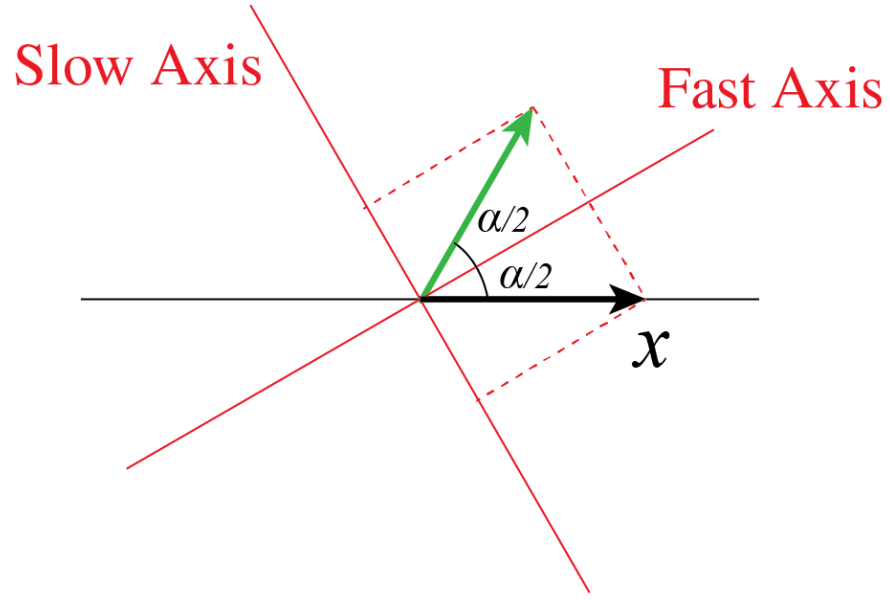


Figure 7: Rotation of horizontally polarized light by an angle α using a half-wave plate.

2.f. Full-Wave Plates

A full-wave plate introduces a phase difference of 2π , which is the same as introducing no phase difference between the two field components. So what can possibly be an application for a full-wave plate? We recall from [Eq. \(15\)](#) that the phase difference is 2π only for a particular wavelength. To obtain wavelength-dependent transmission, orient the incident linear polarization at 45° to the plate's axes and place a crossed analyzer after the plate. At the design wavelength λ_0 , the full-wave plate restores the incident polarization and the analyzer extinguishes it. Nearby wavelengths acquire a different relative phase and can pass partially. Other wavelengths with integer-wave retardance are also extinguished, so this arrangement does not select a unique wavelength.

3. MORE ON JONES MATRICES

If the direction of either the slow or fast axis and the ordinary and extra-ordinary refractive indices n_o and n_e are given, it is easy to write down the Jones matrix of a birefringent plate of given thickness d using the rotation matrices, see [Eq. \(27\)](#). Instead of using the rotation matrices, one can also write down a system of equations for the elements of the Jones matrix. Suppose that $\hat{v}_o = v_{o,x}\hat{x} + v_{o,y}\hat{y}$ and $\hat{v}_e = v_{e,x}\hat{x} + v_{e,y}\hat{y}$ are in the direction of the ordinary and the extra-ordinary axes, respectively. Then if the Jones matrix is

$$\text{cal } M = \begin{pmatrix} a & b \\ c & q \end{pmatrix}, \quad (35)$$

then

$$\begin{aligned} \text{cal } M\hat{v}_o &= e^{ik_0 n_o d} \hat{v}_o, \\ \text{cal } M\hat{v}_e &= e^{ik_0 n_e d} \hat{v}_e \end{aligned} \quad (36)$$

where $k_0 = 2\pi/\lambda_0$. Equating components implies

$$\begin{aligned} av_{o,x} + bv_{o,y} &= e^{ik_0 n_o d} v_{o,x}, \\ cv_{o,x} + qv_{o,y} &= e^{ik_0 n_o d} v_{o,y}, \\ av_{e,x} + bv_{e,y} &= e^{ik_0 n_e d} v_{e,x}, \\ cv_{e,x} + qv_{e,y} &= e^{ik_0 n_e d} v_{e,y}. \end{aligned} \quad (37)$$

Similarly, for a linear polarizer it is simple to write down the Jones matrix if one knows the direction in which the polarizer absorbs or transmits all the light: use Eq. (28) in combination with the rotation matrices. Alternatively, if $\hat{v}$ is in the direction of the linear polarizer and $\hat{w}$ is perpendicular to it, we have

$$\begin{aligned} \text{cal } M\hat{v} &= \hat{v} \\ \text{cal } M\hat{w} &= \mathbf{0}, \end{aligned} \quad (38)$$

which is a system of equations of type Eq. (37) for the elements of the Jones matrix.

Suppose now that the complex (2,2)-matrix Eq. (35) is given. How can one verify whether this matrix corresponds to a linear polarizer or to a wave plate? Note that the elements of a Jones matrix are in general complex.

1. Linear polarizer. An ideal lossless linear polarizer transmits one real linear-polarization direction and blocks its orthogonal direction. Up to an overall phase, its eigenvalues are 1 and 0, with orthogonal eigenvectors that can be chosen real. A Jones matrix with eigenvalues 2 and 0 would amplify the transmitted field, so it is not a passive ideal polarizer. Eigenvectors that cannot be chosen real do not identify two linear-polarization directions.

2. Wave plate. To show that the matrix corresponds to a wave plate, there should exist two **real** orthogonal eigenvectors with, in general, complex eigenvalues of modulus 1. In fact, one of the eigenvectors corresponds to the ordinary axis with refractive index n_o , and the other to the extra-ordinary axis with refractive index n_e . The eigenvalues are then

$$e^{ik_0 n_o d} \quad \text{and} \quad e^{ik_0 n_e d}, \quad (39)$$

where d is the plate thickness and k_0 is the vacuum wave number. Thus, a lossless linear wave plate has two orthogonal real eigenvectors and eigenvalues of unit modulus. A global phase factor does not affect its polarization action.

3. Jones matrix for propagation through sugars. In sugars, left and right circularly polarized light propagate with their own refractive index. Therefore sugars are called **circularly birefringent**. The matrix Eq. (35) corresponds to propagation through sugar when there are two real orthogonal unit vectors $\hat{v}$ and $\hat{w}$ such that the circular polarization states

$$\hat{v} + i\hat{w}, \quad \hat{v} - i\hat{w} \quad (40)$$

are eigenstates of $\text{cal } M$ with complex eigenvalues with modulus 1.

4. DECOMPOSITION OF AN ELLIPTICAL POLARIZATION STATE INTO LINEAR AND CIRCULAR STATES

Any elliptical polarization state can be written as the sum of two perpendicular linearly polarized states:

$$J = \begin{pmatrix} \text{cal } A_x e^{i\phi_x} \\ \text{cal } A_y e^{i\phi_y} \end{pmatrix} = \text{cal } A_x e^{i\phi_x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \text{cal } A_y e^{i\phi_y} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (41)$$

Furthermore, any elliptical polarization state can be written as the sum of two circular polarization states, one right-circularly polarized and the other left-circularly polarized:

$$J = \begin{pmatrix} \text{cal } A_x e^{i\phi_x} \\ \text{cal } A_y e^{i\phi_y} \end{pmatrix} = \frac{1}{2}(\text{cal } A_x e^{i\phi_x} - i \text{cal } A_y e^{i\phi_y}) \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2}(\text{cal } A_x e^{i\phi_x} + i \text{cal } A_y e^{i\phi_y}) \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

We conclude that to study what happens to elliptic polarization, it suffices to consider two orthogonal linear polarizations, or, if that is more convenient, left- and right-circular polarized light. In a birefringent material each of the two linear polarizations, namely parallel to the o-axis and parallel to the e-axis, propagates with its own refractive index. To predict what happens to an arbitrary linear polarization state which is not aligned to either of these axes, or more generally what happens to an elliptical polarization state, we write this polarization state as a linear combination of o- and e-states, i.e. we expand the field on the o- and e-basis.

To see what happens to an arbitrary elliptical polarization state in a circular birefringent material, the incident light is best written as a linear combination of left and right circular polarizations.

5. CHAPTER SUMMARY

- **Polarization** describes the orientation of the electric field vector oscillation in an electromagnetic wave.
- **Linear polarization:** The electric field oscillates in a fixed direction; described by a Jones vector with real components.
- **Circular polarization:** The electric field rotates at constant amplitude; left-circular and right-circular have opposite handedness.
- **Elliptical polarization:** The most general fully polarized state; the electric field traces an ellipse set by component amplitudes and relative phase.
- **Jones vectors** represent polarization states as 2D complex vectors; **Jones matrices** describe how optical elements transform polarization.
- **Polarizers** transmit one polarization and block the orthogonal one; **Malus's Law** gives transmitted intensity as $I = I_0 \cos^2 \theta$.
- **Birefringence** occurs when a material has different refractive indices for different polarization directions.

- **Wave plates:** Quarter-wave plates convert linear to circular polarization (and vice versa); half-wave plates rotate linear polarization.
- **Circular birefringence** gives the two circular components different phase shifts and can rotate linear polarization.
- **Degree of polarization** measures the polarized fraction of a beam's total intensity.

External sources in recommended order

1. [Double Vision - Sixty Symbols](#): Demonstration of double refraction by a calcite crystal due to birefringence.
2. [MIT OCW - Linear Polarizer](#): Demonstration of linear polarizers and linear polarization.
3. [MIT OCW - Polarization Rotation Using Polarizers](#): Demonstration of polarization rotation using linear polarizers.
4. [Demonstration of a QuarterWavePlate](#) by Andrew Berger.
5. [MIT OCW - Quarter-wave Plate](#): Demonstration of the quarter-wave plate to create elliptical (in particular circular) polarization.
6. [Demonstration of a HalfWavePlate](#) by Andrew Berger.
7. [MIT OCW - Half-wave Plate](#): Demonstration of the half-wave plate.

6. REFERENCES

REFERENCES

- Bartholinus, E. (1669). *Experimenta crystalli islandici disdiaclastici quibus mira & insolita refractio detegitur*. Copenhagen: Daniel Paulli.
- Born, M., & Wolf, E. (1999). *Principles of Optics* (7th ed., Chapter 3). Cambridge University Press.
- Fowles, G. R. (1989a). *Introduction to Modern Optics*. Dover Publications.
- Fowles, G. R. (1989b). *Introduction to Modern Optics* (Chapter 3). Dover Publications.
- Goldstein, D. H. (2011). *Polarized Light* (3rd ed., Chapter 4). CRC Press.
- Hecht, E. (2002). *Optics* (4th ed., Chapter 8). Addison Wesley.
- Horvath, G., Barta, A., Pomozi, I., Suhai, B., Hegedus, R., Åkesson, S., Meyer-Rochow, B., & Wehner, R. (2011). Viking navigation using polarized skylight. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2127), 651–660.
- Saleh, B. E. A., & Teich, M. C. (2007). *Fundamentals of Photonics* (2nd ed., Chapter 6). Wiley-Interscience.