Appendix B: of the Right-Hand Rule for the Cross Product

Appendix B: Proof of the Right-Hand Rule for the Cross Product

We will prove the right-hand rule for the cross product of two vectors in $\mathbb{R}^3$.

For any vectors **v** and **w** in $\mathbb{R}^3$, define a new vector, $\mathbf{n}(\mathbf{v}, \mathbf{w})$, as follows:

1. If **v** and **w** are nonzero and not parallel, and θ is the angle between them, then $\mathbf{n}(\mathbf{v}, \mathbf{w})$ is the vector in $\mathbb{R}^3$ such that:
    1. the magnitude of $\mathbf{n}(\mathbf{v}, \mathbf{w})$ is $\|\mathbf{v}\| \, \|\mathbf{w}\| \, \sin\theta$,
    2. $\mathbf{n}(\mathbf{v}, \mathbf{w})$ is perpendicular to the plane containing **v** and **w**, and
    3. **v**, **w**, $\mathbf{n}(\mathbf{v}, \mathbf{w})$ form a right-handed system.
2. If **v** and **w** are nonzero and parallel, then $\mathbf{n}(\mathbf{v}, \mathbf{w}) = \mathbf{0}$.
3. If either **v** or **w** is $\mathbf{0}$, then $\mathbf{n}(\mathbf{v}, \mathbf{w}) = \mathbf{0}$.

The goal is to show that $\mathbf{n}(\mathbf{v}, \mathbf{w}) = \mathbf{v} \times \mathbf{w}$ for all **v**, **w** in $\mathbb{R}^3$, which would prove the right-hand rule for the cross product (by part 1(c) of our definition). To do this, we will perform the following steps:

Step 1: Show that $\mathbf{n}(\mathbf{v}, \mathbf{w}) = \mathbf{v} \times \mathbf{w}$ if **v** and **w** are any two of the basis vectors **i**, **j**, **k**.

This was already shown in Example 1.11 in Section 1.4.

Step 2: Show that $\mathbf{n}(a\mathbf{v}, b\mathbf{w}) = ab(\mathbf{v} \times \mathbf{w})$ for any scalars a, b if **v** and **w** are any two of the basis vectors **i**, **j**, **k**.

If either $a = 0$ or $b = 0$ then $\mathbf{n}(a\mathbf{v}, b\mathbf{w}) = \mathbf{0} = ab(\mathbf{v} \times \mathbf{w})$, so the result holds. So assume that $a \neq 0$ and $b \neq 0$. Let **v** and **w** be any two of the basis vectors **i**, **j**, **k**. For example, we will show that the result holds for $\mathbf{v} = \mathbf{i}$ and $\mathbf{w} = \mathbf{k}$ (the other possibilities follow in a similar fashion).

For $a\mathbf{v} = a\mathbf{i}$ and $b\mathbf{w} = b\mathbf{k}$, the angle θ between $a\mathbf{v}$ and $b\mathbf{w}$ is 90°. Hence the magnitude of $\mathbf{n}(a\mathbf{v}, b\mathbf{w})$, by definition, is $\|a\mathbf{i}\| \, \|b\mathbf{k}\| \, \sin 90^\circ = |ab|$. Also, by definition, $\mathbf{n}(a\mathbf{v}, b\mathbf{w})$ is perpendicular to the plane containing $a\mathbf{i}$ and $b\mathbf{k}$, namely, the xz-plane. Thus, $\mathbf{n}(a\mathbf{v}, b\mathbf{w})$ must be a scalar multiple of **j**. Since its magnitude is $|ab|$, then $\mathbf{n}(a\mathbf{v}, b\mathbf{w})$ must be either $|ab|\mathbf{j}$ or $-|ab|\mathbf{j}$.

There are four possibilities for the combinations of signs for a and b. We will consider the case when $a > 0$ and $b > 0$ (the other three possibilities are handled similarly). In this case, $\mathbf{n}(a\mathbf{v}, b\mathbf{w})$ must be either $ab\mathbf{j}$ or $-ab\mathbf{j}$. Now, since **i**, **j**, **k** form a right-handed system, then **i**, **k**, **j** form a left-handed system, and so

i, **k**, $-\mathbf{j}$ form a right-handed system. Thus, $a\mathbf{i}$, $b\mathbf{k}$, $-ab\mathbf{j}$ form a right-handed system (since $a > 0$, $b > 0$, and $ab > 0$). So since, by definition, $a\mathbf{i}$, $b\mathbf{k}$, $\mathbf{n}(a\mathbf{i}, b\mathbf{k})$ form a right-handed system, and since $\mathbf{n}(a\mathbf{i}, b\mathbf{k})$ has to be either $ab\mathbf{j}$ or $-ab\mathbf{j}$, this means that we must have $\mathbf{n}(a\mathbf{i}, b\mathbf{k}) = -ab\mathbf{j}$.

But we know that $a\mathbf{i} \times b\mathbf{k} = ab(\mathbf{i} \times \mathbf{k}) = ab(-\mathbf{j}) = -ab\mathbf{j}$. Therefore, $\mathbf{n}(a\mathbf{i}, b\mathbf{k}) = ab(\mathbf{i} \times \mathbf{k})$, which is what we needed to show.

$\therefore \quad \mathbf{n}(a\mathbf{v}, b\mathbf{w}) = ab(\mathbf{v} \times \mathbf{w})$ ✓

Step 3: Show that $\mathbf{n}(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \mathbf{n}(\mathbf{u}, \mathbf{v}) + \mathbf{n}(\mathbf{u}, \mathbf{w})$ for any vectors **u**, **v**, **w**.

If $\mathbf{u} = \mathbf{0}$ then the result holds trivially since $\mathbf{n}(\mathbf{u}, \mathbf{v} + \mathbf{w})$, $\mathbf{n}(\mathbf{u}, \mathbf{v})$ and $\mathbf{n}(\mathbf{u}, \mathbf{w})$ are all the zero vector. If $\mathbf{v} = \mathbf{0}$, then the result follows easily since $\mathbf{n}(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \mathbf{n}(\mathbf{u}, \mathbf{0} + \mathbf{w}) = \mathbf{n}(\mathbf{u}, \mathbf{w}) = \mathbf{0} + \mathbf{n}(\mathbf{u}, \mathbf{w}) = \mathbf{n}(\mathbf{u}, \mathbf{0}) + \mathbf{n}(\mathbf{u}, \mathbf{w})$ $= \mathbf{n}(\mathbf{u}, \mathbf{v}) + \mathbf{n}(\mathbf{u}, \mathbf{w})$. A similar argument shows that the result holds if $\mathbf{w} = \mathbf{0}$.

So now assume that **u**, **v** and **w** are all nonzero vectors. We will describe a geometric construction of $\mathbf{n}(\mathbf{u}, \mathbf{v})$, which is shown in the figure below. Let P be a plane perpendicular to **u**. Multiply the vector **v** by the positive scalar $\|\mathbf{u}\|$, then project the vector $\|\mathbf{u}\|\,\mathbf{v}$ straight down onto the plane P. You can think of this projection vector (denoted by $proj_P\|\mathbf{u}\|\,\mathbf{v}$) as the shadow of the vector $\|\mathbf{u}\|\,\mathbf{v}$ on the plane P, with the light source directly overhead the terminal point of $\|\mathbf{u}\|\,\mathbf{v}$. If θ is the angle between **u** and **v**, then we see that $proj_P\|\mathbf{u}\|\,\mathbf{v}$ has magnitude $\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$, which is the magnitude of $\mathbf{n}(\mathbf{u}, \mathbf{v})$. So rotating $proj_P\|\mathbf{u}\|\,\mathbf{v}$ by 90° in a counter-clockwise direction in the plane P gives a vector whose magnitude is the same as that of $\mathbf{n}(\mathbf{u}, \mathbf{v})$ and which is perpendicular to $proj_P\|\mathbf{u}\|\,\mathbf{v}$ (and hence perpendicular to **v**). Since this vector is in P then it is also perpendicular to **u**. And we can see that **u**, **v** and this vector form a right-handed system. Hence this vector must be $\mathbf{n}(\mathbf{u}, \mathbf{v})$. Note that this holds even if $\mathbf{u} \parallel \mathbf{v}$, since in that case $\theta = 0^\circ$ and so $\sin\theta = 0$ which means that $\mathbf{n}(\mathbf{u}, \mathbf{v})$ has magnitude 0, which is what we would expect.

Now apply this same geometric construction to get $\mathbf{n}(\mathbf{u}, \mathbf{w})$ and $\mathbf{n}(\mathbf{u}, \mathbf{v} + \mathbf{w})$. Since $\|\mathbf{u}\|\,(\mathbf{v} + \mathbf{w})$ is the sum of the vectors $\|\mathbf{u}\|\,\mathbf{v}$ and $\|\mathbf{u}\|\,\mathbf{w}$, then the projection vector $proj_P\|\mathbf{u}\|\,(\mathbf{v} + \mathbf{w})$ is the sum of the projection vectors $proj_P\|\mathbf{u}\|\,\mathbf{v}$ and $proj_P\|\mathbf{u}\|\,\mathbf{w}$ (to see this, using the shadow analogy again and the parallelogram rule for vector addition, think of how projecting a parallelogram onto a plane gives you a parallelogram in that plane). So then rotating all three projection vectors by 90° in a counter-clockwise direction in the plane P preserves that sum (see the figure below), which means that $\mathbf{n}(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \mathbf{n}(\mathbf{u}, \mathbf{v}) + \mathbf{n}(\mathbf{u}, \mathbf{w})$. ✓

Step 4: Show that $\mathbf{n}(\mathbf{w}, \mathbf{v}) = -\mathbf{n}(\mathbf{v}, \mathbf{w})$ for any vectors **v**, **w**.

If **v** and **w** are nonzero and parallel, or if either is $\mathbf{0}$, then $\mathbf{n}(\mathbf{w}, \mathbf{v}) = \mathbf{0} = -\mathbf{n}(\mathbf{v}, \mathbf{w})$, so the result holds. So assume that **v** and **w** are nonzero and not parallel. Then $\mathbf{n}(\mathbf{w}, \mathbf{v})$ has magnitude $\|\mathbf{w}\|\,\|\mathbf{v}\|\,\sin\theta$, which is the same as the magnitude of $\mathbf{n}(\mathbf{v}, \mathbf{w})$, and hence is the same as the magnitude of $-\mathbf{n}(\mathbf{v}, \mathbf{w})$. By definition, $\mathbf{n}(\mathbf{v}, \mathbf{w})$ is perpendicular to the plane containing **w** and **v**, and hence so is $-\mathbf{n}(\mathbf{v}, \mathbf{w})$. Also, **v**, **w**, $\mathbf{n}(\mathbf{v}, \mathbf{w})$ form a right-handed system, and so **w**, **v**, $\mathbf{n}(\mathbf{v}, \mathbf{w})$ form a left-handed system, and hence **w**, **v**, $-\mathbf{n}(\mathbf{v}, \mathbf{w})$ form a right-handed system. Thus, we have shown that $-\mathbf{n}(\mathbf{v}, \mathbf{w})$ is a vector with the same magnitude as $\mathbf{n}(\mathbf{w}, \mathbf{v})$ and is perpendicular to the plane containing **w** and **v**, and that **w**, **v**, $-\mathbf{n}(\mathbf{v}, \mathbf{w})$ form a right-handed system. So by definition this means that $-\mathbf{n}(\mathbf{v}, \mathbf{w})$ must be $\mathbf{n}(\mathbf{w}, \mathbf{v})$. ✓

Step 5: Show that $\mathbf{n}(\mathbf{v},\mathbf{w}) = \mathbf{v} \times \mathbf{w}$ for all vectors **v**, **w**.

Write $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$. Then by Steps 3 and 4, we have

$$\begin{aligned}
\mathbf{n}(\mathbf{v},\mathbf{w}) &= \mathbf{n}(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}, w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}) \\
&= \mathbf{n}(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}, w_1\,\mathbf{i}) \;+\; \mathbf{n}(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}, w_2\,\mathbf{j} + w_3\,\mathbf{k}) \\
&= \mathbf{n}(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}, w_1\,\mathbf{i}) \;+\; \mathbf{n}(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}, w_2\,\mathbf{j}) \;+\; \mathbf{n}(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}, w_3\,\mathbf{k}) \\
&= \;-\mathbf{n}(w_1\,\mathbf{i}, v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) \;+\; -\mathbf{n}(w_2\,\mathbf{j}, v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) \;+\; -\mathbf{n}(w_3\,\mathbf{k}, v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}).
\end{aligned}$$

We can use Steps 1 and 2 to evaluate the three terms on the right side of the last equation above:

$$\begin{aligned}
-\mathbf{n}(w_1\,\mathbf{i}, v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) &= \;-\mathbf{n}(w_1\,\mathbf{i}, v_1\,\mathbf{i}) \;+\; -\mathbf{n}(w_1\,\mathbf{i}, v_2\,\mathbf{j}) \;+\; -\mathbf{n}(w_1\,\mathbf{i}, v_3\,\mathbf{k}) \\
&= \;-v_1 w_1 \mathbf{n}(\mathbf{i},\mathbf{i}) \;+\; -v_2 w_1 \mathbf{n}(\mathbf{i},\mathbf{j}) \;+\; -v_3 w_1 \mathbf{n}(\mathbf{i},\mathbf{k}) \\
&= \;-v_1 w_1 (\mathbf{i} \times \mathbf{i}) \;+\; -v_2 w_1 (\mathbf{i} \times \mathbf{j}) \;+\; -v_3 w_1 (\mathbf{i} \times \mathbf{k}) \\
&= \;-v_1 w_1 \mathbf{0} \;+\; -v_2 w_1\,\mathbf{k} \;+\; -v_3 w_1 (-\mathbf{j}) \\
-\mathbf{n}(w_1\,\mathbf{i}, v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) &= \;-v_2 w_1\,\mathbf{k} \;+\; v_3 w_1\,\mathbf{j}
\end{aligned}$$

Similarly, we can calculate

$$\begin{aligned}
-\mathbf{n}(w_2\,\mathbf{j}, v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) &= v_1 w_2\,\mathbf{k} \;-\; v_3 w_2\,\mathbf{i} \\
\text{and} -\mathbf{n}(w_3\,\mathbf{k}, v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) &= \;-v_1 w_3\,\mathbf{j} \;+\; v_2 w_3\,\mathbf{i}\;.
\end{aligned}$$

Thus, putting it all together, we have

$$\begin{aligned}
\mathbf{n}(\mathbf{v},\mathbf{w}) &= \;-v_2 w_1\,\mathbf{k} \;+\; v_3 w_1\,\mathbf{j} \;+\; v_1 w_2\,\mathbf{k} \;-\; v_3 w_2\,\mathbf{i} \;-\; v_1 w_3\,\mathbf{j} \;+\; v_2 w_3\,\mathbf{i} \\
&= (v_2 w_3 - v_3 w_2)\mathbf{i} \;+\; (v_3 w_1 - v_1 w_3)\mathbf{j} \;+\; (v_1 w_2 - v_2 w_1)\mathbf{k} \\
&= \mathbf{v} \times \mathbf{w} \;\text{ by definition of the cross product.}
\end{aligned}$$

$\therefore\;\; \mathbf{n}(\mathbf{v},\mathbf{w}) = \mathbf{v} \times \mathbf{w}$ for all vectors **v**, **w**.

So since **v**, **w**, $\mathbf{n}(\mathbf{v},\mathbf{w})$ form a right-handed system, then **v**, **w**, $\mathbf{v} \times \mathbf{w}$ form a right-handed system, which completes the proof.