1. Right Triangle Trigonometry

Right Triangle Trigonometry

Trigonometry is the study of the relations between the sides and angles of triangles. The word “trigonometry” is derived from the Greek words *trigono* ($\tau\rho\acute{\iota}\gamma\omega\nu$o), meaning “triangle”, and *metro* ($\mu\epsilon\tau\rho\acute{\omega}$), meaning “measure”. Though the ancient Greeks, such as Hipparchus and Ptolemy, used trigonometry in their study of astronomy between roughly 150 B.C. - A.D. 200, its history is much older. For example, the Egyptian scribe Ahmes recorded some rudimentary trigonometric calculations (concerning ratios of sides of pyramids) in the famous Rhind Papyrus sometime around 1650 B.C.[1]

Trigonometry is distinguished from elementary geometry in part by its extensive use of certain functions of angles, known as the *trigonometric functions*. Before discussing those functions, we will review some basic terminology about angles.

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1. Ahmes claimed that he copied the papyrus from a work that may date as far back as 3000 B.C. ↩

1.1 Angles

Recall the following definitions from elementary geometry:

1. An angle is **acute** if it is between 0° and 90°.
2. An angle is a **right angle** if it equals 90°.
3. An angle is **obtuse** if it is between 90° and 180°.
4. An angle is a **straight angle** if it equals 180°.

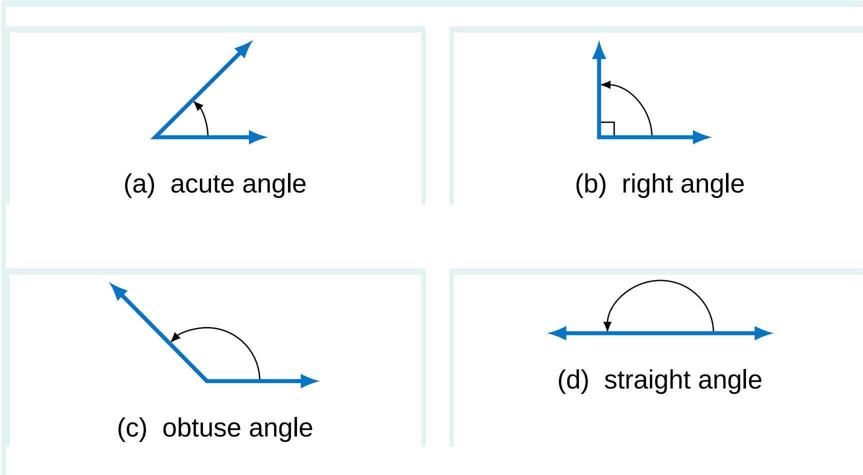


Figure 1.1.1*Types of angles*

In elementary geometry, angles are always considered to be positive and not larger than $360°$. For now we will only consider such angles.[1] The following definitions will be used throughout the text:

1. Two acute angles are **complementary** if their sum equals $90°$. In other words, if $0° \leq \angle A\,,\, \angle B \leq 90°$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90°$.
2. Two angles between $0°$ and $180°$ are **supplementary** if their sum equals $180°$. In other words, if $0° \leq \angle A\,,\, \angle B \leq 180°$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180°$.
3. Two angles between $0°$ and $360°$ are **conjugate** (or **explementary**) if their sum equals $360°$. In other words, if $0° \leq \angle A\,,\, \angle B \leq 360°$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360°$.

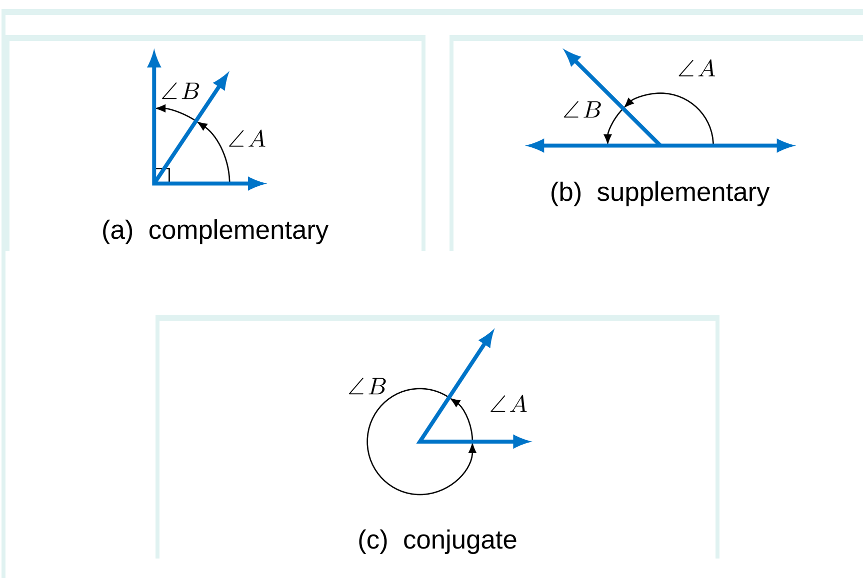


Figure 1.1.2*Types of pairs of angles*

Instead of using the angle notation $\angle A$ to denote an angle, we will sometimes use just a capital letter by itself (e.g. A, B, C) or a lowercase variable name (e.g. x, y, t). It is also common to use letters (either uppercase or lowercase) from the Greek alphabet, shown in the table below, to represent angles:

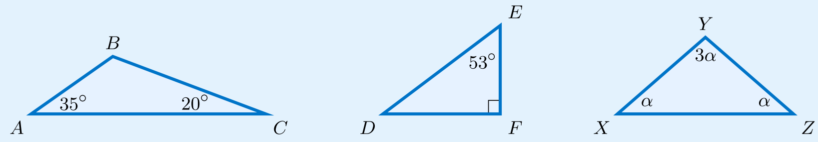
Table 1.1The Greek alphabet

Letters		**Name**	**Letters**		**Name**	**Letters**		**Name**
A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

In elementary geometry you learned that the sum of the angles in a triangle equals 180°, and that an **isosceles triangle** is a triangle with two sides of equal length. Recall that in a **right triangle** one of the angles is a right angle. Thus, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

Example 1.1

For each triangle below, determine the unknown angle(s):



Note: We will sometimes refer to the angles of a triangle by their vertex points. For example, in the first triangle above we will simply refer to the angle $\angle BAC$ as angle A.

Solution: For triangle $\triangle ABC$, $A = 35^\circ$ and $C = 20^\circ$, and we know that $A + B + C = 180^\circ$, so

$$35^\circ \;+\; B \;+\; 20^\circ \;=\; 180^\circ \quad\Rightarrow\quad B \;=\; 180^\circ \;-\; 35^\circ \;-\; 20^\circ \quad\Rightarrow\quad \boxed{B \;=\; 125^\circ}\,.$$

For the right triangle $\triangle DEF$, $E = 53^\circ$ and $F = 90^\circ$, and we know that the two acute angles D and E are complementary, so

$$D \;+\; E \;=\; 90^\circ \quad\Rightarrow\quad D \;=\; 90^\circ \;-\; 53^\circ \quad\Rightarrow\quad \boxed{D \;=\; 37^\circ}\,.$$

For triangle $\triangle XYZ$, the angles are in terms of an unknown number α, but we do know that $X + Y + Z = 180^\circ$, which we can use to solve for α and then use that to solve for X, Y, and Z:

$$\alpha \;+\; 3\alpha \;+\; \alpha \;=\; 180^\circ \quad\Rightarrow\quad 5\alpha \;=\; 180^\circ \quad\Rightarrow\quad \alpha \;=\; 36^\circ \quad\Rightarrow\quad \boxed{X = 36^\circ\;,\;\; Y = 3 \times 36^\circ = 108^\circ\;,\;\; Z = 36^\circ}$$

Example 1.2

Thales' Theorem states that if A, B, and C are (distinct) points on a circle such that the line segment $\overline{AB}$ is a diameter of the circle, then the angle $\angle ACB$ is a right angle (see Figure 1.1.3(a)). In other words, the triangle $\triangle ABC$ is a right triangle.

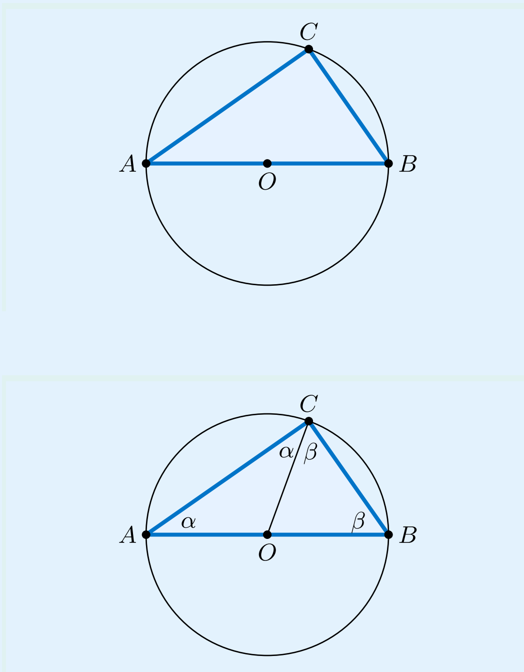


Figure 1.1.3*Thales' Theorem:* $\angle ACB = 90^\circ$

To prove this, let O be the center of the circle and draw the line segment $\overline{OC}$, as in Figure 1.1.3(b). Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Since $\overline{AB}$ is a diameter of the circle, $\overline{OA}$ and $\overline{OC}$ have the same length (namely, the circle's radius). This means that $\triangle OAC$ is an isosceles triangle, and so $\angle OCA = \angle OAC = \alpha$. Likewise, $\triangle OBC$ is an isosceles triangle and $\angle OCB = \angle OBC = \beta$. So we see that $\angle ACB = \alpha + \beta$. And since the angles of $\triangle ABC$ must add up to 180°, we see that $180^\circ = \alpha + (\alpha + \beta) + \beta = 2\,(\alpha + \beta)$, so $\alpha + \beta = 90^\circ$. Thus, $\angle ACB = 90^\circ$. qed

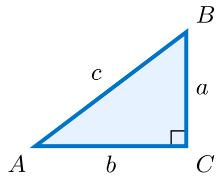


Figure 1.1.4

In a right triangle, the side opposite the right angle is called the **hypotenuse**, and the other two sides are called its **legs**. For example, in Figure 1.1.4 the right angle is C, the hypotenuse is the line segment $\overline{AB}$, which has length c, and $\overline{BC}$ and $\overline{AC}$ are the legs, with lengths a and b, respectively. The hypotenuse is always the longest side of a right triangle (see Exercise 11).

By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

Theorem 1.1

Pythagorean Theorem: The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

Thus, if a right triangle has a hypotenuse of length c and legs of lengths a and b, as in Figure 1.1.4, then the Pythagorean Theorem says:

$$\setlength\fboxsep 2mm\setlength\fboxrule 1pt\boxed{a^2 \;+\; b^2 \;=\; c^2} \tag{1.1}$$

Let us prove this. In the right triangle $\triangle ABC$ in Figure 1.1.5(a) below, if we draw a line segment from the vertex C to the point D on the hypotenuse such that $\overline{CD}$ is **perpendicular** to $\overline{AB}$ (that is, $\overline{CD}$ forms a right angle with $\overline{AB}$), then this divides $\triangle ABC$ into two smaller triangles $\triangle CBD$ and $\triangle ACD$, which are both similar to $\triangle ABC$.

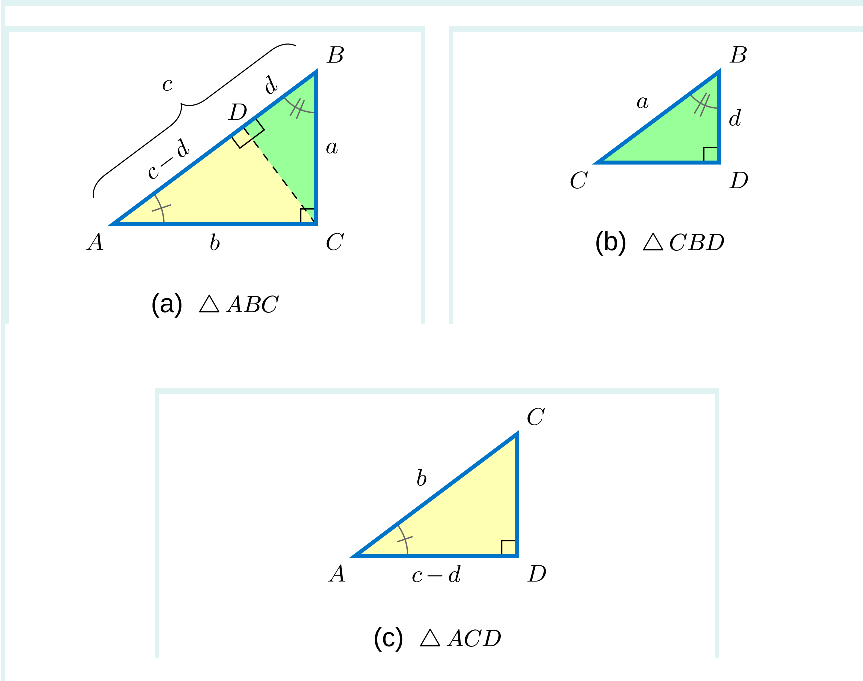


Figure 1.1.5
Similar triangles $\triangle\,ABC$, $\triangle\,CBD$, $\triangle\,ACD$

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle\,ABC$ is similar to $\triangle\,CBD$, by proportionality of corresponding sides we see that

$$\overline{AB}\text{ is to }\overline{CB}\text{ (hypotenuses)}\backslash\mathrm{enskip}\,\text{as}\backslash\mathrm{enskip}\,\overline{BC}\text{ is to }\overline{BD}\text{ (vertical legs)} \quad\Rightarrow\quad \frac{c}{a} = \frac{a}{d} \quad\Rightarrow\quad cd = a^2 \,.$$

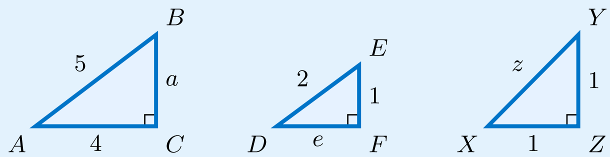
Since $\triangle\,ABC$ is similar to $\triangle\,ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d} = \frac{c}{b} \quad\Rightarrow\quad b^2 = c^2 - cd = c^2 - a^2 \quad\Rightarrow\quad a^2 + b^2 = c^2 \,.\text{qed}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD} \perp \overline{AB}$ and $\triangle\,ABC \sim \triangle\,CBD \sim \triangle\,ACD$.

Example 1.3

For each right triangle below, determine the length of the unknown side:



Solution: For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad\Rightarrow\quad a^2 = 25 - 16 = 9 \quad\Rightarrow\quad \boxed{a = 3}\,.$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

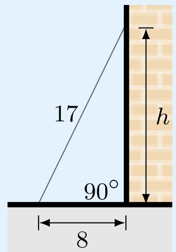
$$e^2 + 1^2 = 2^2 \quad\Rightarrow\quad e^2 = 4 - 1 = 3 \quad\Rightarrow\quad \boxed{e = \sqrt{3}}\,.$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad\Rightarrow\quad z^2 = 2 \quad\Rightarrow\quad \boxed{z = \sqrt{2}}\,.$$

Example 1.4

[r]



A 17 ft ladder leaning against a wall has its foot 8 ft from the base of the wall. At what height is the top of the ladder touching the wall?

Solution: Let h be the height at which the ladder touches the wall. We can assume that the ground makes a right angle with the wall, as in the picture on the right. Then we see that the ladder, ground, and wall form a right triangle with a hypotenuse of length 17 ft (the length of the ladder) and legs with lengths 8 ft and h ft. So by the Pythagorean Theorem, we have

$$h^2 + 8^2 = 17^2 \quad \Rightarrow \quad h^2 = 289 - 64 = 225 \quad \Rightarrow \quad \boxed{h = 15 \text{ ft}}\,.$$

EXERCISES

For Exercises 1-4, find the numeric value of the indicated angle(s) for the triangle $\triangle ABC$.

2

1. Find B if $A = 15^\circ$ and $C = 50^\circ$.
2. Find C if $A = 110^\circ$ and $B = 31^\circ$.

   2

3. Find A and B if $C = 24^\circ$, $A = \alpha$, and $B = 2\alpha$.
4. Find A, B, and C if $A = \beta$ and $B = C = 4\beta$.

For Exercises 5-8, find the numeric value of the indicated angle(s) for the right triangle $\triangle ABC$, with C being the right angle.

2

5. Find B if $A = 45°$.

6. Find A and B if $A = \alpha$ and $B = 2\alpha$.

   2

7. Find A and B if $A = \phi$ and $B = \phi^2$.

8. Find A and B if $A = \theta$ and $B = 1/\theta$.

9. A car goes 24 miles due north then 7 miles due east. What is the straight distance between the car's starting point and end point?

10. One end of a rope is attached to the top of a pole 100 ft high. If the rope is 150 ft long, what is the maximum distance along the ground from the base of the pole to where the other end can be attached? You may assume that the pole is perpendicular to the ground.

11. Prove that the hypotenuse is the longest side in every right triangle. (*Hint: Is* $a^2 + b^2 > a^2$*?*)

12. Can a right triangle have sides with lengths 2, 5, and 6? Explain your answer.

13. If the lengths a, b, and c of the sides of a right triangle are positive integers, with $a^2 + b^2 = c^2$, then they form what is called a **Pythagorean triple**. The triple is normally written as (a,b,c). For example, (3,4,5) and (5,12,13) are well-known Pythagorean triples.

    1. Show that (6,8,10) is a Pythagorean triple.
    2. Show that if (a,b,c) is a Pythagorean triple then so is (ka,kb,kc) for any integer $k > 0$. How would you interpret this geometrically?
    3. Show that ($2mn$,$m^2 - n^2$,$m^2 + n^2$) is a Pythagorean triple for all integers $m > n > 0$.
    4. The triple in part(c) is known as **Euclid's formula** for generating Pythagorean triples. Write down the first ten Pythagorean triples generated by this formula, i.e. use: $m = 2$ and $n = 1$; $m = 3$ and $n = 1, 2$; $m = 4$ and $n = 1, 2, 3$; $m = 5$ and $n = 1, 2, 3, 4$.

14. This exercise will describe how to draw a line through any point outside a circle such that the line intersects the circle at only one point. This is called a *tangent line* to the circle (see the picture on the left in Figure 1.1.6), a notion which we will use throughout the text.

inside

Figure 1.1.6

On a sheet of paper draw a circle of radius 1 inch, and call the center of that circle O. Pick a point P which is 2.5 inches away from O. Draw the circle which has $\overline{OP}$ as a diameter, as in the picture on the right in Figure 1.1.6. Let A be one of the points where this circle intersects the first circle. Draw the line through P and A. In general the tangent line through a point on a circle is perpendicular to the line joining that point to the center of the circle (why?). Use this fact to explain why the line you drew is the tangent line through A and to calculate the length of $\overline{PA}$. Does it match the physical measurement of $\overline{PA}$? [r]

15. Suppose that $\triangle\, ABC$ is a triangle with side $\overline{AB}$ a diameter of a circle with center O, as in the picture on the right, and suppose that the vertex C lies on the circle. Now imagine that you rotate the circle 180° around its center, so that $\triangle\, ABC$ is in a new position, as indicated by the dashed lines in the picture. Explain how this picture proves Thales' Theorem.

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1. Later in the text we will discuss negative angles and angles larger than 360°. ↩

1.2 Trigonometric Functions of an Acute Angle

[r]

Consider a right triangle $\triangle\, ABC$, with the right angle at C and with lengths a, b, and c, as in the figure on the right. For the acute angle A, call the leg $\overline{BC}$ its **opposite side**, and call the leg $\overline{AC}$ its **adjacent side**. Recall that the hypotenuse of the triangle is the side $\overline{AB}$. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we define the six **trigonometric functions** of A as follows:

Table 1.2The six trigonometric functions of A

Name of function		Abbreviation			Definition		
sine A			$\sin\ A$		$=$	$\dfrac{\text{opposite side}}{\text{hypotenuse}}$	$=$
cosine A			$\cos\ A$		$=$	$\dfrac{\text{adjacent side}}{\text{hypotenuse}}$	$=$
tangent A			$\tan\ A$		$=$	$\dfrac{\text{opposite side}}{\text{adjacent side}}$	$=$
cosecant A			$\csc\ A$		$=$	$\dfrac{\text{hypotenuse}}{\text{opposite side}}$	$=$
secant A			$\sec\ A$		$=$	$\dfrac{\text{hypotenuse}}{\text{adjacent side}}$	$=$
cotangent A			$\cot\ A$		$=$	$\dfrac{\text{adjacent side}}{\text{opposite side}}$	$=$

We will usually use the abbreviated names of the functions. Notice from Table 1.2 that the pairs $\sin\ A$ and $\csc\ A$, $\cos\ A$ and $\sec\ A$, and $\tan\ A$ and $\cot\ A$ are reciprocals:

$$\csc\ A = \frac{1}{\sin\ A} \qquad \sec\ A = \frac{1}{\cos\ A} \qquad \cot\ A = \frac{1}{\tan\ A}$$

$$\sin\ A = \frac{1}{\csc\ A} \qquad \cos\ A = \frac{1}{\sec\ A} \qquad \tan\ A = \frac{1}{\cot\ A}$$

Example 1.5

[r]

For the right triangle $\triangle ABC$ shown on the right, find the values of all six trigonometric functions of the acute angles A and B.

Solution: The hypotenuse of $\triangle ABC$ has length 5. For angle A, the opposite side $\overline{BC}$ has length 3 and the adjacent side $\overline{AC}$ has length 4. Thus:

$$\sin\ A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos\ A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \tan\ A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}\text{\vspac}$$

$$\csc\ A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \qquad \sec\ A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} \qquad \cot\ A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

For angle B, the opposite side $\overline{AC}$ has length 4 and the adjacent side $\overline{BC}$ has length 3. Thus:

$$\sin\ B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos\ B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan\ B = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}\text{\vspac}$$

$$\csc\ B = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} \qquad \sec\ B = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} \qquad \cot\ B = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

Notice in Example 1.5 that we did not specify the units for the lengths. This raises the possibility that our answers depended on a triangle of a specific physical size.

For example, suppose that two different students are reading this textbook: one in the United States and one in Germany. The American student thinks that the lengths 3, 4, and 5 in Example 1.5 are measured in inches, while the German student thinks that they are measured in centimeters. Since 1 in ≈ 2.54 cm, the students are using triangles of different physical sizes (see Figure 1.2.1 below, not drawn to scale).

(a) Inches

(b) Centimeters

(c) Similar triangles

Figure 1.2.1 $\triangle\,ABC \;\sim\; \triangle\,A'B'C'$

If the American triangle is $\triangle\,ABC$ and the German triangle is $\triangle\,A'B'C'$, then we see from Figure 1.2.1 that $\triangle\,ABC$ is similar to $\triangle\,A'B'C'$, and hence the corresponding angles are equal and the ratios of the corresponding sides are equal. In fact, we know that common ratio: the sides of $\triangle\,ABC$ are approximately 2.54 times longer than the corresponding sides of $\triangle\,A'B'C'$. So when the American student calculates $\sin\;A$ and the German student calculates $\sin\;A'$, they get the same answer:[1]

$$\triangle\,ABC \;\sim\; \triangle\,A'B'C' \quad\Rightarrow\quad \frac{BC}{B'C'} = \frac{AB}{A'B'} \quad\Rightarrow\quad \frac{BC}{AB} = \frac{B'C'}{A'B'} \quad\Rightarrow\quad \sin\;A \;=\; \sin\;A'$$

Likewise, the other values of the trigonometric functions of A and A' are the same. In fact, our argument was general enough to work with any similar right triangles. This leads us to the following conclusion:

When calculating the trigonometric functions of an acute angle A, you may use *any* right triangle which has A as one of the angles.

Since we defined the trigonometric functions in terms of ratios of sides, you can think of the units of measurement for those sides as canceling out in those ratios. This means that *the values of the trigonometric functions are unitless numbers*. So when the American student calculated $3/5$ as the value of $\sin\;A$ in Example 1.5, that is the same as the $3/5$ that the German student calculated, despite the different units for the lengths of the sides.

Example 1.6

[r]

Find the values of all six trigonometric functions of 45°.

Solution: Since we may use any right triangle which has 45° as one of the angles, use the simplest one: take a square whose sides are all 1 unit long and divide it in half diagonally, as in the figure on the right. Since the two legs of the triangle $\triangle\,ABC$ have the same length, $\triangle\,ABC$ is an isosceles triangle, which means that the angles A and B are equal. So since $A+B=90^\circ$, this means that we must have $A=B=45^\circ$. By the Pythagorean Theorem, the length c of the hypotenuse is given by

$$c^2 \;=\; 1^2 \;+\; 1^2 \;=\; 2 \quad\Rightarrow\quad c \;=\; \sqrt{2}\;.$$

Thus, using the angle A we get:

$$\sin\;45^\circ \;=\; \frac{\text{opposite}}{\text{hypotenuse}} \;=\; \frac{1}{\sqrt{2}} \qquad \cos\;45^\circ \;=\; \frac{\text{adjacent}}{\text{hypotenuse}} \;=\; \frac{1}{\sqrt{2}} \qquad \tan\;45^\circ \;=\; \frac{\text{opposite}}{\text{adjacent}} \;=\; \frac{1}{1} \;=\; 1\backslash\mathrm{v}$$

$$\csc\;45^\circ \;=\; \frac{\text{hypotenuse}}{\text{opposite}} \;=\; \sqrt{2} \qquad \sec\;45^\circ \;=\; \frac{\text{hypotenuse}}{\text{adjacent}} \;=\; \sqrt{2} \qquad \cot\;45^\circ \;=\; \frac{\text{adjacent}}{\text{opposite}} \;=\; \frac{1}{1} \;=\; 1\backslash\mathrm{vs}]$$

Note that we would have obtained the same answers if we had used any right triangle similar to $\triangle\,ABC$. For example, if we multiply each side of $\triangle\,ABC$ by $\sqrt{2}$, then we would have a similar triangle with legs of length $\sqrt{2}$ and hypotenuse of length 2. This would give us $\sin\;45^\circ = \frac{\sqrt{2}}{2}$, which equals $\frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$ as before. The same goes for the other functions.

Example 1.7

[r]

Find the values of all six trigonometric functions of 60°.

Solution: Since we may use any right triangle which has 60° as one of the angles, we will use a simple one: take a triangle whose sides are all 2 units long and divide it in half by drawing the bisector from one vertex to the opposite side, as in the figure on the right. Since the original triangle was an *equilateral triangle* (i.e. all three sides had the same length), its three angles were all the same, namely 60°. Recall from elementary geometry that the bisector from the vertex angle of an equilateral triangle to its opposite side bisects both the vertex angle and the opposite side. So as in the figure on the right, the triangle $\triangle\,ABC$ has angle $A = 60^\circ$ and angle $B = 30^\circ$, which forces the angle C to be 90°. Thus, $\triangle\,ABC$ is a right triangle. We see that the hypotenuse has length $c = AB = 2$ and the leg $\overline{AC}$ has length $b = AC = 1$. By the Pythagorean Theorem, the length a of the leg $\overline{BC}$ is given by

$$a^2 \;+\; b^2 \;=\; c^2 \quad\Rightarrow\quad a^2 \;=\; 2^2 \;-\; 1^2 \;=\; 3 \quad\Rightarrow\quad a \;=\; \sqrt{3}\;.$$

Thus, using the angle A we get:

$$\sin\;60^\circ \;=\; \frac{\text{opposite}}{\text{hypotenuse}} \;=\; \frac{\sqrt{3}}{2} \qquad \cos\;60^\circ \;=\; \frac{\text{adjacent}}{\text{hypotenuse}} \;=\; \frac{1}{2} \qquad \tan\;60^\circ \;=\; \frac{\text{opposite}}{\text{adjacent}} \;=\; \frac{\sqrt{3}}{1} \;=\; \sqrt{3}\backslash\text{v}$$

$$\csc\;60^\circ \;=\; \frac{\text{hypotenuse}}{\text{opposite}} \;=\; \frac{2}{\sqrt{3}} \qquad \sec\;60^\circ \;=\; \frac{\text{hypotenuse}}{\text{adjacent}} \;=\; 2 \qquad \cot\;60^\circ \;=\; \frac{\text{adjacent}}{\text{opposite}} \;=\; \frac{1}{\sqrt{3}} \qquad \backslash\text{vspa}$$

Notice that, as a bonus, we get the values of all six trigonometric functions of 30°, by using angle $B = 30^\circ$ in the same triangle $\triangle\,ABC$ above:

$$\sin\;30^\circ \;=\; \frac{\text{opposite}}{\text{hypotenuse}} \;=\; \frac{1}{2} \qquad \cos\;30^\circ \;=\; \frac{\text{adjacent}}{\text{hypotenuse}} \;=\; \frac{\sqrt{3}}{2} \qquad \tan\;30^\circ \;=\; \frac{\text{opposite}}{\text{adjacent}} \;=\; \frac{1}{\sqrt{3}} \qquad \backslash\text{vspa}$$

$$\csc\;30^\circ \;=\; \frac{\text{hypotenuse}}{\text{opposite}} \;=\; 2 \qquad \sec\;30^\circ \;=\; \frac{\text{hypotenuse}}{\text{adjacent}} \;=\; \frac{2}{\sqrt{3}} \qquad \cot\;30^\circ \;=\; \frac{\text{adjacent}}{\text{opposite}} \;=\; \frac{\sqrt{3}}{1} \;=\; \sqrt{3}\backslash\text{vs}$$

Example 1.8

[r]

A is an acute angle such that $\sin\ A = \frac{2}{3}$. Find the values of the other trigonometric functions of A.

Solution: In general it helps to draw a right triangle to solve problems of this type. The reason is that the trigonometric functions were defined in terms of ratios of sides of a right triangle, and you are given one such function (the sine, in this case) already in terms of a ratio: $\sin\ A = \frac{2}{3}$. Since $\sin\ A$ is defined as $\frac{\text{opposite}}{\text{hypotenuse}}$, use 2 as the length of the side opposite A and use 3 as the length of the hypotenuse in a right triangle $\triangle ABC$ (see the figure above), so that $\sin\ A = \frac{2}{3}$. The adjacent side to A has unknown length b, but we can use the Pythagorean Theorem to find it:

$$2^2 + b^2 = 3^2 \quad \Rightarrow \quad b^2 = 9 - 4 = 5 \quad \Rightarrow \quad b = \sqrt{5}$$

We now know the lengths of all sides of the triangle $\triangle ABC$, so we have:

$$\cos\ A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} \qquad \tan\ A = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{\sqrt{5}} \qquad \color{red}{\backslash\text{vspace}}2mm$$

$$\csc\ A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{2} \qquad \sec\ A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{\sqrt{5}} \qquad \cot\ A = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{5}}{2} \color{red}{\backslash\text{vspace}}2mm$$

You may have noticed the connections between the sine and cosine, secant and cosecant, and tangent and cotangent of the complementary angles in Examples 1.5 and 1.7. Generalizing those examples gives us the following theorem:

Theorem 1.2

Cofunction Theorem: If A and B are the complementary acute angles in a right triangle $\triangle ABC$, then the following relations hold:

$$\sin\ A = \cos\ B \qquad \sec\ A = \csc\ B \qquad \tan\ A = \cot\ B$$

$$\sin\ B = \cos\ A \qquad \sec\ B = \csc\ A \qquad \tan\ B = \cot\ A$$

We say that the pairs of functions $\{\,\sin, \cos\,\}$, $\{\,\sec, \csc\,\}$, and $\{\,\tan, \cot\,\}$ are **cofunctions**.

So sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. That is how the functions cosine, cosecant, and cotangent got the “co” in their names. The Cofunction Theorem says that any trigonometric function of an acute angle is equal to its cofunction of the complementary angle.

Example 1.9

Write each of the following numbers as trigonometric functions of an angle less than $45°$: **(a)** $\sin\ 65°$; **(b)** $\cos\ 78°$; **(c)** $\tan\ 59°$.

Solution: (a) The complement of $65°$ is $90° - 65° = 25°$ and the cofunction of $\sin$ is $\cos$, so by the Cofunction Theorem we know that $\sin\ 65° = \cos\ 25°$.

(b) The complement of $78°$ is $90° - 78° = 12°$ and the cofunction of $\cos$ is $\sin$, so $\cos\ 78° = \sin\ 12°$.

(c) The complement of $59°$ is $90° - 59° = 31°$ and the cofunction of $\tan$ is $\cot$, so $\tan\ 59° = \cot\ 31°$.

(a) $45 - 45 - 90$

(b) $30 - 60 - 90$

Figure 1.2.2*Two general right triangles (any $a > 0$)*

The angles $30°$, $45°$, and $60°$ arise often in applications. We can use the Pythagorean Theorem to generalize the right triangles in Examples 1.6 and 1.7 and see what *any* $45 - 45 - 90$ and $30 - 60 - 90$ right triangles look like, as in Figure 1.2.2 above.

Example 1.10

[r]

Find the sine, cosine, and tangent of 75°.

Solution: Since $75^\circ = 45^\circ + 30^\circ$, place a $30-60-90$ right triangle $\triangle ADB$ with legs of length $\sqrt{3}$ and 1 on top of the hypotenuse of a $45-45-90$ right triangle $\triangle ABC$ whose hypotenuse has length $\sqrt{3}$, as in the figure on the right. From Figure 1.2.2(a) we know that the length of each leg of $\triangle ABC$ is the length of the hypotenuse divided by $\sqrt{2}$. So $AC = BC = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$. Draw $\overline{DE}$ perpendicular to $\overline{AC}$, so that $\triangle ADE$ is a right triangle. Since $\angle BAC = 45^\circ$ and $\angle DAB = 30^\circ$, we see that $\angle DAE = 75^\circ$ since it is the sum of those two angles. Thus, we need to find the sine, cosine, and tangent of $\angle DAE$. Notice that $\angle ADE = 15^\circ$, since it is the complement of $\angle DAE$. And $\angle ADB = 60^\circ$, since it is the complement of $\angle DAB$. Draw $\overline{BF}$ perpendicular to $\overline{DE}$, so that $\triangle DFB$ is a right triangle. Then $\angle BDF = 45^\circ$, since it is the difference of $\angle ADB = 60^\circ$ and $\angle ADE = 15^\circ$. Also, $\angle DBF = 45^\circ$ since it is the complement of $\angle BDF$. The hypotenuse $\overline{BD}$ of $\triangle DFB$ has length 1 and $\triangle DFB$ is a $45-45-90$ right triangle, so we know that $DF = FB = \frac{1}{\sqrt{2}}$.

Now, we know that $\overline{DE} \perp \overline{AC}$ and $\overline{BC} \perp \overline{AC}$, so $\overline{FE}$ and $\overline{BC}$ are parallel. Likewise, $\overline{FB}$ and $\overline{EC}$ are both perpendicular to $\overline{DE}$ and hence $\overline{FB}$ is parallel to $\overline{EC}$. Thus, $FBCE$ is a rectangle, since $\angle BCE$ is a right angle. So $EC = FB = \frac{1}{\sqrt{2}}$ and $FE = BC = \sqrt{\frac{3}{2}}$. Hence,

$$DE = DF + FE = \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}} = \frac{\sqrt{3}+1}{\sqrt{2}} \quad \text{and} \quad AE = AC - EC = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{2}}. \quad \text{Thus,}$$

$$\sin\ 75^\circ = \frac{DE}{AD} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}\ , \quad \cos\ 75^\circ = \frac{AE}{AD} = \frac{\frac{\sqrt{3}-1}{\sqrt{2}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}\ , \quad \text{and} \quad \tan\ 75^\circ = \frac{DE}{AE} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{\frac{\sqrt{3}-1}{\sqrt{2}}} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}\ .$$

Note: Taking reciprocals, we get $\csc\ 75^\circ = \frac{4}{\sqrt{6}+\sqrt{2}}$, $\sec\ 75^\circ = \frac{4}{\sqrt{6}-\sqrt{2}}$, and $\cot\ 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.

EXERCISES

Figure 1.2.3

For Exercises 1-10, find the values of all six trigonometric functions of angles A and B in the right triangle $\triangle ABC$ in Figure 1.2.3.

2

1. $a = 5$, $b = 12$, $c = 13$
2. $a = 8$, $b = 15$, $c = 17$

   2

3. $a = 7$, $b = 24$, $c = 25$
4. $a = 20$, $b = 21$, $c = 29$

   3

5. $a = 9$, $b = 40$, $c = 41$
6. $a = 1$, $b = 2$, $c = \sqrt{5}$
7. $a = 1$, $b = 3$

   3

8. $a = 2$, $b = 5$
9. $a = 5$, $c = 6$
10. $b = 7$, $c = 8$

For Exercises 11-18, find the values of the other five trigonometric functions of the acute angle A given the indicated value of one of the functions.

4

11. $\sin\ A = \frac{3}{4}$
12. $\cos\ A = \frac{2}{3}$
13. $\cos\ A = \frac{2}{\sqrt{10}}$
14. $\sin\ A = \frac{2}{4}$

    4

15. $\tan\ A = \frac{5}{9}$
16. $\tan\ A = 3$
17. $\sec\ A = \frac{7}{3}$
18. $\csc\ A = 3$

For Exercises 19-23, write the given number as a trigonometric function of an acute angle less than 45°.

5

19. $\sin\ 87^\circ$
20. $\sin\ 53^\circ$
21. $\cos\ 46^\circ$
22. $\tan\ 66^\circ$
23. $\sec\ 77^\circ$

For Exercises 24-28, write the given number as a trigonometric function of an acute angle greater than 45°.

5

24. $\sin\ 1^\circ$

25. $\cos\ 13^\circ$

26. $\tan\ 26^\circ$

27. $\cot\ 10^\circ$

28. $\csc\ 43^\circ$

29. In Example 1.7 we found the values of all six trigonometric functions of 60° and 30°.

    2

    1. Does $\sin\ 30^\circ + \sin\ 30^\circ = \sin\ 60^\circ$?
    2. Does $\cos\ 30^\circ + \cos\ 30^\circ = \cos\ 60^\circ$?

       2

    3. Does $\tan\ 30^\circ + \tan\ 30^\circ = \tan\ 60^\circ$?
    4. Does $2\ \sin\ 30^\circ\ \cos\ 30^\circ = \sin\ 60^\circ$?

30. For an acute angle A, can $\sin\ A$ be larger than 1? Explain your answer.

31. For an acute angle A, can $\cos\ A$ be larger than 1? Explain your answer.

32. For an acute angle A, can $\sin\ A$ be larger than $\tan\ A$? Explain your answer.

33. If A and B are acute angles and $A < B$, explain why $\sin\ A < \sin\ B$.

34. If A and B are acute angles and $A < B$, explain why $\cos\ A > \cos\ B$.

35. Prove the Cofunction Theorem (Theorem 1.2). (*Hint: Draw a right triangle and label the angles and sides.*)

36. Use Example 1.10 to find all six trigonometric functions of 15°.

Figure 1.2.4

37. In Figure 1.2.4, $\overline{CB}$ is a diameter of a circle with a radius of 2 cm and center O, $\triangle ABC$ is a right triangle, and $\overline{CD}$
    has length $\sqrt{3}$ cm.

    1. Find $\sin\ A$. (*Hint: Use Thales' Theorem.*)
    2. Find the length of $\overline{AC}$.
    3. Find the length of $\overline{AD}$.
    4. Figure 1.2.4 is drawn to scale. Use a protractor to measure the angle A, then use your calculator to find the sine of that angle. Is the calculator result close to your answer from part(a)?
       Note: Make sure that your calculator is in degree mode.

38. In Exercise 37, verify that the area of $\triangle\, ABC$ equals $\frac{1}{2} AB \cdot CD$. Why does this make sense?

39. In Exercise 37, verify that the area of $\triangle\, ABC$ equals $\frac{1}{2} AB \cdot AC \, \sin\, A$.

40. In Exercise 37, verify that the area of $\triangle\, ABC$ equals $\frac{1}{2} (BC)^2 \, \cot\, A$.

---

1. We will use the notation AB to denote the length of a line segment $\overline{AB}$. ↩

1.3 Applications and Solving Right Triangles

Throughout its early development, trigonometry was often used as a means of indirect measurement, e.g. determining large distances or lengths by using measurements of angles and small, known distances. Today, trigonometry is widely used in physics, astronomy, engineering, navigation, surveying, and various fields of mathematics and other disciplines. In this section we will see some of the ways in which trigonometry can be applied. Your calculator should be in degree mode for these examples.

Example 1.11

A person stands 150 ft away from a flagpole and measures an *angle of elevation* of 32° from his horizontal line of sight to the top of the flagpole. Assume that the person's eyes are a vertical distance of 6 ft from the ground. What is the height of the flagpole? [r]

Solution: The picture on the right describes the situation. We see that the height of the flagpole is $h + 6$ ft, where

$$\frac{h}{150} ~=~ \tan\, 32^\circ \quad\Rightarrow\quad h ~=~ 150\, \tan\, 32^\circ ~=~ 150\, (0.6249) ~=~ 94 ~.$$

How did we know that $\tan\, 32^\circ = 0.6249$? By using a calculator. And since none of the numbers we were given had decimal places, we rounded off the answer for h to the nearest integer. Thus, the height of the flagpole is $h + 6 = 94 + 6 = \boxed{100 \text{ ft}}$.

Example 1.12

A person standing 400 ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be $25°$. The person then walks 500 ft straight back and measures the angle of elevation to now be $20°$. How tall is the mountain? [r]

Solution: We will assume that the ground is flat and not inclined relative to the base of the mountain. Let h be the height of the mountain, and let x be the distance from the base of the mountain to the point directly beneath the top of the mountain, as in the picture on the right. Then we see that

$$\begin{aligned} \frac{h}{x+400} &= \tan\ 25^\circ \quad \Rightarrow \quad h \;=\; (x+400)\ \tan\ 25^\circ\ ,\ \text{and} \\ \frac{h}{x+400+500} &= \tan\ 20^\circ \quad \Rightarrow \quad h \;=\; (x+900)\ \tan\ 20^\circ\ ,\ \text{so} \end{aligned}$$

$(x+400)\ \tan\ 25^\circ \;=\; (x+900)\ \tan\ 20^\circ$, since they both equal h. Use that equation to solve for x:

$$x\ \tan\ 25^\circ \;-\; x\ \tan\ 20^\circ \;=\; 900\ \tan\ 20^\circ \;-\; 400\ \tan\ 25^\circ \quad \Rightarrow \quad x \;=\; \frac{900\ \tan\ 20^\circ \;-\; 400\ \tan\ 25^\circ}{\tan\ 25^\circ \;-\; \tan\ 20^\circ} \;=\; 1378\ \text{f}$$

Finally, substitute x into the first formula for h to get the height of the mountain:

$$h \;=\; (1378+400)\ \tan\ 25^\circ \;=\; 1778\ (0.4663) \;=\; \boxed{829\ \text{ft}}$$

Example 1.13

A blimp 4280 ft above the ground measures an *angle of depression* of 24° from its horizontal line of sight to the base of a house on the ground. Assuming the ground is flat, how far away along the ground is the house from the blimp? [r]

Solution: Let x be the distance along the ground from the blimp to the house, as in the picture to the right. Since the ground and the blimp's horizontal line of sight are parallel, we know from elementary geometry that the angle of elevation θ from the base of the house to the blimp is equal to the angle of depression from the blimp to the base of the house, i.e. $\theta = 24^\circ$. Hence,

$$\frac{4280}{x} = \tan\ 24^\circ \quad \Rightarrow \quad x = \frac{4280}{\tan\ 24^\circ} = \boxed{9613\ \text{ft}}\ .$$

Example 1.14

An observer at the top of a mountain 3 miles above sea level measures an angle of depression of 2.23° to the ocean horizon. Use this to estimate the radius of the earth.

Figure 1.3.1

Solution: We will assume that the earth is a sphere.[1] Let r be the radius of the earth. Let the point A represent the top of the mountain, and let H be the ocean horizon in the line of sight from A, as in Figure 1.3.1. Let O be the center of the earth, and let B be a point on the horizontal line of sight from A (i.e. on the line perpendicular to $\overline{OA}$). Let θ be the angle $\angle AOH$. Since A is 3 miles above sea level, we have $OA = r + 3$. Also, $OH = r$. Now since $\overline{AB} \perp \overline{OA}$, we have $\angle OAB = 90^\circ$, so we see that $\angle OAH = 90^\circ - 2.23^\circ = 87.77^\circ$. We see that the line through A and H is a tangent line to the surface of the earth (considering the surface as the circle of radius r through H as in the picture). So by Exercise 14 in Section 1.1, $\overline{AH} \perp \overline{OH}$ and hence $\angle OHA = 90^\circ$. Since the angles in the triangle $\triangle OAH$ add up to 180°, we have $\theta = 180^\circ - 90^\circ - 87.77^\circ = 2.23^\circ$. Thus,

$$\cos\,\theta \;=\; \frac{OH}{OA} \;=\; \frac{r}{r+3} \quad\Rightarrow\quad \frac{r}{r+3} \;=\; \cos\,2.23^\circ\;,$$

so solving for r we get

$$\begin{aligned} r \;=\; (r\;+\;3)\;\cos\;2.23^\circ \quad &\Rightarrow\quad r\;-\;r\;\cos\;2.23^\circ \;=\; 3\;\cos\;2.23^\circ \\ &\Rightarrow\quad r \;=\; \frac{3\;\cos\;2.23^\circ}{1\;-\;\cos\;2.23^\circ} \\ &\Rightarrow\quad \boxed{r \;=\; 3958.3\text{ miles}}\;. \end{aligned}$$

Note: This answer is very close to the earth's actual (mean) radius of 3956.6 miles.

Example 1.15

[r]

As another application of trigonometry to astronomy, we will find the distance from the earth to the sun. Let O be the center of the earth, let A be a point on the equator, and let B represent an object (e.g. a star) in space, as in the picture on the right. If the earth is positioned in such a way that the angle $\angle OAB = 90°$, then we say that the angle $\alpha = \angle OBA$ is the *equatorial parallax* of the object. The equatorial parallax of the sun has been observed to be approximately $\alpha = 0.00244°$. Use this to estimate the distance from the center of the earth to the sun.

Solution: Let B be the position of the sun. We want to find the length of $\overline{OB}$. We will use the actual radius of the earth, mentioned at the end of Example 1.14, to get $OA = 3956.6$ miles. Since $\angle OAB = 90°$, we have

$$\frac{OA}{OB} = \sin\ \alpha \quad \Rightarrow \quad OB = \frac{OA}{\sin\ \alpha} = \frac{3956.6}{\sin\ 0.00244°} = 92908394\ ,$$

so the distance from the center of the earth to the sun is approximately 93 million miles .
Note: The earth's orbit around the sun is an ellipse, so the actual distance to the sun varies.

In the above example we used a very small angle ($0.00244°$). A degree can be divided into smaller units: a **minute** is one-sixtieth of a degree, and a **second** is one-sixtieth of a minute. The symbol for a minute is $'$ and the symbol for a second is $''$. For example, $4.5° = 4°\ 30'$. And $4.505° = 4°\ 30'\ 18''$:

$$4°\ 30'\ 18'' = 4 + \frac{30}{60} + \frac{18}{3600}\ \text{degrees} = 4.505°$$

In Example 1.15 we used $\alpha = 0.00244° \approx 8.8''$, which we mention only because some angle measurement devices do use minutes and seconds.

Example 1.16

[r]

An observer on earth measures an angle of $32' \ 4''$ from one visible edge of the sun to the other (opposite) edge, as in the picture on the right. Use this to estimate the radius of the sun.

Solution: Let the point E be the earth and let S be the center of the sun. The observer's lines of sight to the visible edges of the sun are tangent lines to the sun's surface at the points A and B. Thus, $\angle EAS = \angle EBS = 90°$. The radius of the sun equals AS. Clearly $AS = BS$. So since $EB = EA$ (why?), the triangles $\triangle EAS$ and $\triangle EBS$ are similar. Thus, $\angle AES = \angle BES = \frac{1}{2} \angle AEB = \frac{1}{2} (32' \ 4'')$ $= 16' \ 2'' = (16/60) + (2/3600) = 0.26722°$.

Now, ES is the distance from the *surface* of the earth (where the observer stands) to the center of the sun. In Example 1.15 we found the distance from the *center* of the earth to the sun to be $92,908,394$ miles. Since we treated the sun in that example as a point, then we are justified in treating that distance as the distance between the centers of the earth and sun. So $ES = 92908394$ $- \ \text{radius of earth} = 92908394 - 3956.6 = 92904437.4$ miles. Hence,

$$\sin\,(\angle AES) \ = \ \frac{AS}{ES} \quad \Rightarrow \quad AS \ = \ ES \ \sin\ 0.26722° \ = \ (92904437.4)\ \sin\ 0.26722° \ = \ \boxed{433,293 \text{ miles}}\ .$$

Note: This answer is close to the sun's actual (mean) radius of $432,200$ miles.

You may have noticed that the solutions to the examples we have shown required at least one right triangle. In applied problems it is not always obvious which right triangle to use, which is why these sorts of problems can be difficult. Often no right triangle will be immediately evident, so you will have to create one. There is no general strategy for this, but remember that a right triangle requires a right angle, so look for places where you can form perpendicular line segments. When the problem contains a circle, you can create right angles by using the perpendicularity of the tangent line to the circle at a point[2] with the line that joins that point to the center of the circle. We did exactly that in Examples 1.14, 1.15, and 1.16.

Example 1.17

[r]

The machine tool diagram on the right shows a symmetric *V-block*, in which one circular roller sits on top of a smaller circular roller. Each roller touches both slanted sides of the V-block. Find the diameter d of the large roller, given the information in the diagram.

Solution: The diameter d of the large roller is twice the radius OB, so we need to find OB. To do this, we will show that $\triangle OBC$ is a right triangle, then find the angle $\angle BOC$, and then find BC. The length OB will then be simple to determine.

Since the slanted sides are tangent to each roller, $\angle ODA = \angle PEC = 90^\circ$. By symmetry, since the vertical line through the centers of the rollers makes a 37° angle with each slanted side, we have $\angle OAD = 37^\circ$. Hence, since $\triangle ODA$ is a right triangle, $\angle DOA$ is the complement of $\angle OAD$. So $\angle DOA = 53^\circ$.

Since the horizontal line segment $\overline{BC}$ is tangent to each roller, $\angle OBC = \angle PBC = 90^\circ$. Thus, $\triangle OBC$ is a right triangle. And since $\angle ODA = 90^\circ$, we know that $\triangle ODC$ is a right triangle. Now, $OB = OD$ (since they each equal the radius of the large roller), so by the Pythagorean Theorem we have $BC = DC$:

$$BC^2 = OC^2 - OB^2 = OC^2 - OD^2 = DC^2 \quad \Rightarrow \quad BC = DC$$

Thus, $\triangle OBC$ and $\triangle ODC$ are *congruent triangles* (which we denote by $\triangle OBC \cong \triangle ODC$), since their corresponding sides are equal. Thus, their corresponding angles are equal. So in particular, $\angle BOC = \angle DOC$. We know that $\angle DOB = \angle DOA = 53^\circ$. Thus,

$$53^\circ = \angle DOB = \angle BOC + \angle DOC = \angle BOC + \angle BOC = 2\angle BOC \quad \Rightarrow \quad \angle BOC = 26.5^\circ .$$

Likewise, since $BP = EP$ and $\angle PBC = \angle PEC = 90^\circ$, $\triangle BPC$ and $\triangle EPC$ are congruent right triangles. Thus, $BC = EC$. But we know that $BC = DC$, and we see from the diagram that $EC + DC = 1.38$. Thus, $BC + BC = 1.38$ and so $BC = 0.69$. We now have all we need to find OB:

$$\frac{BC}{OB} = \tan \angle BOC \quad \Rightarrow \quad OB = \frac{BC}{\tan \angle BOC} = \frac{0.69}{\tan 26.5^\circ} = 1.384$$

Hence, the diameter of the large roller is $d = 2 \times OB = 2\,(1.384) = \boxed{2.768}$.

Example 1.18

A *slider-crank mechanism* is shown in Figure 1.3.2 below. As the piston moves downward the connecting rod rotates the crank in the clockwise direction, as indicated.

Figure 1.3.2*Slider-crank mechanism*

The point A is the center of the connecting rod's *wrist pin* and only moves vertically. The point B is the center of the *crank pin* and moves around a circle of radius r centered at the point O, which is directly below A and does not move. As the crank rotates it makes an angle θ with the line $\overline{OA}$. The *instantaneous center of rotation* of the connecting rod at a given time is the point C where the horizontal line through A intersects the extended line through O and B. From Figure 1.3.2 we see that $\angle OAC = 90^\circ$, and we let $a = AC$, $b = AB$, and $c = BC$. In the exercises you will show that for $0^\circ < \theta < 90^\circ$,

$$c \;=\; \frac{\sqrt{b^2 \;-\; r^2\;(\sin\;\theta)^2}}{\cos\;\theta} \qquad \text{and} \qquad a \;=\; r\;\sin\;\theta \;+\; \sqrt{b^2 \;-\; r^2\;(\sin\;\theta)^2}\;\tan\;\theta\;.$$

[r]

For some problems it may help to remember that when a right triangle has a hypotenuse of length r and an acute angle θ, as in the picture on the right, the adjacent side will have length $r\;\cos\;\theta$ and the opposite side will have length $r\;\sin\;\theta$. You can think of those lengths as the horizontal and vertical "components" of the hypotenuse.

Notice in the above right triangle that we were given two pieces of information: one of the acute angles and the length of the hypotenuse. From this we determined the lengths of the other two sides, and the other acute angle is just the complement of the known acute angle. In general, a triangle has six parts: three sides and three angles. **Solving a triangle** means finding the unknown parts based on the known parts. In the case of a right triangle, one part is always known: one of the angles is 90°.

Example 1.19

Figure 1.3.3

Solve the right triangle in Figure 1.3.3 using the given information:

1. $c = 10,\ A = 22^\circ$

   Solution: The unknown parts are a, b, and B. Solving yields:

$$\begin{aligned} a &= c\ \sin\ A &&= 10\ \sin\ 22^\circ &&= 3.75 \\ b &= c\ \cos\ A &&= 10\ \cos\ 22^\circ &&= 9.27 \\ B &= 90^\circ\ -\ A &&= 90^\circ\ -\ 22^\circ &&= 68^\circ \end{aligned}$$

2. $b = 8,\ A = 40^\circ$

   Solution: The unknown parts are a, c, and B. Solving yields:

$$\begin{aligned} \frac{a}{b} &= \tan\ A \quad \Rightarrow \quad a = b\ \tan\ A = 8\ \tan\ 40^\circ = 6.71 \\ \frac{b}{c} &= \cos\ A \quad \Rightarrow \quad c = \frac{b}{\cos\ A} = \frac{8}{\cos\ 40^\circ} = 10.44 \end{aligned}$$

$$B = 90^\circ\ -\ A = 90^\circ\ -\ 40^\circ = 50^\circ$$

3. $a = 3,\ b = 4$

   Solution: The unknown parts are c, A, and B. By the Pythagorean Theorem,

$$c = \sqrt{a^2\ +\ b^2} = \sqrt{3^2\ +\ 4^2} = \sqrt{25} = 5\ .$$

   Now, $\tan\ A = \frac{a}{b} = \frac{3}{4} = 0.75$. So how do we find A? There should be a key labeled 1pt $\tan^{-1}$ on your calculator, which works like this: give it a number x and it will tell you the angle θ such that $\tan\ \theta = x$. In our case, we want the angle A such that $\tan\ A = 0.75$:

$$\text{Enter: } 0.75 \quad \text{Press: } \textcolor{red}{\backslash\text{setlength}\backslash\text{fboxsep}}1\text{pt}\textcolor{red}{\backslash\text{ovalbox}}\tan^{-1} \quad \text{Answer: } 36.86989765$$

   This tells us that $A = 36.87^\circ$, approximately. Thus $B = 90^\circ - A = 90^\circ - 36.87^\circ = 53.13^\circ$. Note: The 1pt $\sin^{-1}$ and 1pt $\cos^{-1}$ keys work similarly for sine and cosine, respectively. These keys use the *inverse trigonometric functions*, which we will discuss in Chapter 5.

EXERCISES

[r]

1. From a position 150 ft above the ground, an observer in a building measures angles of depression of 12° and 34° to the top and bottom, respectively, of a smaller building, as in the picture on the right. Use this to find the height h of the smaller building.

[r]

2. Generalize Example 1.12: A person standing a ft from the base of a mountain measures an angle of elevation α from the ground to the top of the mountain. The person then walks b ft straight back and measures an angle of elevation β to the top of the mountain, as in the picture on the right. Assuming the ground is level, find a formula for the height h of the mountain in terms of a, b, α, and β.
3. As the angle of elevation from the top of a tower to the sun decreases from 64° to 49° during the day, the length of the shadow of the tower increases by 92 ft along the ground. Assuming the ground is level, find the height of the tower. [r]
4. Two banks of a river are parallel, and the distance between two points A and B along one bank is 500 ft. For a point C on the opposite bank, $\angle BAC = 56^\circ$ and $\angle ABC = 41^\circ$, as in the picture on the right. What is the width w of the river?
(*Hint: Divide* $\overline{AB}$ *into two pieces.*)

[r]

5. A tower on one side of a river is directly east and north of points A and B, respectively, on the other side of the river. The top of the tower has angles of elevation α and β from A and B, respectively, as in the picture on the right. Let d be the distance between A and B. Assuming that both sides of the river are at the same elevation, show that the height h of the tower is

$$h = \frac{d}{\sqrt{(\cot\,\alpha)^2 + (\cot\,\beta)^2}} \,.$$

6. The equatorial parallax of the moon has been observed to be approximately $57'$. Taking the radius of the earth to be 3956.6 miles, estimate the distance from the center of the earth to the moon. (*Hint: See Example 1.15.*)
7. An observer on earth measures an angle of $31'\,7''$ from one visible edge of the moon to the other (opposite) edge. Use this to estimate the radius of the moon. (*Hint: Use Exercise 6 and see Example 1.16.*) [r]
8. A ball bearing sits between two metal grooves, with the top groove having an angle of 120° and the bottom groove having an angle of 90°, as in the picture on the right. What must the diameter of the ball bearing be for the distance between the vertexes of the grooves to be half an inch? You may assume that the top vertex is directly above the bottom vertex.

[r]

9. The machine tool diagram on the right shows a symmetric *worm thread*, in which a circular roller of diameter 1.5 inches sits. Find the amount d that the top of the roller rises above the top of the thread, given the information in the diagram. (*Hint: Extend the slanted sides of the thread until they meet at a point.*)

10. Repeat Exercise 9 using 1.8 inches as the distance across the top of the worm thread.

11. In Exercise 9, what would the distance across the top of the worm thread have to be to make d equal to 0 inches?

12. For $0^\circ < \theta < 90^\circ$ in the slider-crank mechanism in Example 1.18, show that

$$c \;=\; \frac{\sqrt{b^2 \;-\; r^2\,(\sin\,\theta)^2}}{\cos\,\theta} \qquad \text{and} \qquad a \;=\; r\,\sin\,\theta \;+\; \sqrt{b^2 \;-\; r^2\,(\sin\,\theta)^2}\;\tan\,\theta\;.$$

(*Hint: In Figure 1.3.2 draw line segments from B perpendicular to $\overline{OA}$ and $\overline{AC}$.*) [r]

13. The machine tool diagram on the right shows a symmetric *die punch*. In this view, the rounded tip is part of a circle of radius r, and the slanted sides are tangent to that circle and form an angle of 54°. The top and bottom sides of the die punch are horizontal. Use the information in the diagram to find the radius r.

[r]

14. In the figure on the right, $\angle\,BAC = \theta$ and $BC = a$. Use this to find AB, AC, AD, DC, CE, and DE in terms of θ and a.
(*Hint: What is the angle $\angle\,ACD$?*)

Figure 1.3.4

For Exercises 15-23, solve the right triangle in Figure 1.3.4 using the given information.

3

15. $a = 5$, $b = 12$
16. $c = 6$, $B = 35^\circ$
17. $b = 2$, $A = 8^\circ$

3

18. $a = 2$, $c = 7$
19. $a = 3$, $A = 26^\circ$
20. $b = 1$, $c = 2$

3

21. $b = 3$, $B = 26^\circ$
22. $a = 2$, $B = 8^\circ$
23. $c = 2$, $A = 45^\circ$

24. In Example 1.10 in Section 1.2, we found the exact values of all six trigonometric functions of 75°. For example, we showed that $\cot\ 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$. So since $\tan\ 15^\circ = \cot\ 75^\circ$ by the Cofunction Theorem, this means that $\tan\ 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$. We will now describe another method for finding the exact values of the trigonometric functions of 15°. In fact, it can be used to find the exact values for the trigonometric functions of $\frac{\theta}{2}$ when those for θ are known, for any $0^\circ < \theta < 90^\circ$. The method is illustrated in Figure 1.3.5 and is described below.

    inside

    Figure 1.3.5

    Draw a semicircle of radius 1 centered at a point O on a horizontal line. Let P be the point on the semicircle such that $\overline{OP}$ makes an angle of 60° with the horizontal line, as in Figure 1.3.5. Draw a line straight down from P to the horizontal line at the point Q. Now create a second semicircle as follows: Let A be the left endpoint of the first semicircle, then draw a new semicircle centered at A with radius equal to AP. Then create a third semicircle in the same way: Let B be the left endpoint of the second semicircle, then draw a new semicircle centered at B with radius equal to BP.
    This procedure can be continued indefinitely to create more semicircles. In general, it can be shown that the line segment from the center of the new semicircle to P makes an angle with the horizontal line equal to half the angle from the previous semicircle's center to P.

    1. Explain why $\angle PAQ = 30^\circ$. (*Hint: What is the supplement of* 60°*?*)
    2. Explain why $\angle PBQ = 15^\circ$ and $\angle PCQ = 7.5^\circ$.
    3. Use Figure 1.3.5 to find the exact values of $\sin\ 15^\circ$, $\cos\ 15^\circ$, and $\tan\ 15^\circ$. (*Hint: To start, you will need to use* $\angle POQ = 60^\circ$ *and* $OP = 1$ *to find the exact lengths of* $\overline{PQ}$ *and* $\overline{OQ}$*.*)
    4. Use Figure 1.3.5 to calculate the exact value of $\tan\ 7.5^\circ$.
    5. Use the same method but with an initial angle of $\angle POQ = 45^\circ$ to find the exact values of $\sin\ 22.5^\circ$, $\cos\ 22.5^\circ$, and $\tan\ 22.5^\circ$.

    [r]

25. A manufacturer needs to place ten identical ball bearings against the inner side of a circular container such that each ball bearing touches two other ball bearings, as in the picture on the right. The (inner) radius of the container is 4 cm.

    1. Find the common radius r of the ball bearings.
    2. The manufacturer needs to place a circular ring inside the container. What is the largest possible (outer) radius of the ring such that it is not on top of the ball bearings and its base is level with the base of the container?

[r]

26. A circle of radius 1 is *inscribed* inside a polygon with eight sides of equal length, called a *regular octagon*. That is, each of the eight sides is tangent to the circle, as in the picture on the right.

    1. Calculate the area of the octagon.
    2. If you were to increase the number of sides of the polygon, would the area inside it increase or decrease? What number would the area approach, if any? Explain.
    3. Inscribe a regular octagon inside the same circle. That is, draw a regular octagon such that each of its eight vertexes touches the circle. Calculate the area of this octagon.

[r]

27. The picture on the right shows a cube whose sides are of length $a > 0$.

    1. Find the length of the diagonal line segment $\overline{AB}$.
    2. Find the angle θ that $\overline{AB}$ makes with the base of the cube.

Figure 1.3.6

28. In Figure 1.3.6, suppose that α, β, and AD are known. Show that:

    1. $BC = \dfrac{AD}{\cot\ \alpha - \cot\ \beta}$
    2. $AC = \dfrac{AD\ \cdot\ \tan\ \beta}{\tan\ \beta - \tan\ \alpha}$
    3. $BD = \dfrac{AD\ \cdot\ \sin\ \alpha}{\sin\ (\beta - \alpha)}$
       (*Hint: What is the measure of the angle $\angle ABD$?*)

29. Persons A and B are at the beach, their eyes are 5 ft and 6 ft, respectively, above sea level. How many miles farther out is Person B's horizon than Person A's? (Note: 1 mile = 5280 ft)

---

1. Of course it is not perfectly spherical. The earth is an *ellipsoid*, i.e. egg-shaped, with an observed *ellipticity* of $1/297$ (a sphere has ellipticity 0). See pp. 26-27 in W.H. Munk and G.J.F MacDonald, *The Rotation of the Earth: A Geophysical Discussion*, London: Cambridge University Press, 1960. ↩
2. This will often be worded as *the line that is tangent to the circle*. ↩

1.4 Trigonometric Functions of Any Angle

To define the trigonometric functions of *any* angle - including angles less than 0° or greater than 360° - we need a more general definition of an angle. We say that an **angle** is formed by rotating a ray $\overrightarrow{OA}$ about the endpoint O (called the **vertex**), so that the ray is in a new position, denoted by the ray $\overrightarrow{OB}$.

The ray $\overrightarrow{OA}$ is called the **initial side** of the angle, and $\overrightarrow{OB}$ is the **terminal side** of the angle (see Figure 1.4.1(a)).

(a) angle $\angle AOB$

(b) positive and negative angles

Figure 1.4.1*Definition of a general angle*

We denote the angle formed by this rotation as $\angle AOB$, or simply $\angle O$, or even just O. If the rotation is counter-clockwise then we say that the angle is **positive**, and the angle is **negative** if the rotation is clockwise (see Figure 1.4.1(b)).

One full counter-clockwise rotation of $\overrightarrow{OA}$ back onto itself (called a **revolution**), so that the terminal side coincides with the initial side, is an angle of 360°; in the clockwise direction this would be -360°.[1] Not rotating $\overrightarrow{OA}$ constitutes an angle of 0°. More than one full rotation creates an angle greater than 360°. For example, notice that 30° and 390° have the same terminal side in Figure 1.4.2, since $30 + 360 = 390$.

Figure 1.4.2*Angle greater than* 360°

We can now define the trigonometric functions of any angle in terms of **Cartesian coordinates**. Recall that the **xy-coordinate plane** consists of points denoted by pairs (x, y) of real numbers. The first number, x, is the point's **x coordinate**, and the second number, y, is its **y coordinate**. The x and y coordinates are measured by their positions along the **x-axis** and **y-axis**, respectively, which determine the point's position in the plane. This divides the xy-coordinate plane into four **quadrants** (denoted by QI, QII, QIII, QIV), based on the signs of x and y (see Figure 1.4.3(a)-(b)).

(a) Quadrants I-IV

(b) Points in the plane

(c) Angle θ in the plane

Figure 1.4.3xy*-coordinate plane*

Now let θ be any angle. We say that θ is in **standard position** if its initial side is the positive x-axis and its vertex is the origin $(0, 0)$. Pick *any* point (x, y) on the terminal side of θ a distance $r > 0$ from the origin (see Figure 1.4.3(c)). (Note that $r = \sqrt{x^2 + y^2}$. Why?) We then define the trigonometric functions of θ as follows:

$$\sin\ \theta = \frac{y}{r} \qquad \cos\ \theta = \frac{x}{r} \qquad \tan\ \theta = \frac{y}{x} \tag{1.2}$$

$$\csc\ \theta\ =\ \frac{r}{y} \qquad \sec\ \theta\ =\ \frac{r}{x} \qquad \cot\ \theta\ =\ \frac{x}{y} \tag{1.3}$$

As in the acute case, by the use of similar triangles these definitions are well-defined (i.e. they do not depend on which point (x, y) we choose on the terminal side of θ). Also, notice that $|\sin\ \theta| \le 1$ and $|\cos\ \theta| \le 1$, since $|y| \le r$ and $|x| \le r$ in the above definitions. Notice that in the case of an acute angle these definitions are equivalent to our earlier definitions in terms of right triangles: draw a right triangle with angle θ such that $x = \text{adjacent side}$, $y = \text{opposite side}$, and $r = \text{hypotenuse}$. For example, this would give us $\sin\ \theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\ \theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$, just as before (see Figure 1.4.4(a)).

(a) Acute angle θ

(b) Angles by quadrant

Figure 1.4.4

In Figure 1.4.4(b) we see in which quadrants or on which axes the terminal side of an angle $0° \le \theta < 360°$ may fall. From Figure 1.4.3(a) and formulas (1.2) and (1.3), we see that we can get negative values for a trigonometric function. For example, $\sin\ \theta < 0$ when $y < 0$. Figure 1.4.5 summarizes the signs (positive or negative) for the trigonometric functions based on the angle's quadrant:

Figure 1.4.5*Signs of the trigonometric functions by quadrant*

Example 1.20

[r]

Find the exact values of all six trigonometric functions of 120°.

Solution: We know $120^\circ = 180^\circ - 60^\circ$. By Example 1.7 in Section 1.2, we see that we can use the point $(-1, \sqrt{3})$ on the terminal side of the angle 120° in QII, since we saw in that example that a basic right triangle with a 60° angle has adjacent side of length 1, opposite side of length $\sqrt{3}$, and hypotenuse of length 2, as in the figure on the right. Drawing that triangle in QII so that the hypotenuse is on the terminal side of 120° makes $r = 2$, $x = -1$, and $y = \sqrt{3}$. Hence:

$$\sin\ 120^\circ \ = \ \frac{y}{r} \ = \ \frac{\sqrt{3}}{2} \qquad \cos\ 120^\circ \ = \ \frac{x}{r} \ = \ \frac{-1}{2} \qquad \tan\ 120^\circ \ = \ \frac{y}{x} \ = \ \frac{\sqrt{3}}{-1} \ = -\sqrt{3}\backslash\mathrm{vspace}1mm$$

$$\csc\ 120^\circ \ = \ \frac{r}{y} \ = \ \frac{2}{\sqrt{3}} \qquad \sec\ 120^\circ \ = \ \frac{r}{x} \ = \ \frac{2}{-1} \ = \ -2 \qquad \cot\ 120^\circ \ = \ \frac{x}{y} \ = \ \frac{-1}{\sqrt{3}}\backslash\mathrm{vspace}1mm$$

Example 1.21

[r]

Find the exact values of all six trigonometric functions of 225°.

Solution: We know that $225^\circ = 180^\circ + 45^\circ$. By Example 1.6 in Section 1.2, we see that we can use the point $(-1, -1)$ on the terminal side of the angle 225° in QIII, since we saw in that example that a basic right triangle with a 45° angle has adjacent side of length 1, opposite side of length 1, and hypotenuse of length $\sqrt{2}$, as in the figure on the right. Drawing that triangle in QIII so that the hypotenuse is on the terminal side of 225° makes $r = \sqrt{2}$, $x = -1$, and $y = -1$. Hence:

$$\sin\ 225^\circ \ = \ \frac{y}{r} \ = \ \frac{-1}{\sqrt{2}} \qquad \cos\ 225^\circ \ = \ \frac{x}{r} \ = \ \frac{-1}{\sqrt{2}} \qquad \tan\ 225^\circ \ = \ \frac{y}{x} \ = \ \frac{-1}{-1} \ = \ 1\backslash\mathrm{vspace}1mm$$

$$\csc\ 225^\circ \ = \ \frac{r}{y} \ = \ -\sqrt{2} \qquad \sec\ 225^\circ \ = \ \frac{r}{x} \ = \ -\sqrt{2} \qquad \cot\ 225^\circ \ = \ \frac{x}{y} \ = \ \frac{-1}{-1} \ = \ 1\backslash\mathrm{vspace}1mm$$

Example 1.22

[r]

Find the exact values of all six trigonometric functions of 330°.

Solution: We know that $330^\circ = 360^\circ - 30^\circ$. By Example 1.7 in Section 1.2, we see that we can use the point $(\sqrt{3}, -1)$ on the terminal side of the angle 225° in QIV, since we saw in that example that a basic right triangle with a 30° angle has adjacent side of length $\sqrt{3}$, opposite side of length 1, and hypotenuse of length 2, as in the figure on the right. Drawing that triangle in QIV so that the hypotenuse is on the terminal side of 330° makes $r = 2$, $x = \sqrt{3}$, and $y = -1$. Hence:

$$\sin\ 330^\circ \ = \ \frac{y}{r} \ = \ \frac{-1}{2} \qquad \cos\ 330^\circ \ = \ \frac{x}{r} \ = \ \frac{\sqrt{3}}{2} \qquad \tan\ 330^\circ \ = \ \frac{y}{x} \ = \ \frac{-1}{\sqrt{3}}\textcolor{red}{\backslash\text{vspace}}1mm$$

$$\csc\ 330^\circ \ = \ \frac{r}{y} \ = \ -2 \qquad \sec\ 330^\circ \ = \ \frac{r}{x} \ = \ \frac{2}{\sqrt{3}} \qquad \cot\ 330^\circ \ = \ \frac{x}{y} \ = \ -\sqrt{3}\textcolor{red}{\backslash\text{vspace}}1mm$$

Example 1.23

Figure 1.4.6

Find the exact values of all six trigonometric functions of 0°, 90°, 180°, and 270°.

Solution: These angles are different from the angles we have considered so far, in that the terminal sides lie along either the x-axis or the y-axis. So unlike the previous examples, we do not have any right triangles to draw. However, the values of the trigonometric functions are easy to calculate by picking the simplest points on their terminal sides and then using the definitions in formulas (1.2) and (1.3).

For instance, for the angle 0° use the point $(1,0)$ on its terminal side (the positive x-axis), as in Figure 1.4.6. You could think of the line segment from the origin to the point $(1,0)$ as sort of a degenerate right triangle whose height is 0 and whose hypotenuse and base have the same length 1. Regardless, in the formulas we would use $r=1$, $x=1$, and $y=0$. Hence:

$$\sin\ 0^\circ \;=\; \frac{y}{r} \;=\; \frac{0}{1} \;=\; 0 \qquad \cos\ 0^\circ \;=\; \frac{x}{r} \;=\; \frac{1}{1} \;=\; 1 \qquad \tan\ 0^\circ \;=\; \frac{y}{x} \;=\; \frac{0}{1} \;=\; 0\backslash\text{vspace}1mm$$

$$\csc\ 0^\circ \;=\; \frac{r}{y} \;=\; \frac{1}{0} \;=\; \text{undefined} \qquad \sec\ 0^\circ \;=\; \frac{r}{x} \;=\; \frac{1}{1} \;=\; 1 \qquad \cot\ 0^\circ \;=\; \frac{x}{y} \;=\; \frac{1}{0} \;=\; \text{undefined}\backslash\text{vspace}1r$$

Note that $\csc\ 0^\circ$ and $\cot\ 0^\circ$ are undefined, since division by 0 is not allowed.

Similarly, from Figure 1.4.6 we see that for 90° the terminal side is the positive y-axis, so use the point $(0,1)$. Again, you could think of the line segment from the origin to $(0,1)$ as a degenerate right triangle whose base has length 0 and whose height equals the length of the hypotenuse. We have $r=1$, $x=0$, and $y=1$, and hence:

$$\sin\ 90^\circ \;=\; \frac{y}{r} \;=\; \frac{1}{1} \;=\; 1 \qquad \cos\ 90^\circ \;=\; \frac{x}{r} \;=\; \frac{0}{1} \;=\; 0 \qquad \tan\ 90^\circ \;=\; \frac{y}{x} \;=\; \frac{1}{0} \;=\; \text{undefined}\backslash\text{vspace}1mm$$

$$\csc\ 90^\circ \;=\; \frac{r}{y} \;=\; \frac{1}{1} \;=\; 1 \qquad \sec\ 90^\circ \;=\; \frac{r}{x} \;=\; \frac{1}{0} \;=\; \text{undefined} \qquad \cot\ 90^\circ \;=\; \frac{x}{y} \;=\; \frac{0}{1} \;=\; 0\backslash\text{vspace}1mm$$

Likewise, for 180° use the point $(-1,0)$ so that $r=1$, $x=-1$, and $y=0$. Hence:

$$\sin\ 180^\circ \;=\; \frac{y}{r} \;=\; \frac{0}{1} \;=\; 0 \qquad \cos\ 180^\circ \;=\; \frac{x}{r} \;=\; \frac{-1}{1} \;=\; -1 \qquad \tan\ 180^\circ \;=\; \frac{y}{x} \;=\; \frac{0}{-1} \;=\; 0\backslash\text{vspace}1mm$$

$$\csc\ 180^\circ \ = \ \frac{r}{y} \ = \ \frac{1}{0} \ = \ \text{undefined} \qquad \sec\ 180^\circ \ = \ \frac{r}{x} \ = \ \frac{1}{-1} \ = \ -1 \qquad \cot\ 180^\circ \ = \ \frac{x}{y} \ = \ \frac{-1}{0} \ = \ \text{undefine}$$

Lastly, for 270° use the point $(0,-1)$ so that $r = 1$, $x = 0$, and $y = -1$. Hence:

$$\sin\ 270^\circ \ = \ \frac{y}{r} \ = \ \frac{-1}{1} \ = \ -1 \qquad \cos\ 270^\circ \ = \ \frac{x}{r} \ = \ \frac{0}{1} \ = \ 0 \qquad \tan\ 270^\circ \ = \ \frac{y}{x} \ = \ \frac{-1}{0} \ = \ \text{undefined}\text{\textbackslash vspa}$$

$$\csc\ 270^\circ \ = \ \frac{r}{y} \ = \ \frac{1}{-1} \ = \ -1 \qquad \sec\ 270^\circ \ = \ \frac{r}{x} \ = \ \frac{1}{0} \ = \ \text{undefined} \qquad \cot\ 270^\circ \ = \ \frac{x}{y} \ = \ \frac{0}{-1} \ = \ 0\text{\textbackslash vspa}$$

The following table summarizes the values of the trigonometric functions of angles between 0° and 360° which are integer multiples of 30° or 45°:

Table 1.3Table of trigonometric function values

1.3

Angle	mm $\sin$ mm	m $\cos$ mm	$\tan$	$\csc$	$\sec$
$0°$	0	1	0	undefined	1
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
$90°$	1	0	undefined	1	undefined
$120°$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2
$135°$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	$\sqrt{2}$	$-\sqrt{2}$
$150°$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$
$180°$	0	-1	0	undefined	-1
$210°$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$
$225°$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$
$240°$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2
$270°$	-1	0	undefined	-1	undefined
$300°$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2
$315°$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\sqrt{2}$	$\sqrt{2}$
$330°$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$

Since $360°$ represents one full revolution, the trigonometric function values repeat every $360°$. For example, $\sin\ 360° = \sin\ 0°$, $\cos\ 390° = \cos\ 30°$, $\tan\ 540° = \tan\ 180°$, $\sin(-45°) = \sin\ 315°$, etc. In general, if two angles differ by an integer multiple of $360°$ then each trigonometric function will have equal values at both angles. Angles such as these, which have the same initial and terminal sides, are called **coterminal**.

In Examples 1.20-1.22, we saw how the values of trigonometric functions of an angle θ larger than $90°$ were found by using a certain acute angle as part of a right triangle. That acute angle has a special name: if θ is a nonacute angle then we say that the **reference angle** for θ is the acute angle formed by the terminal side of θ and either the positive or negative x-axis. So in Example 1.20, we see that $60°$ is the reference angle for the nonacute angle $\theta = 120°$; in Example 1.21, $45°$ is the reference angle for $\theta = 225°$; and in Example 1.22, $30°$ is the reference angle for $\theta = 330°$.

Example 1.24

Figure 1.4.7

Let $\theta = 928^\circ$.

1. Which angle between 0° and 360° has the same terminal side (and hence the same trigonometric function values) as θ?
2. What is the reference angle for θ?

Solution: (a) Since $928^\circ = 2 \times 360^\circ + 208^\circ$, then θ has the same terminal side as 208°, as in Figure 1.4.7.

(b) 928° and 208° have the same terminal side in QIII, so the reference angle for $\theta = 928^\circ$ is $208^\circ - 180^\circ = 28^\circ$.

Example 1.25

Suppose that $\cos\ \theta = -\frac{4}{5}$. Find the exact values of $\sin\ \theta$ and $\tan\ \theta$.

Solution: We can use a method similar to the one used to solve Example 1.8 in Section 1.2. That is, draw a right triangle and interpret $\cos\ \theta$ as the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ of two of its sides. Since $\cos\ \theta = -\frac{4}{5}$, we can use 4 as the length of the adjacent side and 5 as the length of the hypotenuse. By the Pythagorean Theorem, the length of the opposite side must then be 3. Since $\cos\ \theta$ is negative, we know from Figure 1.4.5 that θ must be in either QII or QIII. Thus, we have *two* possibilities, as shown in Figure 1.4.8 below:

(a) θ in QII

(b) θ in QIII

Figure 1.4.8$\cos\ \theta = -\frac{4}{5}$

When θ is in QII, we see from Figure 1.4.8(a) that the point $(-4, 3)$ is on the terminal side of θ, and so we have $x = -4$, $y = 3$, and $r = 5$. Thus, $\sin\ \theta = \frac{y}{r} = \frac{3}{5}$ and $\tan\ \theta = \frac{y}{x} = \frac{3}{-4}$.

When θ is in QIII, we see from Figure 1.4.8(b) that the point $(-4, -3)$ is on the terminal side of θ, and so we have $x = -4$, $y = -3$, and $r = 5$. Thus, $\sin\ \theta = \frac{y}{r} = \frac{-3}{5}$ and $\tan\ \theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$.

Thus, either $\sin\ \theta = \frac{3}{5}$ and $\tan\ \theta = -\frac{3}{4}$ or $\sin\ \theta = -\frac{3}{5}$ and $\tan\ \theta = \frac{3}{4}$.

Since reciprocals have the same sign, $\csc\ \theta$ and $\sin\ \theta$ have the same sign, $\sec\ \theta$ and $\cos\ \theta$ have the same sign, and $\cot\ \theta$ and $\tan\ \theta$ have the same sign. So it suffices to remember the signs of $\sin\ \theta$, $\cos\ \theta$, and $\tan\ \theta$:

For an angle θ in standard position and a point (x, y) on its terminal side:

1. $\sin\ \theta$ has the same sign as y
2. $\cos\ \theta$ has the same sign as x
3. $\tan\ \theta$ is positive when x and y have the same sign
4. $\tan\ \theta$ is negative when x and y have opposite signs

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EXERCISES

For Exercises 1-10, state in which quadrant or on which axis the given angle lies.

5

1. 127°
2. -127°
3. 313°
4. -313°
5. -90°

   5

6. 621°
7. 230°
8. 2009°
9. 1079°
10. -514°
11. In which quadrant(s) do sine and cosine have the same sign?
12. In which quadrant(s) do sine and cosine have the opposite sign?
13. In which quadrant(s) do sine and tangent have the same sign?
14. In which quadrant(s) do sine and tangent have the opposite sign?
15. In which quadrant(s) do cosine and tangent have the same sign?
16. In which quadrant(s) do cosine and tangent have the opposite sign?

For Exercises 17-21, find the reference angle for the given angle.

5

17. 317°
18. 63°
19. -126°
20. 696°
21. 275°

For Exercises 22-26, find the exact values of $\sin\,\theta$ and $\tan\,\theta$ when $\cos\,\theta$ has the indicated value.

5

22. $\cos\ \theta = \frac{1}{2}$
23. $\cos\ \theta = -\frac{1}{2}$
24. $\cos\ \theta = 0$
25. $\cos\ \theta = \frac{2}{5}$
26. $\cos\ \theta = 1$

For Exercises 27-31, find the exact values of $\cos\ \theta$ and $\tan\ \theta$ when $\sin\ \theta$ has the indicated value.

5

27. $\sin\ \theta = \frac{1}{2}$
28. $\sin\ \theta = -\frac{1}{2}$
29. $\sin\ \theta = 0$
30. $\sin\ \theta = -\frac{2}{3}$
31. $\sin\ \theta = 1$

For Exercises 32-36, find the exact values of $\sin\ \theta$ and $\cos\ \theta$ when $\tan\ \theta$ has the indicated value.

5

32. $\tan\ \theta = \frac{1}{2}$
33. $\tan\ \theta = -\frac{1}{2}$
34. $\tan\ \theta = 0$
35. $\tan\ \theta = \frac{5}{12}$
36. $\tan\ \theta = 1$

For Exercises 37-40, use Table 1.4 to answer the following questions.

2

37. Does $\sin\ 180^\circ\ +\ \sin\ 45^\circ\ =\ \sin\ 225^\circ$?
38. Does $\tan\ 300^\circ\ -\ \tan\ 30^\circ\ =\ \tan\ 270^\circ$?

    2

39. Does $\cos\ 180^\circ\ -\ \cos\ 60^\circ\ =\ \cos\ 120^\circ$?
40. Does $\cos\ 240^\circ\ =\ (\cos\ 120^\circ)^2\ -\ (\sin\ 120^\circ)^2$?
41. Expand Table 1.4 to include all integer multiples of 15°. See Example 1.10 in Section 1.2.

1. The system of measuring angles in degrees, such that 360° is one revolution, originated in ancient Babylonia. It is often assumed that the number 360 was used because the Babylonians (supposedly) thought that there were 360 days in a year (a year, of course, is one full revolution of the Earth around the Sun). However, there is another, perhaps more likely, explanation which says that in ancient times a person could travel 12 *Babylonian miles* in one day (i.e. one full rotation of the Earth about its axis). The Babylonian mile was large enough (approximately 7 of our miles) to be divided into 30 equal parts for convenience, thus giving $12 \times 30 = 360$ equal parts in a full rotation. See p.26 in H. Eves, *An Introduction to the History of Mathematics*, 5th ed., New York: Saunders College Publishing, 1983. ↩

1.5 Rotations and Reflections of Angles

Now that we know how to deal with angles of any measure, we will take a look at how certain geometric operations can help simplify the use of trigonometric functions of any angle, and how some basic relations between those functions can be made. The two operations on which we will concentrate in this section are *rotation* and *reflection*.

To **rotate an angle** means to rotate its terminal side around the origin when the angle is in standard position. For example, suppose we rotate an angle θ around the origin by 90° in the counterclockwise direction. In Figure 1.5.1 we see an angle θ in QI which is rotated by 90°, resulting in the angle $\theta + 90^\circ$ in QII. Notice that the complement of θ in the right triangle in QI is the same as the supplement of the angle $\theta + 90^\circ$ in QII, since the sum of θ, its complement, and 90° equals 180°. This forces the other angle of the right triangle in QII to be θ.

Figure 1.5.1*Rotation of an angle θ by 90°*

Thus, the right triangle in QI is similar to the right triangle in QII, since the triangles have the same angles. The rotation of θ by 90° does not change the length r of its terminal side, so the hypotenuses of the similar right triangles are equal, and hence by similarity the remaining corresponding sides are also equal. Using Figure 1.5.1 to match up those corresponding sides shows that the point $(-y, x)$ is on the terminal side of $\theta + 90^\circ$ when (x, y) is on the terminal side of θ. Hence, by definition,

$$\sin\,(\theta + 90^\circ) ~=~ \frac{x}{r} ~=~ \cos\,\theta ~, \quad \cos\,(\theta + 90^\circ) ~=~ \frac{-y}{r} ~=~ -\sin\,\theta ~, \quad \tan\,(\theta + 90^\circ) ~=~ \frac{x}{-y} ~=~ -\cot\,\theta ~.$$

Though we showed this for θ in QI, it is easy (see Exercise 4) to use similar arguments for the other quadrants. In general, the following relations hold for all angles θ:

$$\sin\,(\theta + 90^\circ) ~=~ \cos\,\theta \tag{1.4}$$

$$\cos\,(\theta + 90^\circ) ~=~ -\sin\,\theta \tag{1.5}$$

$$\tan(\theta + 90^\circ) \;=\; -\cot\,\theta \tag{1.6}$$

Example 1.26

Recall that any nonvertical line in the xy-coordinate plane can be written as $y = mx + b$, where m is the *slope* of the line (defined as $m = \frac{\text{rise}}{\text{run}}$) and b is the y-*intercept*, i.e. where the line crosses the y-axis (see Figure 1.5.2(a)). We will show that the slopes of perpendicular lines are negative reciprocals. That is, if $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are nonvertical and nonhorizontal perpendicular lines, then $m_2 = -\frac{1}{m_1}$ (see Figure 1.5.2(b)).

(a) Slope of a line

(b) Perpendicular lines

Figure 1.5.2

First, suppose that a line $y = mx + b$ has nonzero slope. The line crosses the x-axis somewhere, so let θ be the angle that the positive x-axis makes with the part of the line above the x-axis, as in Figure 1.5.3. For $m > 0$ we see that θ is acute and $\tan\ \theta = \frac{\text{rise}}{\text{run}} = m$.

(a) $m > 0$, θ acute, rise > 0

(b) $m < 0$, θ obtuse, rise < 0

Figure 1.5.3

If $m < 0$, then we see that θ is obtuse and the rise is negative. Since the run is always positive, our definition of $\tan\ \theta$ from Section 1.4 means that $\tan\ \theta = \frac{-\text{rise}}{-\text{run}} = \frac{\text{rise}}{\text{run}} = m$ (just imagine in Figure 1.5.3(b) the entire line being shifted horizontally to go through the origin, so that θ is unchanged and the point $(-\text{run}, -\text{rise})$ is on the terminal side of θ). Hence:

> For a line $y = mx + b$ with $m \neq 0$, the slope is given by $m = \tan\ \theta$, where θ is the angle formed by the positive x-axis and the part of the line above the x-axis.

Now, in Figure 1.5.2(b) we see that if two lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are perpendicular then rotating one line counterclockwise by 90° around the point of intersection gives us the second line. So if θ is the angle that the line $y = m_1 x + b_1$ makes with the positive x-axis, then $\theta + 90^\circ$ is the angle that the line $y = m_2 x + b_2$ makes with the positive x-axis. So by what we just showed, $m_1 = \tan\ \theta$ and $m_2 = \tan\ (\theta + 90°)$. But by formula (1.6) we know that $\tan\ (\theta + 90°) = -\cot\ \theta$. Hence, $m_2 = -\cot\ \theta = -\frac{1}{\tan\ \theta} = -\frac{1}{m_1}$. qed

Rotating an angle θ by $90°$ in the clockwise direction results in the angle $\theta - 90°$. We could use another geometric argument to derive trigonometric relations involving $\theta - 90°$, but it is easier to use a simple trick: since formulas (1.4)–(1.6) hold for *any* angle θ, just replace θ by $\theta - 90°$ in each formula. Since $(\theta - 90°) + 90° = \theta$, this gives us:

$$\sin\ (\theta - 90°) \ = \ -\cos\ \theta \tag{1.7}$$

$$\cos\ (\theta - 90°) \ = \ \sin\ \theta \tag{1.8}$$

$$\tan\ (\theta - 90°) \ = \ -\cot\ \theta \tag{1.9}$$

We now consider rotating an angle θ by $180°$. Notice from Figure 1.5.4 that the angles $\theta \pm 180°$ have the same terminal side, and are in the quadrant opposite θ.

(a) QI and QIII

(b) QII and QIV

Figure 1.5.4*Rotation of θ by $\pm 180°$*

Since $(-x, -y)$ is on the terminal side of $\theta \pm 180°$ when (x, y) is on the terminal side of θ, we get the following relations, which hold for all θ:

$$\sin\ (\theta \pm 180°) \ = \ -\sin\ \theta \tag{1.10}$$

$$\cos\ (\theta \pm 180°) \ = \ -\cos\ \theta \tag{1.11}$$

$$\tan\,(\theta \pm 180^\circ) \;=\; \tan\,\theta \tag{1.12}$$

Figure 1.5.5

A **reflection** is simply the mirror image of an object. For example, in Figure 1.5.5 the original object is in QI, its reflection around the y-axis is in QII, and its reflection around the x-axis is in QIV. Notice that if we first reflect the object in QI around the y-axis and then follow that with a reflection around the x-axis, we get an image in QIII. That image is the *reflection around the origin* of the original object, and it is equivalent to a rotation of 180° around the origin. Notice also that a reflection around the y-axis is equivalent to a reflection around the x-axis followed by a rotation of 180° around the origin. Applying this to angles, we see that the reflection of an angle θ around the x-axis is the angle $-\theta$, as in Figure 1.5.6.

(a) QI and QIV

(b) QII and QIII

Figure 1.5.6*Reflection of θ around the x-axis*

So we see that reflecting a point (x, y) around the x-axis just replaces y by $-y$. Hence:

$$\sin\,(-\theta) \;=\; -\sin\,\theta \tag{1.13}$$

$$\cos\,(-\theta) \;=\; \cos\,\theta \tag{1.14}$$

$$\tan\,(-\theta) \;=\; -\tan\,\theta \tag{1.15}$$

Notice that the cosine function does not change in formula (1.14) because it depends on x, and not on y, for a point (x, y) on the terminal side of θ.

In general, a function $f(x)$ is an **even function** if $f(-x) = f(x)$ for all x, and it is called an **odd function** if $f(-x) = -f(x)$ for all x. Thus, the cosine function is even, while the sine and tangent functions are odd.

Replacing θ by $-\theta$ in formulas (1.4)$-$(1.6), then using formulas (1.13)$-$(1.15), gives:

$$\sin\,(90^\circ - \theta) \;=\; \cos\,\theta \tag{1.16}$$

$$\cos\left(90^\circ - \theta\right) \;=\; \sin\;\theta \tag{1.17}$$

$$\tan\left(90^\circ - \theta\right) \;=\; \cot\;\theta \tag{1.18}$$

Note that formulas (1.16)–(1.18) extend the Cofunction Theorem from Section 1.2 to *all* θ, not just acute angles. Similarly, formulas (1.10)–(1.12) and (1.13)–(1.15) give:

$$\sin\left(180^\circ - \theta\right) \;=\; \sin\;\theta \tag{1.19}$$

$$\cos\left(180^\circ - \theta\right) \;=\; -\cos\;\theta \tag{1.20}$$

$$\tan\left(180^\circ - \theta\right) \;=\; -\tan\;\theta \tag{1.21}$$

Notice that reflection around the y-axis is equivalent to reflection around the x-axis ($\theta \mapsto -\theta$) followed by a rotation of 180° ($-\theta \mapsto -\theta + 180^\circ = 180^\circ - \theta$), as in Figure 1.5.7.

Figure 1.5.7*Reflection of θ around the y-axis $= 180^\circ - \theta$*

It may seem that these geometrical operations and formulas are not necessary for evaluating the trigonometric functions, since we could just use a calculator. However, there are two reasons for why they are useful. First, the formulas work for any angles, so they are often used to prove general formulas in mathematics and other fields, as we will see later in the text. Second, they can help in determining which angles have a given trigonometric function value.

Example 1.27

Find all angles $0^\circ \leq \theta < 360^\circ$ such that $\sin\ \theta = -0.682$.

Solution: Using the 1pt $\sin^{-1}$ button on a calculator with -0.682 as the input, we get $\theta = -43^\circ$, which is not between 0° and 360°.[1] Since $\theta = -43^\circ$ is in QIV, its reflection $180^\circ - \theta$ around the y-axis will be in QIII and have the same sine value. But $180^\circ - \theta = 180^\circ - (-43^\circ) = 223^\circ$ (see Figure 1.5.8). Also, we know that -43° and $-43^\circ + 360^\circ = 317^\circ$ have the same trigonometric function values. So since angles in QI and QII have positive sine values, we see that the only angles between 0° and 360° with a sine of -0.682 are $\boxed{\theta = 223^\circ \text{ and } 317^\circ}$.

Figure 1.5.8*Reflection around the y-axis: -43° and 223°*

EXERCISES

1. Let $\theta = 32^\circ$. Find the angle between 0° and 360° which is the
   1. reflection of θ around the x-axis
   2. reflection of θ around the y-axis
   3. reflection of θ around the origin

   2
2. Repeat Exercise 1 with $\theta = 248^\circ$.
3. Repeat Exercise 1 with $\theta = -248^\circ$.
4. We proved formulas (1.4)-(1.6) for any angle θ in QI. Mimic that proof to show that the formulas hold for θ in QII.
5. Verify formulas (1.4)-(1.6) for θ on the coordinate axes, i.e. for $\theta = 0^\circ$, 90°, 180°, 270°.
6. In Example 1.26 we used the formulas involving $\theta + 90^\circ$ to prove that the slopes of perpendicular lines are negative reciprocals. Show that this result can also be proved using the formulas involving $\theta - 90^\circ$. (*Hint: Only the last paragraph in that example needs to be modified.*)

For Exercises 7 - 14, find all angles $0^\circ \leq \theta < 360^\circ$ which satisfy the given equation:

4

7. $\sin\ \theta = 0.4226$
8. $\sin\ \theta = 0.1909$
9. $\cos\ \theta = 0.4226$
10. $\sin\ \theta = 0$

    4

11. $\tan\ \theta = 0.7813$
12. $\sin\ \theta = -0.6294$
13. $\cos\ \theta = -0.9816$
14. $\tan\ \theta = -9.514$

    [r]

15. In our proof of the Pythagorean Theorem in Section 1.2, we claimed that in a right triangle $\triangle ABC$ it was possible to draw a line segment $\overline{CD}$ from the right angle vertex C to a point D on the hypotenuse $\overline{AB}$ such that $\overline{CD} \perp \overline{AB}$. Use the picture on the right to prove that claim. (*Hint: Notice how $\triangle ABC$ is placed on the xy-coordinate plane. What is the slope of the hypotenuse? What would be the slope of a line perpendicular to it?*) Also, find the (x,y) coordinates of the point D in terms of a and b.

16. It can be proved without using trigonometric functions that the slopes of perpendicular lines are negative reciprocals. Let $y = m_1 x + b_1$ and $y = m_2 x + b_2$ be perpendicular lines (with nonzero slopes), as in the picture below. Use the picture to show that $m_2 = -\frac{1}{m_1}$.
    (*Hint: Think of similar triangles and the definition of slope.*)

17. Prove formulas (1.19)-(1.21) by using formulas (1.10)$-$(1.12) and (1.13)$-$(1.15).

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1. In Chapter 5 we will discuss why the 1pt $\sin^{-1}$ button returns that value. ↩