9 Linear Higher Order Equations

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IN THIS CHAPTER we extend the results obtained in Chapter 5 for linear second order equations to linear higher order equations.

SECTION 9.1 presents a theoretical introduction to linear higher order equations.

SECTION 9.2 discusses higher order constant coefficient homogeneous equations.

SECTION 9.3 presents the method of undetermined coefficients for higher order equations.

SECTION 9.4 extends the method of variation of parameters to higher order equations.

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Answers to selected exercises in this chapter

9.1 Introduction to Linear Higher Order Equations

An nth order differential equation is said to be linear if it can be written in the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x). \tag{9.1.1}$$

We considered equations of this form with $n = 1$ in Section 2.1 and with $n = 2$ in Chapter 5. In this chapter n is an arbitrary positive integer.

In this section we sketch the general theory of linear nth order equations. Since this theory has already been discussed for $n = 2$ in Sections 5.1 and 5.3, we'll omit proofs.

For convenience, we consider linear differential equations written as

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x), \tag{9.1.2}$$

which can be rewritten as (9.1.1) on any interval on which P_0 has no zeros, with $p_1 = P_1/P_0$, …, $p_n = P_n/P_0$ and $f = F/P_0$. For simplicity, throughout this chapter we'll abbreviate the left side of (9.1.2) by Ly; that is,

$$Ly = P_0y^{(n)} + P_1y^{(n-1)} + \cdots + P_ny.$$

We say that the equation $Ly = F$ is normal on (a, b) if P_0, P_1, …, P_n and F are continuous on (a, b) and P_0 has no zeros on (a, b). If this is so then $Ly = F$ can be written as (9.1.1) with p_1, …, p_n and f continuous on (a, b).

The next theorem is analogous to Theorem 5.3.1.

THEOREM 9.1.1

Suppose $Ly = F$ is normal on (a, b), let x_0 be a point in (a, b), and let k_0, k_1, …, k_{n-1} be arbitrary real numbers. Then the initial value problem

$$Ly = F, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \dots, \quad y^{(n-1)}(x_0) = k_{n-1}$$

has a unique solution on (a, b).

Homogeneous Equations

Eqn. (9.1.2) is said to be homogeneous if $F \equiv 0$ and nonhomogeneous otherwise. Since $y \equiv 0$ is obviously a solution of $Ly = 0$, we call it the trivial solution. Any other solution is nontrivial.

If y_1, y_2, …, y_n are defined on (a, b) and c_1, c_2, …, c_n are constants, then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \tag{9.1.3}$$

is a linear combination of $\{y_1, y_2 \dots, y_n\}$. It's easy to show that if y_1, y_2, …, y_n are solutions of $Ly = 0$ on (a, b), then so is any linear combination of $\{y_1, y_2, \dots, y_n\}$. (See the proof of Theorem 5.1.2.) We say that $\{y_1, y_2, \dots, y_n\}$ is a fundamental set of solutions of $Ly = 0$ on (a, b) if every solution of $Ly = 0$ on (a, b) can be written as a linear combination of $\{y_1, y_2, \dots, y_n\}$, as in (9.1.3). In this case we say that (9.1.3) is the general solution of $Ly = 0$ on (a, b).

It can be shown (Exercises 14 and 15) that if the equation $Ly = 0$ is normal on (a, b) then it has infinitely many fundamental sets of solutions on (a, b). The next definition will help to identify fundamental sets of solutions of $Ly = 0$.

We say that $\{y_1, y_2, \dots, y_n\}$ is linearly independent on (a, b) if the only constants c_1, c_2, …, c_n such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0, \quad a < x < b, \tag{9.1.4}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If (9.1.4) holds for some set of constants c_1, c_2, …, c_n that are not all zero, then $\{y_1, y_2, \dots, y_n\}$ is linearly dependent on (a, b)

The next theorem is analogous to Theorem 5.1.3.

Theorem 9.1.2

If $Ly=0$ is normal on (a,b), then a set $\{y_1,y_2,\dots,y_n\}$ of n solutions of $Ly=0$ on (a,b) is a fundamental set if and only if it's linearly independent on (a,b).

Example 9.1.1

The equation

$$x^3y'''-x^2y''-2xy'+6y=0 \tag{9.1.5}$$

is normal and has the solutions $y_1=x^2$, $y_2=x^3$, and $y_3=1/x$ on $(-\infty,0)$ and $(0,\infty)$. Show that $\{y_1,y_2,y_3\}$ is linearly independent on $(-\infty,0)$ and $(0,\infty)$. Then find the general solution of (9.1.5) on $(-\infty,0)$ and $(0,\infty)$.

Solution Suppose

$$c_1x^2+c_2x^3+\frac{c_3}{x}=0 \tag{9.1.6}$$

on $(0,\infty)$. We must show that $c_1=c_2=c_3=0$. Differentiating (9.1.6) twice yields the system

$$\begin{array}{rcl} c_1x^2+c_2x^3+\backslash\mathrm{dst}\frac{c_3}{x} & = & 0 \\ 2c_1x+3c_2x^2-\backslash\mathrm{dst}\frac{c_3}{x^2} & = & 0 \\ 2c_1+6c_2x+\backslash\mathrm{dst}\frac{2c_3}{x^3} & = & 0. \end{array} \tag{9.1.7}$$

If (9.1.7) holds for all x in $(0,\infty)$, then it certainly holds at $x=1$; therefore,

$$\begin{array}{rcl} c_1+c_2+c_3 & = & 0 \\ 2c_1+3c_2-c_3 & = & 0 \\ 2c_1+6c_2+2c_3 & = & 0. \end{array} \tag{9.1.8}$$

By solving this system directly, you can verify that it has only the trivial solution $c_1=c_2=c_3=0$; however, for our purposes it's more useful to recall from linear algebra that a homogeneous linear system of n equations in n unknowns has only the trivial solution if its determinant is nonzero. Since the determinant of (9.1.8) is

$$\left|\begin{array}{rrr} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 6 & 2 \end{array}\right| = \left|\begin{array}{rrr} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 2 & 4 & 0 \end{array}\right| = 12,$$

it follows that (9.1.8) has only the trivial solution, so $\{y_1, y_2, y_3\}$ is linearly independent on $(0, \infty)$. Now Theorem 9.1.2 implies that

$$y = c_1x^2 + c_2x^3 + \frac{c_3}{x}$$

is the general solution of (9.1.5) on $(0, \infty)$. To see that this is also true on $(-\infty, 0)$, assume that (9.1.6) holds on $(-\infty, 0)$. Setting $x = -1$ in (9.1.7) yields

$$\begin{aligned} c_1 - c_2 - c_3 &= 0 \\ -2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 - 6c_2 - 2c_3 &= 0. \end{aligned}$$

Since the determinant of this system is

$$\left|\begin{array}{rrr} 1 & -1 & -1 \\ -2 & 3 & -1 \\ 2 & -6 & -2 \end{array}\right| = \left|\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 2 & -4 & 0 \end{array}\right| = -12,$$

it follows that $c_1 = c_2 = c_3 = 0$; that is, $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$.

Example 9.1.2

The equation

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0 \tag{9.1.9}$$

is normal and has the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ on $(-\infty, \infty)$. (Verify.) Show that $\{y_1, y_2, y_3, y_4\}$ is linearly independent on $(-\infty, \infty)$. Then find the general solution of (9.1.9).

Solution Suppose c_1, c_2, c_3, and c_4 are constants such that

$$c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-3x} = 0 \tag{9.1.10}$$

for all x. We must show that $c_1 = c_2 = c_3 = c_4 = 0$. Differentiating (9.1.10) three times yields the system

$$\begin{aligned} c_1e^x + c_2e^{-x} + \ \ c_3e^{2x} + \ \ \ c_4e^{-3x} &= 0 \\ c_1e^x - c_2e^{-x} + 2c_3e^{2x} - \ \ 3c_4e^{-3x} &= 0 \\ c_1e^x + c_2e^{-x} + 4c_3e^{2x} + \ \ 9c_4e^{-3x} &= 0 \\ c_1e^x - c_2e^{-x} + 8c_3e^{2x} - 27c_4e^{-3x} &= 0. \end{aligned} \tag{9.1.11}$$

If (9.1.11) holds for all x, then it certainly holds for $x = 0$. Therefore

$$\begin{aligned} c_1 + c_2 + \ \ c_3 + \ \ \ c_4 &= 0 \\ c_1 - c_2 + 2c_3 - \ \ 3c_4 &= 0 \\ c_1 + c_2 + 4c_3 + \ \ 9c_4 &= 0 \\ c_1 - c_2 + 8c_3 - 27c_4 &= 0. \end{aligned}$$

The determinant of this system is

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 \\ 0 & 0 & 3 & 8 \\ 0 & -2 & 7 & -28 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -4 \\ 0 & 3 & 8 \\ -2 & 7 & -28 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 1 & -4 \\ 0 & 3 & 8 \\ 0 & 6 & -24 \end{vmatrix} = -2 \begin{vmatrix} 3 & 8 \\ 6 & -24 \end{vmatrix} = 240, \end{aligned} \tag{9.1.12}$$

so the system has only the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$. Now Theorem 9.1.2 implies that

$$y = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-3x}$$

is the general solution of (9.1.9).

The Wronskian

We can use the method used in Examples 9.1.1 and 9.1.2 to test n solutions $\{y_1, y_2, \dots, y_n\}$ of any nth order equation $Ly = 0$ for linear independence on an interval (a, b) on which the equation is normal. Thus, if c_1, c_2 ,…, c_n are constants such that

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0, \quad a < x < b,$$

then differentiating $n - 1$ times leads to the $n \times n$ system of equations

$$\begin{array}{rcl} c_1y_1(x)+c_2y_2(x)+ & \cdots & +c_ny_n(x)=0 \\ c_1y_1'(x)+c_2y_2'(x)+ & \cdots & +c_ny_n'(x)=0 \\ & \vdots & \\ c_1y_1^{(n-1)}(x)+c_2y_2^{(n-1)}(x)+ & \cdots & +c_ny_n^{(n-1)}(x)=0 \end{array} \tag{9.1.13}$$

for c_1, c_2, …, c_n. For a fixed x, the determinant of this system is

$$W(x)=\left|\begin{array}{cccc} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{array}\right|.$$

We call this determinant the Wronskian (http://www-history.mcs.st-and.ac.uk/Mathematicians/Wronski.html) of $\{y_1,y_2,\dots,y_n\}$. If $W(x)\neq 0$ for some x in (a,b) then the system (9.1.13) has only the trivial solution $c_1=c_2=\cdots=c_n=0$, and Theorem 9.1.2 implies that

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n$$

is the general solution of $Ly=0$ on (a,b).

The next theorem generalizes Theorem 5.1.4. The proof is sketched in (Exercises 17–20).

Theorem 9.1.3

Suppose the homogeneous linear nth order equation

$$P_0(x)y^{(n)}+P_1(x)y^{n-1}+\cdots+P_n(x)y=0 \tag{9.1.14}$$

is normal on (a,b), let y_1, y_2, …, y_n be solutions of (9.1.14) on (a,b), and let x_0 be in (a,b). Then the Wronskian of $\{y_1,y_2,\dots,y_n\}$ is given by

$$W(x)=W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{P_1(t)}{P_0(t)}\,dt\right\},\quad a<x<b. \tag{9.1.15}$$

Therefore, either W has no zeros in (a,b) or $W\equiv 0$ on (a,b).

Formula (9.1.15) is Abel's formula (http://www-history.mcs.st-and.ac.uk/Mathematicians/Abel.html).

The next theorem is analogous to Theorem 5.1.6..

Theorem 9.1.4

Suppose $Ly=0$ is normal on (a,b) and let y_1, y_2, …, y_n be n solutions of $Ly=0$ on (a,b). Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $Ly=0$ on (a,b) is $y=c_1y_1+c_2y_2+\cdots+c_ny_n$.

b. $\{y_1,y_2,\ldots,y_n\}$ is a fundamental set of solutions of $Ly=0$ on (a,b).

c. $\{y_1,y_2,\ldots,y_n\}$ is linearly independent on (a,b).

d. The Wronskian of $\{y_1,y_2,\ldots,y_n\}$ is nonzero at some point in (a,b).

e. The Wronskian of $\{y_1,y_2,\ldots,y_n\}$ is nonzero at all points in (a,b).

Example 9.1.3

In Example 9.1.1 we saw that the solutions $y_1=x^2$, $y_2=x^3$, and $y_3=1/x$ of

$$x^3y'''-x^2y''-2xy'+6y=0$$

are linearly independent on $(-\infty,0)$ and $(0,\infty)$. Calculate the Wronskian of $\{y_1,y_2,y_3\}$.

Solution If $x\neq 0$, then

$$W(x)=\left|\begin{array}{ccc} x^2 & x^3 & \backslash\mathrm{dst}\frac{1}{x} \\ 2x & 3x^2 & \backslash\mathrm{dst}-\backslash\mathrm{dst}\frac{1}{x^2} \\ 2 & 6x & \backslash\mathrm{dst}\frac{2}{x^3} \end{array}\right| = 2x^3\left|\begin{array}{ccc} 1 & x & \backslash\mathrm{dst}\frac{1}{x^3} \\ 2 & 3x & -\backslash\mathrm{dst}\frac{1}{x^3} \\ 1 & 3x & \backslash\mathrm{dst}\frac{1}{x^3} \end{array}\right|,$$

where we factored x^2, x, and 2 out of the first, second, and third rows of $W(x)$, respectively. Adding the second row of the last determinant to the first and third rows yields

$$W(x)=2x^3\left|\begin{array}{ccc} 3 & 4x & 0 \\ 2 & 3x & -\backslash\mathrm{dst}\frac{1}{x^3} \\ 3 & 6x & 0 \end{array}\right| = 2x^3\left(\frac{1}{x^3}\right)\left|\begin{array}{cc} 3 & 4x \\ 3 & 6x \end{array}\right| = 12x.$$

Therefore $W(x)\neq 0$ on $(-\infty,0)$ and $(0,\infty)$.

Example 9.1.4

In Example 9.1.2 we saw that the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

are linearly independent on every open interval. Calculate the Wronskian of $\{y_1, y_2, y_3, y_4\}$.

Solution For all x,

$$W(x) = \begin{vmatrix} e^x & e^{-x} & e^{2x} & e^{-3x} \\ e^x & -e^{-x} & 2e^{2x} & -3e^{-3x} \\ e^x & e^{-x} & 4e^{2x} & 9e^{-3x} \\ e^x & -e^{-x} & 8e^{2x} & -27e^{-3x} \end{vmatrix}.$$

Factoring the exponential common factor from each row yields

$$W(x) = e^{-x} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} = 240e^{-x},$$

from (9.1.12).

Remark

Under the assumptions of Theorem 9.1.4, it isn't necessary to obtain a formula for $W(x)$. Just evaluate $W(x)$ at a convenient point in (a, b), as we did in Examples 9.1.1 and 9.1.2.

Theorem 9.1.5

Suppose c is in (a, b) and α_1, α_2, …, are real numbers, not all zero. Under the assumptions of Theorem 10.3.3, suppose y_1 and y_2 are solutions of (5.1.35) such that

$$\alpha y_i(c) + y_i'(c) + \cdots + y_i^{(n-1)}(c) = 0, \quad 1 \le i \le n. \tag{9.1.16}$$

Then $\{y_1, y_2, \ldots y_n\}$ isn't linearly independent on (a, b).

Proof Since α_1, α_2, ..., α_n are not all zero, (9.1.14) implies that

$$\left|\begin{array}{cccc} y_1(c) & y_1'(c) & \cdots & y_1^{(n-1)}(c) \\ y_2(c) & y_2'(c) & \cdots & y_2^{(n-1)}(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_n(c) & y_n'(c) & \cdots & y_n^{(n-1)}(c) \end{array}\right| = 0,$$

so

$$\left|\begin{array}{cccc} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c)(c) & \cdots & y_n^{(n-1)}(c)(c) \end{array}\right| = 0$$

and Theorem 9.1.4 implies the stated conclusion.

General Solution of a Nonhomogeneous Equation

The next theorem is analogous to Theorem 5.3.2. It shows how to find the general solution of $Ly = F$ if we know a particular solution of $Ly = F$ and a fundamental set of solutions of the complementary equation $Ly = 0$.

Theorem 9.1.6

Suppose $Ly = F$ is normal on (a, b). Let y_p be a particular solution of $Ly = F$ on (a, b), and let $\{y_1, y_2, \dots, y_n\}$ be a fundamental set of solutions of the complementary equation $Ly = 0$ on (a, b). Then y is a solution of $Ly = F$ on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where $c_1, c_2, \dots, c_n$ are constants.

The next theorem is analogous to Theorem 5.3.2.

Theorem 9.1.7

Suppose for each $i = 1$, 2, ..., r, the function y_{p_i} is a particular solution of $Ly = F_i$ on (a, b). Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_r}$$

is a particular solution of

$$Ly = F_1(x) + F_2(x) + \cdots + F_r(x)$$

on (a, b).

We'll apply Theorems 9.1.6 and 9.1.7 throughout the rest of this chapter.

9.1 Exercises

1. Verify that the given function is the solution of the initial value problem.

(a) $x^3y''' - 3x^2y'' + 6xy' - 6y = -\dst\frac{24}{x}, \quad y(-1) = 0, \; y'(-1) = 0, \quad y''(-1) = 0;$

$\dst y = -6x - 8x^2 - 3x^3 + \frac{1}{x}$

(b) $\dst y''' - \frac{1}{x}y'' - y' + \frac{1}{x}y = \frac{x^2-4}{x^4}, \quad y(1) = \frac{3}{2}, \quad y'(1) = \frac{1}{2}, \; y''(1) = 1;$

$y = x + \dst\frac{1}{2x}$

(c) $xy''' - y'' - xy' + y = x^2, \quad y(1) = 2, \quad y'(1) = 5, \quad y''(1) = -1;$

$y = -x^2 - 2 + 2e^{(x-1)} - e^{-(x-1)} + 4x$

(d) $4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2, \quad y(1) = 5, \quad y'(1) = \dst\frac{17}{2};$

$y''(1) = \dst\frac{63}{4}; \quad y = 2x^2 \ln x - x^{1/2} + 2x^{-1/2} + 4x^2$

(e) $x^4y^{(4)} - 4x^3y''' + 12x^2y'' - 24xy' + 24y = 6x^4, \quad y(1) = -2,$

$y'(1) = -9, \quad y''(1) = -27, \quad y'''(1) = -52;$

$y = x^4 \ln x + x - 2x^2 + 3x^3 - 4x^4$

(f) $xy^{(4)} - y''' - 4xy'' + 4y' = 96x^2, \quad y(1) = -5, \quad y'(1) = -24$

$y''(1) = -36; \quad y'''(1) = -48; \quad y = 9 - 12x + 6x^2 - 8x^3$

2. Solve the initial value problem

$$x^3y''' - x^2y'' - 2xy' + 6y = 0, \quad y(-1) = -4, \quad y'(-1) = -14, \quad y''(-1) = -20.$$

Hint: See Example ???.

Show answer

3. Solve the initial value problem

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0, \quad y(0) = 5, \quad y'(0) = -6, \quad y''(0) = 10, \quad y'''(0) - 36.$$

Hint: See Example ???.

Show answer

4. Find solutions y_1, y_2, …, y_n of the equation $y^{(n)} = 0$ that satisfy the initial conditions

$$y_i^{(j)}(x_0) = \begin{cases} 0, & j \neq i-1, \\ 1, & j = i-1, \end{cases} \quad 1 \leq i \leq n.$$

Show answer

5. a. Verify that the function

$$y = c_1 x^3 + c_2 x^2 + \frac{c_3}{x}$$

satisfies

$$x^3 y''' - x^2 y'' - 2xy' + 6y = 0 \tag{A}$$

if c_1, c_2, and c_3 are constants.

b. Use **(a)** to find solutions y_1, y_2, and y_3 of (A) such that

$$\begin{aligned} y_1(1) &= 1, \quad y_1'(1) = 0, \quad y_1''(1) = 0 \\ y_2(1) &= 0, \quad y_2'(1) = 1, \quad y_2''(1) = 0 \\ y_3(1) &= 0, \quad y_3'(1) = 0, \quad y_3''(1) = 1. \end{aligned}$$

c. Use **(b)** to find the solution of (A) such that

$$y(1) = k_0, \quad y'(1) = k_1, \quad y''(1) = k_2.$$

Show answer

6. Verify that the given functions are solutions of the given equation, and show that they form a fundamental set of solutions of the equation on any interval on which the equation is normal.

(a) $y''' + y'' - y' - y = 0; \quad \{e^x,\, e^{-x},\, xe^{-x}\}$

(b) $y''' - 3y'' + 7y' - 5y = 0; \quad \{e^x,\, e^x\cos 2x,\, e^x\sin 2x\}$.

(c) $xy''' - y'' - xy' + y = 0; \quad \{e^x,\, e^{-x},\, x\}$

(d) $x^2y''' + 2xy'' - (x^2+2)y = 0; \quad \dst\{e^x/x,\, e^{-x}/x,\, 1\}$

(e) $(x^2 - 2x + 2)y''' - x^2y'' + 2xy' - 2y = 0; \quad \{x,\, x^2,\, e^x\}$

(f) $(2x-1)y^{(4)} - 4xy''' + (5-2x)y'' + 4xy' - 4y = 0; \quad \{x,\, e^x,\, e^{-x}, e^{2x}\}$

(g) $xy^{(4)} - y''' - 4xy' + 4y' = 0; \quad \{1, x^2,\, e^{2x},\, e^{-2x}\}$

7. Find the Wronskian W of a set of three solutions of

$$y''' + 2xy'' + e^xy' - y = 0,$$

given that $W(0) = 2$.

Show answer

8. Find the Wronskian W of a set of four solutions of

$$y^{(4)} + (\tan x)y''' + x^2y'' + 2xy = 0,$$

given that $W(\pi/4) = K$.

Show answer

9. a. Evaluate the Wronskian $W\ \{e^x,\, xe^x,\, x^2e^x\}$. Evaluate $W(0)$.

b. Verify that y_1, y_2, and y_3 satisfy

$$y''' - 3y'' + 3y' - y = 0. \tag{A}$$

c. Use $W(0)$ from **(a)** and Abel's formula to calculate $W(x)$.

d. What is the general solution of (A)?

Show answer

10. Compute the Wronskian of the given set of functions.

(a) $\{1,\, e^x,\, e^{-x}\}$	**(b)** $\{e^x,\, e^x\sin x,\, e^x\cos x\}$

(c) $\{2,\, x+1,\, x^2+2\}$	**(d)** $x,\, x\ln x,\, 1/x\}$

(e) \dst$\{1,\, x,\, \frac{x^2}{2!},\, \frac{x^3}{3!},\cdots,\, \frac{x^n}{n!}\}$	**(f)** $\{e^x,\, e^{-x},\, x\}$

(g) $\{e^x/x,\, e^{-x}/x,\, 1\}$	**(h)** $\{x,\, x^2,\, e^x\}$

(i) $\{x,\, x^3,\, 1/x,\, 1/x^2\}$	**(j)** $\{e^x,\, e^{-x},\, x,\, e^{2x}\}$

(k) $\{e^{2x},\, e^{-2x},\, 1,\, x^2\}$

Show answer

11. Suppose $Ly=0$ is normal on (a,b) and x_0 is in (a,b). Use Theorem 9.1.1 to show that $y\equiv 0$ is the only solution of the initial value problem

$$Ly=0,\quad y(x_0)=0,\quad y'(x_0)=0,\ldots,y^{(n-1)}(x_0)=0,$$

on (a,b).

12. Prove: If y_1, y_2, …, y_n are solutions of $Ly=0$ and the functions

$$z_i=\sum_{j=1}^{n}a_{ij}y_j,\quad 1\le i\le n,$$

form a fundamental set of solutions of $Ly=0$, then so do y_1, y_2, …, y_n.

13. Prove: If

$$y=c_1y_1+c_2y_2+\cdots+c_ky_k+y_p$$

is a solution of a linear equation $Ly=F$ for every choice of the constants c_1, c_2 ,…, c_k, then $Ly_i=0$ for $1\le i\le k$.

14. Suppose $Ly = 0$ is normal on (a, b) and let x_0 be in (a, b). For $1 \le i \le n$, let y_i be the solution of the initial value problem

$$Ly_i = 0, \quad y_i^{(j)}(x_0) = \left\{ \begin{array}{ll} 0, & j \ne i-1, \\ [3pt]1, & j = i-1, \end{array} \right. \quad 1 \le i \le n,$$

where x_0 is an arbitrary point in (a, b). Show that any solution of $Ly = 0$ on (a, b), can be written as

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

with $c_j = y^{(j-1)}(x_0)$.

15. Suppose $\{y_1, y_2, \dots, y_n\}$ is a fundamental set of solutions of

$$\backslash\mathrm{dst} P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$$

on (a, b), and let

$$\begin{array}{rl} z_1 & = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ z_2 & = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ \vdots & \quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ z_n & = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n, \end{array}$$

where the $\{a_{ij}\}$ are constants. Show that $\{z_1, z_2, \dots, z_n\}$ is a fundamental set of solutions of (A) if and only if the determinant

$$\left| \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right|$$

is nonzero.*Hint: The determinant of a product of $n \times n$ matrices equals the product of the determinants.*

16. Show that $\{y_1, y_2, \dots, y_n\}$ is linearly dependent on (a, b) if and only if at least one of the functions y_1, y_2, …, y_n can be written as a linear combination of the others on (a, b).

Take the following as a hint in Exercises 17–19:

By the definition of determinant,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum \pm a_{1i_1} a_{2i_2}, \dots, a_{ni_n},$$

where the sum is over all permutations $(i_1, i_2, \dots, i_n)$ *of* $(1, 2, \dots, n)$ *and the choice of* $+$ *or* $-$ *in each term depends only on the permutation associated with that term.*

17. Prove: If

$$A(u_1, u_2, \dots, u_n) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ u_1 & u_2 & \cdots & u_n \end{vmatrix},$$

then

$$A(u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) = A(u_1, u_2, \dots, u_n) + A(v_1, v_2, \dots, v_n).$$

18. Let

$$F = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix},$$

where $f_{ij}\ (1 \le i,\ j \le n)$ is differentiable. Show that

$$F' = F_1 + F_2 + \cdots + F_n,$$

where F_i is the determinant obtained by differentiating the ith row of F.

19. Use Exercise 18 to show that if W is the Wronskian of the n-times differentiable functions y_1, y_2, …, y_n, then

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

20. Use Exercises 17 and 19 to show that if W is the Wronskian of solutions $\{y_1, y_2, \dots, y_n\}$ of the normal equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0, \tag{A}$$

then $W' = -P_1W/P_0$. Derive Abel's formula (Eqn. (9.1.15)) from this. *Hint: Use (A) to write* $y^{(n)}$ *in terms of* $y, y', \dots, y^{(n-1)}$.

21. Prove Theorem 9.1.6.

22. Prove Theorem 9.1.7.

23. Show that if the Wronskian of the n-times continuously differentiable functions $\{y_1, y_2, \dots, y_n\}$ has no zeros in (a, b), then the differential equation obtained by expanding the determinant

$$\begin{vmatrix} y & y_1 & y_2 & \cdots & y_n \\ y' & y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = 0,$$

in cofactors of its first column is normal and has $\{y_1, y_2, \dots, y_n\}$ as a fundamental set of solutions on (a, b).

24. Use the method suggested by Exercise 23 to find a linear homogeneous equation such that the given set of functions is a fundamental set of solutions on intervals on which the Wronskian of the set has no zeros.

(a) $\{x,\, x^2-1,\, x^2+1\}$	**(b)** $\{e^x,\, e^{-x},\, x\}$
(c) $\{e^x,\, xe^{-x},\, 1\}$	**(d)** $\{x,\, x^2,\, e^x\}$
(e) $\{x,\, x^2,\, 1/x\}$	**(f)** $\{x+1,\, e^x,\, e^{3x}\}$
(g) $\{x,\, x^3,\, 1/x,\, 1/x^2\}$	**(h)** $\{x,\, x\ln x,\, 1/x,\, x^2\}$
(i) $\{e^x,\, e^{-x},\, x,\, e^{2x}\}$	**(j)** $\{e^{2x},\, e^{-2x},\, 1,\, x^2\}$

Show answer

9.2 Higher Order Constant Coefficient Homogeneous Equations

If a_0, a_1, …, a_n are constants and $a_0 \neq 0$, then

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(x)$$

is said to be a constant coefficient equation. In this section we consider the homogeneous constant coefficient equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0. \tag{9.2.1}$$

Since (9.2.1) is normal on $(-\infty, \infty)$, the theorems in Section 9.1 all apply with $(a, b) = (-\infty, \infty)$.

As in Section 5.2, we call

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n \tag{9.2.2}$$

the characteristic polynomial of (9.2.1). We saw in Section 5.2 that when $n = 2$ the solutions of (9.2.1) are determined by the zeros of the characteristic polynomial. This is also true when $n > 2$, but the situation is more complicated in this case. Consequently, we take a different approach here than in Section 5.2.

If k is a positive integer, let D^k stand for the k-th derivative operator; that is

$$D^k y = y^{(k)}.$$

If

$$q(r) = b_0 r^m + b_1 r^{m-1} + \cdots + b_m$$

is an arbitrary polynomial, define the operator

$$q(D) = b_0 D^m + b_1 D^{m-1} + \cdots + b_m$$

such that

$$q(D)y = (b_0 D^m + b_1 D^{m-1} + \cdots + b_m)y = b_0 y^{(m)} + b_1 y^{(m-1)} + \cdots + b_m y$$

whenever y is a function with m derivatives. We call $q(D)$ a polynomial operator.

With p as in (9.2.2),

$$p(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_n,$$

so (9.2.1) can be written as $p(D)y = 0$. If r is a constant then

$$\begin{aligned} p(D)e^{rx} &= \left(a_0 D^n e^{rx} + a_1 D^{n-1} e^{rx} + \cdots + a_n e^{rx}\right) \\ &= (a_0 r^n + a_1 r^{n-1} + \cdots + a_n)e^{rx}; \end{aligned}$$

that is

$$p(D)(e^{rx}) = p(r)e^{rx}.$$

This shows that $y = e^{rx}$ is a solution of (9.2.1) if $p(r) = 0$. In the simplest case, where p has n distinct real zeros r_1, r_2,…, r_n, this argument yields n solutions

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}, \dots, \quad y_n = e^{r_n x}.$$

It can be shown (Exercise 39) that the Wronskian of $\{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}$ is nonzero if r_1, r_2, …, r_n are distinct; hence, $\{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}$ is a fundamental set of solutions of $p(D)y = 0$ in this case.

Example 9.2.1

a. Find the general solution of

$$y''' - 6y'' + 11y' - 6y = 0. \tag{9.2.3}$$

b. Solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 5, \quad y''(0) = 9. \tag{9.2.4}$$

Solution The characteristic polynomial of (9.2.3) is

$$p(r) = r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3).$$

Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a set of solutions of (9.2.3). It is a fundamental set, since its Wronskian is

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0.$$

Therefore the general solution of (9.2.3) is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}. \tag{9.2.5}$$

Solution (b) We must determine c_1, c_2 and c_3 in (9.2.5) so that y satisfies the initial conditions in (9.2.4). Differentiating (9.2.5) twice yields

$$\begin{aligned} y' &= c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x} \\ y'' &= c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}. \end{aligned} \tag{9.2.6}$$

Setting $x = 0$ in (9.2.5) and (9.2.6) and imposing the initial conditions yields

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_1 + 2c_2 + 3c_3 &= 5 \\ c_1 + 4c_2 + 9c_3 &= 9. \end{aligned}$$

The solution of this system is $c_1 = 4$, $c_2 = -1$, $c_3 = 1$. Therefore the solution of (9.2.4) is

$$y = 4e^x - e^{2x} + e^{3x}$$

(Figure 9.2.1). ■

Figure 9.2.1. $y = 4e^x - e^{2x} + e^{3x}$

Now we consider the case where the characteristic polynomial (9.2.2) does not have n distinct real zeros. For this purpose it is useful to define what we mean by a factorization of a polynomial operator. We begin with an example.

Example 9.2.2

Consider the polynomial

$$p(r) = r^3 - r^2 + r - 1$$

and the associated polynomial operator

$$p(D) = D^3 - D^2 + D - 1.$$

Since $p(r)$ can be factored as

$$p(r) = (r-1)(r^2+1) = (r^2+1)(r-1),$$

it's reasonable to expect that p(D) can be factored as

$$p(D) = (D-1)(D^2+1) = (D^2+1)(D-1). \tag{9.2.7}$$

However, before we can make this assertion we must define what we mean by saying that two operators are equal, and what we mean by the products of operators in (9.2.7). We say that two operators are equal if they apply to the same functions and always produce the same result. The definitions of the products in (9.2.7) is this: if y is any three-times differentiable function then

a. $(D-1)(D^2+1)y$ is the function obtained by first applying D^2+1 to y and then applying $D-1$ to the resulting function
b. $(D^2+1)(D-1)y$ is the function obtained by first applying $D-1$ to y and then applying D^2+1 to the resulting function.

From **(a)**,

$$\begin{aligned}(D-1)(D^2+1)y &= (D-1)[(D^2+1)y] \\ &= (D-1)(y''+y) = D(y''+y) - (y''+y) \\ &= (y'''+y') - (y''+y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y.\end{aligned} \tag{9.2.8}$$

This implies that

$$(D-1)(D^2+1) = (D^3 - D^2 + D - 1).$$

From **(b)**,

$$\begin{aligned}(D^2+1)(D-1)y &= (D^2+1)[(D-1)y] \\ &= (D^2+1)(y'-y) = D^2(y'-y) + (y'-y) \\ &= (y'''-y'') + (y'-y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y,\end{aligned} \tag{9.2.9}$$

$$(D^2+1)(D-1)=(D^3-D^2+D-1),$$

which completes the justification of (9.2.7).

Example 9.2.3

Use the result of Example 9.2.2 to find the general solution of

$$y'''-y''+y'-y=0. \tag{9.2.10}$$

Solution From (9.2.8), we can rewrite (9.2.10) as

$$(D-1)(D^2+1)y=0,$$

which implies that any solution of $(D^2+1)y=0$ is a solution of (9.2.10). Therefore $y_1=\cos x$ and $y_2=\sin x$ are solutions of (9.2.10).

From (9.2.9), we can rewrite (9.2.10) as

$$(D^2+1)(D-1)y=0,$$

which implies that any solution of $(D-1)y=0$ is a solution of (9.2.10). Therefore $y_3=e^x$ is solution of (9.2.10).

The Wronskian of $\{e^x,\cos x,\sin x\}$ is

$$W(x)=\left|\begin{array}{rrr}\cos x & \sin x & e^x\\ -\sin x & \cos x & e^x\\ -\cos x & -\sin x & e^x\end{array}\right|.$$

Since

$$W(0)=\left|\begin{array}{rrr}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{array}\right|=2,$$

$\{\cos x,\sin x,e^x\}$ is linearly independent and

$$y = c_1 \cos x + c_2 \sin x + c_3 e^x$$

is the general solution of (9.2.10).

Example 9.2.4

Find the general solution of

$$y^{(4)} - 16y = 0. \tag{9.2.11}$$

Solution The characteristic polynomial of (9.2.11) is

$$p(r) = r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r - 2)(r + 2)(r^2 + 4).$$

By arguments similar to those used in Examples 9.2.2 and 9.2.3, it can be shown that (9.2.11) can be written as

$$(D^2 + 4)(D + 2)(D - 2)y = 0$$

or

$$(D^2 + 4)(D - 2)(D + 2)y = 0$$

or

$$(D - 2)(D + 2)(D^2 + 4)y = 0.$$

Therefore y is a solution of (9.2.11) if it's a solution of any of the three equations

$$(D - 2)y = 0, \quad (D + 2)y = 0, \quad (D^2 + 4)y = 0.$$

Hence, $\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is a set of solutions of (9.2.11). The Wronskian of this set is

$$W(x)=\begin{vmatrix} e^{2x} & e^{-2x} & \cos 2x & \sin 2x\\ 2e^{2x} & -2e^{-2x} & -2\sin 2x & 2\cos 2x\\ 4e^{2x} & 4e^{-2x} & -4\cos 2x & -4\sin 2x\\ 8e^{2x} & -8e^{-2x} & 8\sin 2x & -8\cos 2x\end{vmatrix}.$$

Since

$$W(0)=\begin{vmatrix} 1 & 1 & 1 & 0\\ 2 & -2 & 0 & 2\\ 4 & 4 & -4 & 0\\ 8 & -8 & 0 & -8\end{vmatrix}=-512,$$

$\{e^{2x},e^{-2x},\cos 2x,\sin 2x\}$ is linearly independent, and

$$y_1=c_1e^{2x}+c_2e^{-2x}+c_3\cos 2x+c_4\sin 2x$$

is the general solution of (9.2.11).

It is known from algebra that every polynomial

$$p(r)=a_0r^n+a_1r^{n-1}+\cdots+a_n$$

with real coefficients can be factored as

$$p(r)=a_0p_1(r)p_2(r)\cdots p_k(r),$$

where no pair of the polynomials p_1, p_2, …, p_k has a commom factor and each is either of the form

$$p_j(r)=(r-r_j)^{m_j}, \tag{9.2.12}$$

where r_j is real and m_j is a positive integer, or

$$p_j(r)=\left[(r-\lambda_j)^2+\omega_j^2\right]^{m_j}, \tag{9.2.13}$$

where λ_j and ω_j are real, $\omega_j\neq 0$, and m_j is a positive integer. If (9.2.12) holds then r_j is a real zero of p, while if (9.2.13) holds then $\lambda+i\omega$ and $\lambda-i\omega$ are complex conjugate zeros of p. In either case, m_j is the multiplicity of the zero(s).

By arguments similar to those used in our examples, it can be shown that

$$p(D) = a_0 p_1(D) p_2(D) \cdots p_k(D) \tag{9.2.14}$$

and that the order of the factors on the right can be chosen arbitrarily. Therefore, if $p_j(D)y = 0$ for some j then $p(D)y = 0$. To see this, we simply rewrite (9.2.14) so that $p_j(D)$ is applied first. Therefore the problem of finding solutions of $p(D)y = 0$ with p as in (9.2.14) reduces to finding solutions of each of these equations

$$p_j(D)y = 0, \quad 1 \le j \le k,$$

where p_j is a power of a first degree term or of an irreducible quadratic. To find a fundamental set of solutions $\{y_1, y_2, \dots, y_n\}$ of $p(D)y = 0$, we find fundamental set of solutions of each of the equations and take $\{y_1, y_2, \dots, y_n\}$ to be the set of all functions in these separate fundamental sets. In Exercise 40 we sketch the proof that $\{y_1, y_2, \dots, y_n\}$ is linearly independent, and therefore a fundamental set of solutions of $p(D)y = 0$.

To apply this procedure to general homogeneous constant coefficient equations, we must be able to find fundamental sets of solutions of equations of the form

$$(D - a)^m y = 0$$

and

$$\left[(D - \lambda)^2 + \omega^2\right]^m y = 0,$$

where m is an arbitrary positive integer. The next two theorems show how to do this.

Theorem 9.2.1

If m is a positive integer, then

$$\{e^{ax}, xe^{ax}, \dots, x^{m-1}e^{ax}\} \tag{9.2.15}$$

is a fundamental set of solutions of

$$(D - a)^m y = 0. \tag{9.2.16}$$

Proof We'll show that if

$$f(x) = c_1 + c_2 x + \cdots + c_m x^{m-1}$$

is an arbitrary polynomial of degree $\le m-1$, then $y = e^{ax} f$ is a solution of (9.2.16). First note that if g is any differentiable function then

$$(D-a)e^{ax}g = De^{ax}g - ae^{ax}g = ae^{ax}g + e^{ax}g' - ae^{ax}g,$$

so

$$(D-a)e^{ax}g = e^{ax}g'. \tag{9.2.17}$$

Therefore

$$\begin{array}{rcll}
(D-a)e^{ax}f & = & e^{ax}f' & \text{(from (???) with } g=f) \\
(D-a)^2 e^{ax}f & = & (D-a)e^{ax}f' = e^{ax}f'' & \text{(from (???) with } g=f') \\
(D-a)^3 e^{ax}f & = & (D-a)e^{ax}f'' = e^{ax}f''' & \text{(from (???) with } g=f'') \\
 & \vdots & & \\
(D-a)^m e^{ax}f & = & (D-a)e^{ax}f^{(m-1)} = e^{ax}f^{(m)} & \text{(from (???) with } g=f^{(m-1)}).
\end{array}$$

Since $f^{(m)} = 0$, the last equation implies that $y = e^{ax} f$ is a solution of (9.2.16) if f is any polynomial of degree $\le m-1$. In particular, each function in (9.2.15) is a solution of (9.2.16). To see that (9.2.15) is linearly independent (and therefore a fundamental set of solutions of (9.2.16)), note that if

$$c_1 e^{ax} + c_2 x e^{ax} + c \cdots + c_{m-1} x^{m-1} e^{ax} = 0$$

for all x in some interval (a, b), then

$$c_1 + c_2 x + c \cdots + c_{m-1} x^{m-1} = 0$$

for all x in (a, b). However, we know from algebra that if this polynomial has more than $m-1$ zeros then $c_1 = c_2 = \cdots = c_n = 0$.

Example 9.2.5

Find the general solution of

$$y''' + 3y'' + 3y' + y = 0. \tag{9.2.18}$$

Solution The characteristic polynomial of (9.2.18) is

$$p(r) = r^3 + 3r^2 + 3r + 1 = (r+1)^3.$$

Therefore (9.2.18) can be written as

$$(D+1)^3 y = 0,$$

so Theorem 9.2.1 implies that the general solution of (9.2.18) is

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The proof of the next theorem is sketched in Exercise 41.

Theorem 9.2.2

If $\omega \neq 0$ and m is a positive integer, then

$$\begin{aligned} \{e^{\lambda x}\cos\omega x, xe^{\lambda x}\cos\omega x, &\quad \dots, x^{m-1}e^{\lambda x}\cos\omega x, \\ e^{\lambda x}\sin\omega x, xe^{\lambda x}\sin\omega x, &\quad \dots, x^{m-1}e^{\lambda x}\sin\omega x\} \end{aligned}$$

is a fundamental set of solutions of

$$[(D-\lambda)^2 + \omega^2]^m y = 0.$$

Example 9.2.6

Find the general solution of

$$(D^2 + 4D + 13)^3 y = 0. \tag{9.2.19}$$

Solution The characteristic polynomial of (9.2.19) is

$$p(r) = (r^2 + 4r + 13)^3 = \left((r+2)^2 + 9\right)^3.$$

Therefore (9.2.19) can be be written as

$$[(D+2)^2 + 9]^3 y = 0,$$

so Theorem 9.2.2 implies that the general solution of (9.2.19) is

$$y = (a_1 + a_2 x + a_3 x^2)e^{-2x}\cos 3x + (b_1 + b_2 x + b_3 x^2)e^{-2x}\sin 3x.$$

Example 9.2.7

Find the general solution of

$$y^{(4)} + 4y''' + 6y'' + 4y' = 0. \tag{9.2.20}$$

Solution The characteristic polynomial of (9.2.20) is

$$\begin{aligned} p(r) &= r^4 + 4r^3 + 6r^2 + 4r \\ &= r(r^3 + 4r^2 + 6r + 4) \\ &= r(r+2)(r^2 + 2r + 2) \\ &= r(r+2)[(r+1)^2 + 1]. \end{aligned}$$

Therefore (9.2.20) can be written as

$$[(D+1)^2 + 1](D+2)Dy = 0.$$

Fundamental sets of solutions of

$$\left[(D+1)^2 + 1\right] y = 0, \quad (D+2)y = 0, \quad \text{and} \quad Dy = 0.$$

are given by

$$\{e^{-x}\cos x, e^{-x}\sin x\}, \quad \{e^{-2x}\}, \quad \text{and} \quad \{1\},$$

respectively. Therefore the general solution of (9.2.20) is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + c_3 e^{-2x} + c_4.$$

Example 9.2.8

Find a fundamental set of solutions of

$$[(D+1)^2+1]^2(D-1)^3(D+1)D^2y = 0. \tag{9.2.21}$$

Solution A fundamental set of solutions of (9.2.21) can be obtained by combining fundamental sets of solutions of

$$\backslash\mathrm{dst}\ \left[(D+1)^2+1\right]^2 y = 0, \quad (D-1)^3 y = 0,$$
$$\backslash\mathrm{dst}(D+1)y = 0, \quad \text{and} \quad D^2 y = 0.$$

Fundamental sets of solutions of these equations are given by

$$\backslash\mathrm{dst}\{e^{-x}\cos x, xe^{-x}\cos x, e^{-x}\sin x, xe^{-x}\sin x\}, \quad \{e^x, xe^x, x^2e^x\},$$
$$\backslash\mathrm{dst}\{e^{-x}\}, \text{ and } \{1, x\},$$

respectively. These ten functions form a fundamental set of solutions of (9.2.21).

9.2 Exercises

In Exercises 1–14 find the general solution.

1. $y''' - 3y'' + 3y' - y = 0$

Show answer

2. $y^{(4)} + 8y'' - 9y = 0$

Show answer

3. $y''' - y'' + 16y' - 16y = 0$

Show answer

4. $2y''' + 3y'' - 2y' - 3y = 0$

Show answer

5. $y''' + 5y'' + 9y' + 5y = 0$

Show answer

6. $4y''' - 8y'' + 5y' - y = 0$

Show answer

7. $27y''' + 27y'' + 9y' + y = 0$

Show answer

8. $y^{(4)} + y'' = 0$

Show answer

9. $y^{(4)} - 16y = 0$

Show answer

10. $y^{(4)} + 12y'' + 36y = 0$

Show answer

11. $16y^{(4)} - 72y'' + 81y = 0$

Show answer

12. $6y^{(4)} + 5y''' + 7y'' + 5y' + y = 0$

Show answer

13. $4y^{(4)} + 12y''' + 3y'' - 13y' - 6y = 0$

Show answer

14. $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = 0$

Show answer

In Exercises 15–27 solve the initial value problem. Where indicated by **C/G**, *graph the solution.*

15. $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = -2, \; y''(0) = 0$

Show answer

16. $y''' + 3y'' - y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 14, \quad y''(0) = -40$

Show answer

17. C/G $y''' - y'' - y' + y = 0, \quad y(0) = -2, \quad y'(0) = 9, \quad y''(0) = 4$

Show answer

18. C/G $y''' - 2y' - 4y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 22$

Show answer

19. C/G

$3y''' - y'' - 7y' + 5y = 0, \quad y(0) = $ \dst $\frac{14}{5}, \quad y'(0) = 0, \quad y''(0) = 10$

Show answer

20. $y''' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = -4$

Show answer

21. $2y''' - 11y'' + 12y' + 9y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 13$

Show answer

22. $8y''' - 4y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = -3, \quad y''(0) = -1$

Show answer

23. $y^{(4)} - 16y = 0, \quad y(0) = 2, \; y'(0) = 2, \; y''(0) = -2, \; y'''(0) = 0$

Show answer

24. $y^{(4)} - 6y''' + 7y'' + 6y' - 8y = 0, \quad y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -14,$

$y'''(0) = -62$

Show answer

25. $4y^{(4)} - 13y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 3, \quad y''(0) = 1, \quad y'''(0) = 3$

Show answer

26. $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5, \quad y'(0) = -2, \quad y''(0) = 6, \quad y'''(0) = 8$

Show answer

27. C/G $4y^{(4)} + 8y''' + 19y'' + 32y' + 12y = 0, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -\backslash\mathrm{dst}\frac{7}{2},\ y'''(0) = \backslash\mathrm{dst}\frac{31}{4}$

Show answer

28. Find a fundamental set of solutions of the given equation, and verify that it's a fundamental set by evaluating its Wronskian at $x = 0$.

(a) $(D-1)^2(D-2)y = 0$	**(b)** $(D^2+4)(D-3)y = 0$
(c) $(D^2+2D+2)(D-1)y = 0$	**(d)** $D^3(D-1)y = 0$
(e) $(D^2-1)(D^2+1)y = 0$	**(f)** $(D^2-2D+2)(D^2+1)y = 0$

Show answer

In Exercises 29–38 find a fundamental set of solutions.

29. $(D^2+6D+13)(D-2)^2D^3y=0$

Show answer

30. $(D-1)^2(2D-1)^3(D^2+1)y=0$

Show answer

31. $(D^2+9)^3D^2y=0$

Show answer

32. $(D-2)^3(D+1)^2Dy=0$

Show answer

33. $(D^2+1)(D^2+9)^2(D-2)y=0$

Show answer

34. $(D^4-16)^2y=0$

Show answer

35. $(4D^2+4D+9)^3y=0$

Show answer

36. $D^3(D-2)^2(D^2+4)^2y=0$

Show answer

37. $(4D^2+1)^2(9D^2+4)^3y=0$

Show answer

38. $\left[(D-1)^4-16\right]y=0$

Show answer

39. It can be shown that

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i), \tag{A}$$

where the left side is the Vandermonde determinant (http://www-history.mcs.st-and.ac.uk/Mathematicians/Vandermonde.html) and the right side is the product of all factors of the form $(a_j - a_i)$ with i and j between 1 and n and $i < j$.

a. Verify (A) for $n = 2$ and $n = 3$.

b. Find the Wronskian of $\{e^{a_1 x}, \quad e^{a_2 x}, \ldots, e^{a_n x}\}$.

Show answer

40. A theorem from algebra says that if P_1 and P_2 are polynomials with no common factors then there are polynomials Q_1 and Q_2 such that

$$Q_1P_1 + Q_2P_2 = 1.$$

This implies that

$$Q_1(D)P_1(D)y + Q_2(D)P_2(D)y = y$$

for every function y with enough derivatives for the left side to be defined.

a. Use this to show that if P_1 and P_2 have no common factors and

$$P_1(D)y = P_2(D)y = 0$$

then $y = 0$.

b. Suppose P_1 and P_2 are polynomials with no common factors. Let u_1, …, u_r be linearly independent solutions of $P_1(D)y = 0$ and let v_1, …, v_s be linearly independent solutions of $P_2(D)y = 0$. Use **(a)** to show that $\{u_1, \dots, u_r, v_1, \dots, v_s\}$ is a linearly independent set.

c. Suppose the characteristic polynomial of the constant coefficient equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{A}$$

has the factorization

$$p(r) = a_0 p_1(r) p_2(r) \cdots p_k(r),$$

where each p_j is of the form

$$p_j(r) = (r - r_j)^{n_j} \text{ or } p_j(r) = [(r - \lambda_j)^2 + w_j^2]^{m_j} \quad (\omega_j > 0)$$

and no two of the polynomials p_1, p_2, …, p_k have a common factor. Show that we can find a fundamental set of solutions $\{y_1, y_2, \dots, y_n\}$ of (A) by finding a fundamental set of solutions of each of the equations

$$p_j(D)y = 0, \quad 1 \leq j \leq k,$$

and taking $\{y_1, y_2, \dots, y_n\}$ to be the set of all functions in these separate fundamental sets.

41. a. Show that if

$$z = p(x)\cos\omega x + q(x)\sin\omega x, \tag{A}$$

where p and q are polynomials of degree $\le k$, then

$$(D^2 + \omega^2)z = p_1(x)\cos\omega x + q_1(x)\sin\omega x,$$

where p_1 and q_1 are polynomials of degree $\le k-1$.

b. Apply **(a)** m times to show that if z is of the form (A) where p and q are polynomial of degree $\le m-1$, then

$$(D^2 + \omega^2)^m z = 0. \tag{B}$$

c. Use Eqn. (9.2.17) to show that if $y = e^{\lambda x}z$ then

$$[(D-\lambda)^2 + \omega^2]^m y = e^{\lambda x}(D^2+\omega^2)^m z.$$

d. Conclude from **(b)** and **(c)** that if p and q are arbitrary polynomials of degree $\le m-1$ then

$$y = e^{\lambda x}(p(x)\cos\omega x + q(x)\sin\omega x)$$

is a solution of

$$[(D-\lambda)^2 + \omega^2]^m y = 0. \tag{C}$$

e. Conclude from **(d)** that the functions

$$\begin{array}{c} e^{\lambda x}\cos\omega x, xe^{\lambda x}\cos\omega x, \quad \dots, x^{m-1}e^{\lambda x}\cos\omega x, \\ e^{\lambda x}\sin\omega x, xe^{\lambda x}\sin\omega x, \quad \dots, x^{m-1}e^{\lambda x}\sin\omega x \end{array} \tag{D}$$

are all solutions of (C).

f. Complete the proof of Theorem 9.2.2 by showing that the functions in (D) are linearly independent.

42. a. Use the trigonometric identities

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \sin(A+B) &= \cos A\sin B + \sin A\cos B \end{aligned}$$

to show that

$$(\cos A + i\sin A)(\cos B + i\sin B) = \cos(A+B) + i\sin(A+B).$$

b. Apply **(a)** repeatedly to show that if n is a positive integer then

$$\prod_{k=1}^{n}(\cos A_k + i\sin A_k) = \cos(A_1 + A_2 + \cdots + A_n) + i\sin(A_1 + A_2 + \cdots + A_n).$$

c. Infer from **(b)** that if n is a positive integer then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{A}$$

d. Show that (A) also holds if $n = 0$ or a negative integer.

HINT

Verify by direct calculation that

$$(\cos\theta + i\sin\theta)^{-1} = (\cos\theta - i\sin\theta).$$

Then replace θ by $-\theta$ in (A).

e. Now suppose n is a positive integer. Infer from (A) that if

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, \dots, n-1,$$

and

$$\zeta_k = \cos\left(\frac{(2k+1)\pi}{n}\right) + i\sin\left(\frac{(2k+1)\pi}{n}\right), \quad k = 0, 1, \dots, n-1,$$

then

$$z_k^n = 1 \quad \text{and} \quad \zeta_k^n = -1, \quad k = 0, 1, \dots, n-1.$$

(Why don't we also consider other integer values for k?)

f. Let ρ be a positive number. Use **(e)** to show that

$$z^n - \rho = (z - \rho^{1/n} z_0)(z - \rho^{1/n} z_1) \cdots (z - \rho^{1/n} z_{n-1})$$

and

$$z^n + \rho = (z - \rho^{1/n} \zeta_0)(z - \rho^{1/n} \zeta_1) \cdots (z - \rho^{1/n} \zeta_{n-1}).$$

43. Use **(e)** of Exercise 42 to find a fundamental set of solutions of the given equation.

(a) $y''' - y = 0$	**(b)** $y''' + y = 0$

(c) $y^{(4)} + 64y = 0$	**(d)** $y^{(6)} - y = 0$

(e) $y^{(6)} + 64y = 0$	**(f)** $\left[(D-1)^6 - 1\right] y = 0$

(g) $y^{(5)} + y^{(4)} + y''' + y'' + y' + y = 0$

Show answer

44. An equation of the form

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = 0, \quad x > 0, \tag{A}$$

where a_0, a_1, …, a_n are constants, is an Euler or equidimensional equation.

Show that if

$$x = e^t \quad \text{and} \quad Y(t) = y(x(t)), \tag{B}$$

then

$$\begin{aligned}
\backslash\mathrm{dst}x\frac{dy}{dx} &= \backslash\mathrm{dst}\frac{dY}{dt}\\
\backslash\mathrm{dst}x^2\frac{d^2y}{dx^2} &= \backslash\mathrm{dst}\frac{d^2Y}{dt^2} - \frac{dY}{dt}\\
\backslash\mathrm{dst}x^3\frac{d^3y}{dx^3} &= \backslash\mathrm{dst}\frac{d^3Y}{dt^3} - 3\frac{d^2Y}{dt^2} + 2\frac{dY}{dt}.
\end{aligned}$$

In general, it can be shown that if r is any integer ≥ 2 then

$$x^r\frac{d^ry}{dx^r} = \frac{d^rY}{dt^r} + A_{1r}\frac{d^{r-1}Y}{dt^{r-1}} + \cdots + A_{r-1,r}\frac{dY}{dt}$$

where A_{1r}, …, $A_{r-1,r}$ are integers. Use these results to show that the substitution (B) transforms (A) into a constant coefficient equation for Y as a function of t.

45. Use Exercise 44 to show that a function $y = y(x)$ satisfies the equation

$$a_0x^3y''' + a_1x^2y'' + a_2xy' + a_3y = 0, \tag{A}$$

on $(0, \infty)$ if and only if the function $Y(t) = y(e^t)$ satisfies

$$a_0\frac{d^3Y}{dt^3} + (a_1 - 3a_0)\frac{d^2Y}{dt^2} + (a_2 - a_1 + 2a_0)\frac{dY}{dt} + a_3Y = 0.$$

Assuming that a_0, a_1, a_2, a_3 are real and $a_0 \neq 0$, find the possible forms for the general solution of (A).

Show answer

9.3 Undetermined Coefficients for Higher Order Equations

In this section we consider the constant coefficient equation

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny=F(x), \tag{9.3.1}$$

where $n\geq 3$ and F is a linear combination of functions of the form

$$e^{\alpha x}\left(p_0+p_1x+\cdots+p_kx^k\right)$$

or

$$e^{\lambda x}\left[\left(p_0+p_1x+\cdots+p_kx^k\right)\cos\omega x+\left(q_0+q_1x+\cdots+q_kx^k\right)\sin\omega x\right].$$

From Theorem 9.1.5, the general solution of (9.3.1) is $y=y_p+y_c$, where y_p is a particular solution of (9.3.1) and y_c is the general solution of the complementary equation

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny=0.$$

In Section 9.2 we learned how to find y_c. Here we will learn how to find y_p when the forcing function has the form stated above. The procedure that we use is a generalization of the method that we used in Sections 5.4 and 5.5, and is again called method of undetermined coefficients. Since the underlying ideas are the same as those in Sections 5.4 and 5.5, we'll give an informal presentation based on examples.

Forcing Functions of the Form $e^{\alpha x}\left(p_0+p_1x+\cdots+p_kx^k\right)$

We first consider equations of the form

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny=e^{\alpha x}\left(p_0+p_1x+\cdots+p_kx^k\right).$$

Example 9.3.1

Find a particular solution of

$$y'''+3y''+2y'-y=e^x(21+24x+28x^2+5x^3). \tag{9.3.2}$$

Solution Substituting

$$\begin{aligned} y &= ue^x, \\ y' &= e^x(u' + u), \\ y'' &= e^x(u'' + 2u' + u), \\ y''' &= e^x(u''' + 3u'' + 3u' + u) \end{aligned}$$

into (9.3.2) and canceling e^x yields

$$(u''' + 3u'' + 3u' + u) + 3(u'' + 2u' + u) + 2(u' + u) - u = 21 + 24x + 28x^2 + 5x^3,$$

or

$$u''' + 6u'' + 11u' + 5u = 21 + 24x + 28x^2 + 5x^3. \tag{9.3.3}$$

Since the unknown u appears on the left, we can see that (9.3.3) has a particular solution of the form

$$u_p = A + Bx + Cx^2 + Dx^3.$$

Then

$$\begin{aligned} u_p' &= B + 2Cx + 3Dx^2 \\ u_p'' &= 2C + 6Dx \\ u_p''' &= 6D. \end{aligned}$$

Substituting from the last four equations into the left side of (9.3.3) yields

$$\begin{aligned} u_p''' + 6u_p'' + 11u_p' + 5u_p &= 6D + 6(2C + 6Dx) + 11(B + 2Cx + 3Dx^2) \\ &\quad + 5(A + Bx + Cx^2 + Dx^3) \\ &= (5A + 11B + 12C + 6D) + (5B + 22C + 36D)x \\ &\quad + (5C + 33D)x^2 + 5Dx^3. \end{aligned}$$

Comparing coefficients of like powers of x on the right sides of this equation and (9.3.3) shows that u_p satisfies (9.3.3) if

$$\begin{array}{rcl} 5D &=& 5 \\ 5C + 33D &=& 28 \\ 5B + 22C + 36D &=& 24 \\ 5A + 11B + 12C + \ \ 6D &=& 21. \end{array}$$

Solving these equations successively yields $D = 1$, $C = -1$, $B = 2$, $A = 1$. Therefore

$$u_p = 1 + 2x - x^2 + x^3$$

is a particular solution of (9.3.3), so

$$y_p = e^x u_p = e^x(1 + 2x - x^2 + x^3)$$

is a particular solution of (9.3.2) (Figure 9.3.1).

Figure 9.3.1. $y_p = e^x(1 + 2x - x^2 + x^3)$

Example 9.3.2

Find a particular solution of

$$y^{(4)} - y''' - 6y'' + 4y' + 8y = e^{2x}(4 + 19x + 6x^2). \tag{9.3.4}$$

Solution Substituting

$$\begin{aligned} y &= ue^{2x}, \\ y' &= e^{2x}(u' + 2u), \\ y'' &= e^{2x}(u'' + 4u' + 4u), \\ y''' &= e^{2x}(u''' + 6u'' + 12u' + 8u), \\ y^{(4)} &= e^{2x}(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) \end{aligned}$$

into (9.3.4) and canceling e^{2x} yields

$$\begin{aligned} &(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) - (u''' + 6u'' + 12u' + 8u) \\ &- 6(u'' + 4u' + 4u) + 4(u' + 2u) + 8u = 4 + 19x + 6x^2, \end{aligned}$$

or

$$u^{(4)} + 7u''' + 12u'' = 4 + 19x + 6x^2. \tag{9.3.5}$$

Since neither u nor u' appear on the left, we can see that (9.3.5) has a particular solution of the form

$$u_p = Ax^2 + Bx^3 + Cx^4. \tag{9.3.6}$$

Then

$$\begin{aligned} u_p' &= 2Ax + 3Bx^2 + 4Cx^3 \\ u_p'' &= 2A + 6Bx + 12Cx^2 \\ u_p''' &= 6B + 24Cx \\ u_p^{(4)} &= 24C. \end{aligned}$$

Substituting u_p'', u_p''', and $u_p^{(4)}$ into the left side of (9.3.5) yields

$$\begin{aligned} u_p^{(4)} + 7u_p''' + 12u_p'' &= 24C + 7(6B + 24Cx) + 12(2A + 6Bx + 12Cx^2) \\ &= (24A + 42B + 24C) + (72B + 168C)x + 144Cx^2. \end{aligned}$$

Comparing coefficients of like powers of x on the right sides of this equation and (9.3.5) shows that u_p satisfies (9.3.5) if

$$\begin{array}{rcl} 144C &=& 6 \\ 72B + 168C &=& 19 \\ 24A + 42B + \ \ 24C &=& 4. \end{array}$$

Solving these equations successively yields $C = 1/24$, $B = 1/6$, $A = -1/6$. Substituting these into (9.3.6) shows that

$$u_p = \frac{x^2}{24}(-4 + 4x + x^2)$$

is a particular solution of (9.3.5), so

$$y_p = e^{2x} u_p = \frac{x^2 e^{2x}}{24}(-4 + 4x + x^2)$$

is a particular solution of (9.3.4). (Figure 9.3.2).

Figure 9.3.2. $y_p = \dst \frac{x^2e^{2x}}{24}(-4 + 4x + x^2)$

Forcing Functions of the Form $e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$

We now consider equations of the form

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right),$$

where P and Q are polynomials.

Example 9.3.3

Find a particular solution of

$$y''' + y'' - 4y' - 4y = e^x[(5-5x)\cos x + (2+5x)\sin x]. \tag{9.3.7}$$

Solution Substituting

$$\begin{aligned} y &= ue^x, \\ y' &= e^x(u'+u), \\ y'' &= e^x(u''+2u'+u), \\ y''' &= e^x(u'''+3u''+3u'+u) \end{aligned}$$

into (9.3.7) and canceling e^x yields

$$(u'''+3u''+3u'+u) + (u''+2u'+u) - 4(u'+u) - 4u = (5-5x)\cos x + (2+5x)\sin x,$$

or

$$u''' + 4u'' + u' - 6u = (5-5x)\cos x + (2+5x)\sin x. \tag{9.3.8}$$

Since $\cos x$ and $\sin x$ are not solutions of the complementary equation

$$u''' + 4u'' + u' - 6u = 0,$$

a theorem analogous to Theorem 5.5.1 implies that (9.3.8) has a particular solution of the form

$$u_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x. \tag{9.3.9}$$

Then

$$\begin{aligned} u_p' &= (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x, \\ u_p'' &= (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x, \\ u_p''' &= -(3A_1 + B_0 + B_1 x)\cos x - (3B_1 - A_0 - A_1 x)\sin x, \end{aligned}$$

so

$$\begin{aligned} u_p''' + 4u_p'' + u_p' - 6u_p \quad &= \quad -[10A_0 + 2A_1 - 8B_1 + 10A_1 x]\cos x \\ & \quad\quad -[10B_0 + 2B_1 + 8A_1 + 10B_1 x]\sin x. \end{aligned}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (9.3.8) shows that u_p is a solution of (9.3.8) if

$$\begin{aligned} -10A_1 &= -5 \\ -10B_1 &= 5 \\ -10A_0 - 2A_1 + 8B_1 &= 5 \\ -10B_0 - 2B_1 - 8A_1 &= 2. \end{aligned}$$

Solving the first two equations yields $A_1 = 1/2$, $B_1 = -1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -10A_0 &= 5 + 2A_1 - 8B_1 = 10 \\ -10B_0 &= 2 + 2B_1 + 8A_1 = 5, \end{aligned}$$

so $A_0 = -1$, $B_0 = -1/2$. Substituting $A_0 = -1$, $A_1 = 1/2$, $B_0 = -1/2$, $B_1 = -1/2$ into (9.3.9) shows that

$$u_p = -\backslash\mathrm{dst}\frac{1}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$$

is a particular solution of (9.3.8), so

$$y_p = e^x u_p = -\frac{e^x}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$$

is a particular solution of (9.3.7) (Figure 9.3.3).

Figure 9.3.3. $y_p = \backslash\mathrm{dst}\, e^x u_p = -\frac{e^x}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$

Example 9.3.4

Find a particular solution of

$$y''' + 4y'' + 6y' + 4y = e^{-x}\left[(1-6x)\cos x - (3+2x)\sin x\right]. \tag{9.3.10}$$

Solution Substituting

$$\begin{aligned} y &= ue^{-x}, \\ y' &= e^{-x}(u'-u), \\ y'' &= e^{-x}(u''-2u'+u), \\ y''' &= e^{-x}(u'''-3u''+3u'-u) \end{aligned}$$

into (9.3.10) and canceling e^{-x} yields

$$(u'''-3u''+3u'-u)+4(u''-2u'+u)+6(u'-u)+4u = (1-6x)\cos x - (3+2x)\sin x,$$

or

$$u''' + u'' + u' + u = (1-6x)\cos x - (3+2x)\sin x. \tag{9.3.11}$$

Since $\cos x$ and $\sin x$ are solutions of the complementary equation

$$u''' + u'' + u' + u = 0,$$

a theorem analogous to Theorem 5.5.1 implies that (9.3.11) has a particular solution of the form

$$u_p = (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x. \tag{9.3.12}$$

Then

$$\begin{aligned} u_p' &= \left[A_0 + (2A_1+B_0)x + B_1x^2\right]\cos x + \left[B_0 + (2B_1-A_0)x - A_1x^2\right]\sin x, \\ u_p'' &= \left[2A_1 + 2B_0 - (A_0-4B_1)x - A_1x^2\right]\cos x \\ &\quad + \left[2B_1 - 2A_0 - (B_0+4A_1)x - B_1x^2\right]\sin x, \\ u_p''' &= -\left[3A_0 - 6B_1 + (6A_1+B_0)x + B_1x^2\right]\cos x \\ &\quad - \left[3B_0 + 6A_1 + (6B_1-A_0)x - A_1x^2\right]\sin x, \end{aligned}$$

so

$$\begin{aligned} u_p''' + u_p'' + u_p' + u_p &= -\left[2A_0 - 2B_0 - 2A_1 - 6B_1 + (4A_1-4B_1)x\right]\cos x \\ &\quad -\left[2B_0 + 2A_0 - 2B_1 + 6A_1 + (4B_1+4A_1)x\right]\sin x. \end{aligned}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (9.3.11) shows that u_p is a solution of (9.3.11) if

$$\begin{array}{rcr} -4A_1+4B_1 & = & -6 \\ -4A_1-4B_1 & = & -2 \\ -2A_0+2B_0+2A_1+6B_1 & = & 1 \\ -2A_0-2B_0-6A_1+2B_1 & = & -3. \end{array}$$

Solving the first two equations yields $A_1=1$, $B_1=-1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -2A_0+2B_0 &= \quad 1-2A_1-6B_1=2 \\ -2A_0-2B_0 &= -3+6A_1-2B_1=4, \end{aligned}$$

Figure 9.3.4. $\dst y_p=-\frac{xe^{-x}}{2}\left[(3-2x)\cos x+(1+x)\sin x\right]$

so $A_0=-3/2$ and $B_0=-1/2$. Substituting $A_0=-3/2$, $A_1=1$, $B_0=-1/2$, $B_1=-1/2$ into (9.3.12) shows that

$$u_p=-\dst\frac{x}{2}\left[(3-2x)\cos x+(1+x)\sin x\right]$$

is a particular solution of (9.3.11), so

$$y_p=e^{-x}u_p=-\frac{xe^{-x}}{2}\left[(3-2x)\cos x+(1+x)\sin x\right]$$

(Figure 9.3.4) is a particular solution of (9.3.10).

9.3 Exercises

In Exercises 1–59 find a particular solution.

1. $y''' - 6y'' + 11y' - 6y = -e^{-x}(4 + 76x - 24x^2)$

Show answer

2. $y''' - 2y'' - 5y' + 6y = e^{-3x}(32 - 23x + 6x^2)$

Show answer

3. $4y''' + 8y'' - y' - 2y = -e^{x}(4 + 45x + 9x^2)$

Show answer

4. $y''' + 3y'' - y' - 3y = e^{-2x}(2 - 17x + 3x^2)$

Show answer

5. $y''' + 3y'' - y' - 3y = e^{x}(-1 + 2x + 24x^2 + 16x^3)$

Show answer

6. $y''' + y'' - 2y = e^{x}(14 + 34x + 15x^2)$

Show answer

7. $4y''' + 8y'' - y' - 2y = -e^{-2x}(1 - 15x)$

Show answer

8. $y''' - y'' - y' + y = e^{x}(7 + 6x)$

Show answer

9. $2y''' - 7y'' + 4y' + 4y = e^{2x}(17 + 30x)$

Show answer

10. $y''' - 5y'' + 3y' + 9y = 2e^{3x}(11 - 24x^2)$

Show answer

11. $y''' - 7y'' + 8y' + 16y = 2e^{4x}(13 + 15x)$

Show answer

12. $8y''' - 12y'' + 6y' - y = e^{x/2}(1 + 4x)$

Show answer

13. $y^{(4)} + 3y''' - 3y'' - 7y' + 6y = -e^{-x}(12 + 8x - 8x^2)$

Show answer

14. $y^{(4)} + 3y''' + y'' - 3y' - 2y = -3e^{2x}(11 + 12x)$

Show answer

15. $y^{(4)} + 8y''' + 24y'' + 32y' = -16e^{-2x}(1 + x + x^2 - x^3)$

Show answer

16. $4y^{(4)} - 11y'' - 9y' - 2y = -e^{x}(1 - 6x)$

Show answer

17. $y^{(4)} - 2y''' + 3y' - y = e^{x}(3 + 4x + x^2)$

Show answer

18. $y^{(4)} - 4y''' + 6y'' - 4y' + 2y = e^{2x}(24 + x + x^4)$

Show answer

19. $2y^{(4)} + 5y''' - 5y' - 2y = 18e^{x}(5 + 2x)$

Show answer

20. $y^{(4)} + y''' - 2y'' - 6y' - 4y = -e^{2x}(4 + 28x + 15x^2)$

Show answer

21. $2y^{(4)} + y''' - 2y' - y = 3e^{-x/2}(1 - 6x)$

Show answer

22. $y^{(4)} - 5y'' + 4y = e^{x}(3 + x - 3x^2)$

Show answer

23. $y^{(4)} - 2y''' - 3y'' + 4y' + 4y = e^{2x}(13 + 33x + 18x^2)$

Show answer

24. $y^{(4)} - 3y''' + 4y' = e^{2x}(15 + 26x + 12x^2)$

Show answer

25. $y^{(4)} - 2y''' + 2y' - y = e^{x}(1 + x)$

Show answer

26. $2y^{(4)} - 5y''' + 3y'' + y' - y = e^{x}(11 + 12x)$

Show answer

27. $y^{(4)} + 3y''' + 3y'' + y' = e^{-x}(5 - 24x + 10x^2)$

Show answer

28. $y^{(4)} - 7y''' + 18y'' - 20y' + 8y = e^{2x}(3 - 8x - 5x^2)$

Show answer

29. $y''' - y'' - 4y' + 4y = e^{-x}\left[(16 + 10x)\cos x + (30 - 10x)\sin x\right]$

Show answer

30. $y''' + y'' - 4y' - 4y = e^{-x}\left[(1 - 22x)\cos 2x - (1 + 6x)\sin 2x\right]$

Show answer

31. $y''' - y'' + 2y' - 2y = e^{2x}[(27 + 5x - x^2)\cos x + (2 + 13x + 9x^2)\sin x]$

Show answer

32. $y''' - 2y'' + y' - 2y = -e^{x}[(9 - 5x + 4x^2)\cos 2x - (6 - 5x - 3x^2)\sin 2x]$

Show answer

33. $y''' + 3y'' + 4y' + 12y = 8\cos 2x - 16\sin 2x$

Show answer

34. $y''' - y'' + 2y = e^{x}[(20 + 4x)\cos x - (12 + 12x)\sin x]$

Show answer

35. $y''' - 7y'' + 20y' - 24y = -e^{2x}[(13 - 8x)\cos 2x - (8 - 4x)\sin 2x]$

Show answer

36. $y''' - 6y'' + 18y' = -e^{3x}[(2 - 3x)\cos 3x - (3 + 3x)\sin 3x]$

Show answer

37. $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = e^{x}(8\cos x + 16\sin x)$

Show answer

38. $y^{(4)} - 3y''' + 2y'' + 2y' - 4y = e^{x}(2\cos 2x - \sin 2x)$

Show answer

39. $y^{(4)} - 8y''' + 24y'' - 32y' + 15y = e^{2x}(15x\cos 2x + 32\sin 2x)$

Show answer

40. $y^{(4)}+6y'''+13y''+12y'+4y=e^{-x}[(4-x)\cos x-(5+x)\sin x]$

Show answer

41. $y^{(4)}+3y'''+2y''-2y'-4y=-e^{-x}(\cos x-\sin x)$

Show answer

42. $y^{(4)}-5y'''+13y''-19y'+10y=e^{x}(\cos 2x+\sin 2x)$

Show answer

43. $y^{(4)}+8y'''+32y''+64y'+39y=e^{-2x}[(4-15x)\cos 3x-(4+15x)\sin 3x]$

Show answer

44. $y^{(4)}-5y'''+13y''-19y'+10y=e^{x}[(7+8x)\cos 2x+(8-4x)\sin 2x]$

Show answer

45. $y^{(4)}+4y'''+8y''+8y'+4y=-2e^{-x}(\cos x-2\sin x)$

Show answer

46. $y^{(4)}-8y'''+32y''-64y'+64y=e^{2x}(\cos 2x-\sin 2x)$

Show answer

47. $y^{(4)}-8y'''+26y''-40y'+25y=e^{2x}[3\cos x-(1+3x)\sin x]$

Show answer

48. $y'''-4y''+5y'-2y=e^{2x}-4e^{x}-2\cos x+4\sin x$

Show answer

49. $y'''-y''+y'-y=5e^{2x}+2e^{x}-4\cos x+4\sin x$

Show answer

50. $y'''-y'=-2(1+x)+4e^{x}-6e^{-x}+96e^{3x}$

Show answer

51. $y'''-4y''+9y'-10y=10e^{2x}+20e^{x}\sin 2x-10$

Show answer

52. $y'''+3y''+3y'+y=12e^{-x}+9\cos 2x-13\sin 2x$

Show answer

53. $y''' + y'' - y' - y = 4e^{-x}(1 - 6x) - 2x\cos x + 2(1 + x)\sin x$

Show answer

54. $y^{(4)} - 5y'' + 4y = -12e^{x} + 6e^{-x} + 10\cos x$

Show answer

55. $y^{(4)} - 4y''' + 11y'' - 14y' + 10y = -e^{x}(\sin x + 2\cos 2x)$

Show answer

56. $y^{(4)} + 2y''' - 3y'' - 4y' + 4y = 2e^{x}(1 + x) + e^{-2x}$

Show answer

57. $y^{(4)} + 4y = \sinh x\cos x - \cosh x\sin x$

Show answer

58. $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-x}(30 + 24x) - e^{-2x}$

Show answer

59. $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = e^{x}(12x - 2\cos x + 2\sin x)$

Show answer

In Exercises 60–68 find the general solution.

60. $y''' - y'' - y' + y = e^{2x}(10 + 3x)$

Show answer

61. $y''' + y'' - 2y = -e^{3x}(9 + 67x + 17x^2)$

Show answer

62. $y''' - 6y'' + 11y' - 6y = e^{2x}(5 - 4x - 3x^2)$

Show answer

63. $y''' + 2y'' + y' = -2e^{-x}(7 - 18x + 6x^2)$

Show answer

64. $y''' - 3y'' + 3y' - y = e^{x}(1 + x)$

Show answer

65. $y^{(4)} - 2y'' + y = -e^{-x}(4 - 9x + 3x^2)$

Show answer

66. $y''' + 2y'' - y' - 2y = e^{-2x}\left[(23 - 2x)\cos x + (8 - 9x)\sin x\right]$

Show answer

67. $y^{(4)} - 3y''' + 4y'' - 2y' = e^{x}\left[(28 + 6x)\cos 2x + (11 - 12x)\sin 2x\right]$

Show answer

68. $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = e^{x}\left[(2 + 6x)\cos 2x + 3\sin 2x\right]$

Show answer

In Exercises 69–74 solve the initial value problem and graph the solution.

69. C/G $y''' - 2y'' - 5y' + 6y = 2e^x(1 - 6x), \quad y(0) = 2, \quad y'(0) = 7, \quad y''(0) = 9$

Show answer

70. C/G $y''' - y'' - y' + y = -e^{-x}(4 - 8x), \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0$

Show answer

71. C/G $4y''' - 3y' - y = e^{-x/2}(2 - 3x), \quad y(0) = -1, \quad y'(0) = 15, \quad y''(0) = -17$

Show answer

72. C/G $y^{(4)} + 2y''' + 2y'' + 2y' + y = e^{-x}(20 - 12x),\ y(0) = 3,\ y'(0) = -4,\ y''(0) = 7,\ y'''(0) = -22$

Show answer

73. C/G $y''' + 2y'' + y' + 2y = 30\cos x - 10\sin x, \quad y(0) = 3, \quad y'(0) = -4, \quad y''(0) = 16$

Show answer

74. C/G Missing dimension or its units for \hspace

Show answer

75. Prove: A function y is a solution of the constant coefficient nonhomogeneous equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = e^{\alpha x}G(x) \tag{A}$$

if and only if $y = ue^{\alpha x}$, where u satisfies the differential equation

$$a_0u^{(n)} + \frac{p^{(n-1)}(\alpha)}{(n-1)!}u^{(n-1)} + \frac{p^{(n-2)}(\alpha)}{(n-2)!}u^{(n-2)} + \cdots + p(\alpha)u = G(x) \tag{B}$$

and

$$p(r) = a_0r^n + a_1r^{n-1} + \cdots + a_n$$

is the characteristic polynomial of the complementary equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0.$$

76. Prove:

a. The equation

$$\begin{aligned} a_0u^{(n)} \quad &+ \quad \backslash\mathrm{dst}\frac{p^{(n-1)}(\alpha)}{(n-1)!}u^{(n-1)} + \backslash\mathrm{dst}\frac{p^{(n-2)}(\alpha)}{(n-2)!}u^{(n-2)} + \cdots + p(\alpha)u \\ &= \quad \left(p_0 + p_1x + \cdots + p_kx^k\right)\cos\omega x \\ &\quad\; + \left(q_0 + q_1x + \cdots + q_kx^k\right)\sin\omega x \end{aligned} \tag{A}$$

has a particular solution of the form

$$u_p = x^m\left(u_0 + u_1x + \cdots + u_kx^k\right)\cos\omega x + \left(v_0 + v_1x + \cdots + v_kx^k\right)\sin\omega x.$$

b. If $\lambda + i\omega$ is a zero of p with multiplicity $m \geq 1$, then (A) can be written as

$$a(u'' + \omega^2u) = \left(p_0 + p_1x + \cdots + p_kx^k\right)\cos\omega x + \left(q_0 + q_1x + \cdots + q_kx^k\right)\sin\omega x,$$

which has a particular solution of the form

$$u_p = U(x)\cos\omega x + V(x)\sin\omega x,$$

where

$$U(x) = u_0x + u_1x^2 + \cdots + u_kx^{k+1},\ V(x) = v_0x + v_1x^2 + \cdots + v_kx^{k+1}$$

and

$$\begin{aligned} a(U''(x) + 2\omega V'(x)) \quad &= \quad p_0 + p_1x + \cdots + p_kx^k \\ a(V''(x) - 2\omega U'(x)) \quad &= \quad q_0 + q_1x + \cdots + q_kx^k. \end{aligned}$$

9.4 Variation of Parameters for Higher Order Equations

Derivation of the method

We assume throughout this section that the nonhomogeneous linear equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x) \tag{9.4.1}$$

is normal on an interval (a,b). We'll abbreviate this equation as $Ly=F$, where

$$Ly=P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y.$$

When we speak of solutions of this equation and its complementary equation $Ly=0$, we mean solutions on (a,b). We'll show how to use the method of variation of parameters to find a particular solution of $Ly=F$, provided that we know a fundamental set of solutions $\{y_1,y_2,\dots,y_n\}$ of $Ly=0$.

We seek a particular solution of $Ly=F$ in the form

$$y_p=u_1y_1+u_2y_2+\cdots+u_ny_n \tag{9.4.2}$$

where $\{y_1,y_2,\dots,y_n\}$ is a known fundamental set of solutions of the complementary equation

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=0$$

and u_1, u_2, …, u_n are functions to be determined. We begin by imposing the following $n-1$ conditions on $u_1,u_2,\dots,u_n$:

$$\begin{array}{rcl}
u_1'y_1+u_2'y_2+ & \cdots & +u_n'y_n=0\\
u_1'y_1'+u_2'y_2'+ & \cdots & +u_n'y_n'=0\\
 & \vdots & \\
u_1'y_1^{(n-2)}+u_2'y_2^{(n-2)}+ & \cdots & +u_n'y_n^{(n-2)}=0.
\end{array} \tag{9.4.3}$$

These conditions lead to simple formulas for the first $n-1$ derivatives of y_p:

$$y_p^{(r)}=u_1y_1^{(r)}+u_2y_2^{(r)}\cdots+u_ny_n^{(r)},\ 0\le r\le n-1. \tag{9.4.4}$$

These formulas are easy to remember, since they look as though we obtained them by differentiating (9.4.2) $n-1$ times while treating u_1, u_2, …, u_n as constants. To see that (9.4.3) implies (9.4.4), we first differentiate (9.4.2) to obtain

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'+u_1'y_1+u_2'y_2+\cdots+u_n'y_n,$$

which reduces to

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'$$

because of the first equation in (9.4.3). Differentiating this yields

$$y_p'' = u_1y_1'' + u_2y_2'' + \cdots + u_ny_n'' + u_1'y_1' + u_2'y_2' + \cdots + u_n'y_n',$$

which reduces to

$$y_p'' = u_1y_1'' + u_2y_2'' + \cdots + u_ny_n''$$

because of the second equation in (9.4.3). Continuing in this way yields (9.4.4).

The last equation in (9.4.4) is

$$y_p^{(n-1)} = u_1y_1^{(n-1)} + u_2y_2^{(n-1)} + \cdots + u_ny_n^{(n-1)}.$$

Differentiating this yields

$$y_p^{(n)} = u_1y_1^{(n)} + u_2y_2^{(n)} + \cdots + u_ny_n^{(n)} + u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)}.$$

Substituting this and (9.4.4) into (9.4.1) yields

$$u_1Ly_1 + u_2Ly_2 + \cdots + u_nLy_n + P_0(x)\left(u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)}\right) = F(x).$$

Since $Ly_i = 0$ $(1 \le i \le n)$, this reduces to

$$u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)} = \frac{F(x)}{P_0(x)}.$$

Combining this equation with (9.4.3) shows that

$$y_p = u_1y_1 + u_2y_2 + \cdots + u_ny_n$$

is a solution of (9.4.1) if

$$\begin{array}{rcl} u_1'y_1 + u_2'y_2 + & \cdots & + u_n'y_n = 0 \\ u_1'y_1' + u_2'y_2' + & \cdots & + u_n'y_n' = 0 \\ & \vdots & \\ u_1'y_1^{(n-2)} + u_2'y_2^{(n-2)} + & \cdots & + u_n'y_n^{(n-2)} = 0 \\ u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + & \cdots & + u_n'y_n^{(n-1)} = F/P_0, \end{array}$$

which can be written in matrix form as

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F/P_0 \end{bmatrix}. \tag{9.4.5}$$

The determinant of this system is the Wronskian W of the fundamental set of solutions $\{y_1, y_2, \dots, y_n\}$, which has no zeros on (a, b), by Theorem 9.1.4. Solving (9.4.5) by Cramer's rule yields

$$u_j' = (-1)^{n-j} \frac{FW_j}{P_0 W}, \quad 1 \le j \le n, \tag{9.4.6}$$

where W_j is the Wronskian of the set of functions obtained by deleting y_j from $\{y_1, y_2, \dots, y_n\}$ and keeping the remaining functions in the same order. Equivalently, W_j is the determinant obtained by deleting the last row and j-th column of W.

Having obtained u_1', u_2', … ,u_n', we can integrate to obtain u_1, u_2, … , u_n. As in Section 5.7, we take the constants of integration to be zero, and we drop any linear combination of $\{y_1, y_2, \dots, y_n\}$ that may appear in y_p.

Remark

For efficiency, it's best to compute W_1, W_2, …, W_n first, and then compute W by expanding in cofactors of the last row; thus,

$$W = \sum_{j=1}^{n} (-1)^{n-j} y_j^{(n-1)} W_j.$$

Third Order Equations

If $n = 3$, then

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

Therefore

$$W_1 = \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

and (9.4.6) becomes

$$u_1' = \frac{FW_1}{P_0W}, \quad u_2' = -\frac{FW_2}{P_0W}, \quad u_3' = \frac{FW_3}{P_0W}. \tag{9.4.7}$$

Example 9.4.1

Find a particular solution of

$$xy''' - y'' - xy' + y = 8x^2e^x, \tag{9.4.8}$$

given that $y_1 = x$, $y_2 = e^x$, and $y_3 = e^{-x}$ form a fundamental set of solutions of the complementary equation. Then find the general solution of (9.4.8).

Solution We seek a particular solution of (9.4.8) of the form

$$y_p = u_1x + u_2e^x + u_3e^{-x}.$$

The Wronskian of $\{y_1, y_2, y_3\}$ is

$$W(x) = \begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix},$$

so

$$W_1 = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2,$$

$$W_2 = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -e^{-x}(x+1),$$

$$W_3 = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1).$$

Expanding W by cofactors of the last row yields

$$W = 0W_1 - e^x W_2 + e^{-x} W_3 = 0(-2) - e^x \left(-e^{-x}(x+1)\right) + e^{-x} e^x (x-1) = 2x.$$

Since $F(x) = 8x^2 e^x$ and $P_0(x) = x$,

$$\frac{F}{P_0 W} = \frac{8x^2 e^x}{x \cdot 2x} = 4e^x.$$

Therefore, from (9.4.7)

$$\begin{aligned} u_1' &= \quad 4e^x W_1 = \quad 4e^x(-2) = -8e^x, \\ u_2' &= -4e^x W_2 = -4e^x \left(-e^{-x}(x+1)\right) = 4(x+1), \\ u_3' &= \quad 4e^x W_3 = \quad 4e^x \left(e^x(x-1)\right) = 4e^{2x}(x-1). \end{aligned}$$

Integrating and taking the constants of integration to be zero yields

$$u_1 = -8e^x, \quad u_2 = 2(x+1)^2, u_3 = e^{2x}(2x-3).$$

Hence,

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= (-8e^x)x + e^x(2(x+1)^2) + e^{-x}\left(e^{2x}(2x-3)\right) \\ &= e^x(2x^2 - 2x - 1). \end{aligned}$$

Since $-e^x$ is a solution of the complementary equation, we redefine

$$y_p = 2xe^x(x-1).$$

Therefore the general solution of (9.4.8) is

$$y = 2xe^x(x-1) + c_1 x + c_2 e^x + c_3 e^{-x}.$$

Fourth Order Equations

If $n = 4$, then

$$W = \left| \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{array} \right|,$$

Therefore

$$W_1 = \left| \begin{array}{ccc} y_2 & y_3 & y_4 \\ y_2' & y_3' & y_4' \\ y_2'' & y_3'' & y_4'' \end{array} \right|, \quad W_2 = \left| \begin{array}{ccc} y_1 & y_3 & y_4 \\ y_1' & y_3' & y_4' \\ y_1'' & y_3'' & y_4'' \end{array} \right|,$$

$$W_3 = \left| \begin{array}{ccc} y_1 & y_2 & y_4 \\ y_1' & y_2' & y_4' \\ y_1'' & y_2'' & y_4'' \end{array} \right|, \quad W_4 = \left| \begin{array}{ccc} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{array} \right|,$$

and (9.4.6) becomes

$$u_1' = -\frac{FW_1}{P_0W}, \quad u_2' = \frac{FW_2}{P_0W}, \quad u_3' = -\frac{FW_3}{P_0W}, \quad u_4' = \frac{FW_4}{P_0W}. \tag{9.4.9}$$

Example 9.4.2

Find a particular solution of

$$x^4y^{(4)} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = 12x^2, \tag{9.4.10}$$

given that $y_1 = x$, $y_2 = x^2$, $y_3 = 1/x$ and $y_4 = 1/x^2$ form a fundamental set of solutions of the complementary equation. Then find the general solution of (9.4.10) on $(-\infty, 0)$ and $(0, \infty)$.

Solution We seek a particular solution of (9.4.10) of the form

$$y_p = u_1x + u_2x^2 + \frac{u_3}{x} + \frac{u_4}{x^2}.$$

The Wronskian of $\{y_1, y_2, y_3, y_4\}$ is

$$W(x)=\left|\begin{array}{cccc} x & x^2 & 1/x & -1/x^2 \\ 1 & 2x & -1/x^2 & -2/x^3 \\ 0 & 2 & 2/x^3 & 6/x^4 \\ 0 & 0 & -6/x^4 & -24/x^5 \end{array}\right|,$$

so

$$W_1=\left|\begin{array}{ccc} x^2 & 1/x & 1/x^2 \\ 2x & -1/x^2 & -2/x^3 \\ 2 & 2/x^3 & 6/x^4 \end{array}\right|=-\frac{12}{x^4},$$

$$W_2=\left|\begin{array}{ccc} x & 1/x & 1/x^2 \\ 1 & -1/x^2 & -2/x^3 \\ 0 & 2/x^3 & 6/x^4 \end{array}\right|=-\frac{6}{x^5},$$

$$W_3=\left|\begin{array}{ccc} x & x^2 & 1/x^2 \\ 1 & 2x & -2/x^3 \\ 0 & 2 & 6/x^4 \end{array}\right|=\frac{12}{x^2},$$

$$W_4=\left|\begin{array}{ccc} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{array}\right|=\frac{6}{x}.$$

Expanding W by cofactors of the last row yields

$$\begin{aligned} W= & -0W_1+0W_2-\left(-\frac{6}{x^4}\right)W_3+\left(-\frac{24}{x^5}\right)W_4 \\ = & \frac{6}{x^4}\frac{12}{x^2}-\frac{24}{x^5}\frac{6}{x}=-\frac{72}{x^6}. \end{aligned}$$

Since $F(x)=12x^2$ and $P_0(x)=x^4$,

$$\frac{F}{P_0W}=\frac{12x^2}{x^4}\left(-\frac{x^6}{72}\right)=-\frac{x^4}{6}.$$

Therefore, from (9.4.9),

$$\begin{aligned}
u_1' &= -\left(-\frac{x^4}{6}\right) W_1 = \frac{x^4}{6}\left(-\frac{12}{x^4}\right) = -2, \\
u_2' &= \quad -\frac{x^4}{6} W_2 = -\frac{x^4}{6}\left(-\frac{6}{x^5}\right) = \frac{1}{x}, \\
u_3' &= -\left(-\frac{x^4}{6}\right) W_3 = \frac{x^4}{6}\frac{12}{x^2} = 2x^2, \\
u_4' &= \quad -\frac{x^4}{6} W_4 = -\frac{x^4}{6}\frac{6}{x} = -x^3.
\end{aligned}$$

Integrating these and taking the constants of integration to be zero yields

$$u_1 = -2x, \quad u_2 = \ln|x|, \quad u_3 = \frac{2x^3}{3}, u_4 = -\frac{x^4}{4}.$$

Hence,

$$\begin{aligned}
y_p &= u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4 \\
&= (-2x)x + (\ln|x|)x^2 + \frac{2x^3}{3}\frac{1}{x} + \left(-\frac{x^4}{4}\right)\frac{1}{x^2} \\
&= x^2\ln|x| - \frac{19x^2}{12}.
\end{aligned}$$

Since $-19x^2/12$ is a solution of the complementary equation, we redefine

$$y_p = x^2\ln|x|.$$

Therefore

$$y = x^2\ln|x| + c_1x + c_2x^2 + \frac{c_3}{x} + \frac{c_4}{x^2}$$

is the general solution of (9.4.10) on $(-\infty, 0)$ and $(0, \infty)$.

9.4 Exercises

In Exercises 1–21 find a particular solution, given the fundamental set of solutions of the complementary equation.

1. $x^3y''' - x^2(x+3)y'' + 2x(x+3)y' - 2(x+3)y = -4x^4; \quad \{x,\, x^2,\, xe^x\}$

Show answer

2. $y''' + 6xy'' + (6+12x^2)y' + (12x+8x^3)y = x^{1/2}e^{-x^2}; \quad \{e^{-x^2},\, xe^{-x^2},\, x^2e^{-x^2}\}$

Show answer

3. $x^3y''' - 3x^2y'' + 6xy' - 6y = 2x; \quad \{x, x^2, x^3\}$

Show answer

4. $x^2y''' + 2xy'' - (x^2+2)y' = 2x^2; \quad \{1,\, e^x/x,\, e^{-x}/x\}$

Show answer

5. $x^3y''' - 3x^2(x+1)y'' + 3x(x^2+2x+2)y' - (x^3+3x^2+6x+6)y = x^4e^{-3x};$

$\{xe^x,\, x^2e^x,\, x^3e^x\}$

Show answer

6. $x(x^2-2)y''' + (x^2-6)y'' + x(2-x^2)y' + (6-x^2)y = 2(x^2-2)^2; \quad \{e^x,\, e^{-x},\, 1/x\}$

Show answer

7. $xy''' - (x-3)y'' - (x+2)y' + (x-1)y = -4e^{-x}; \quad \{e^x,\, e^x/x,\, e^{-x}/x\}$

Show answer

8. $4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2; \quad \{\sqrt{x},\, 1/\sqrt{x},\, x^2\}$

Show answer

9. $x(x^2-1)y''' + (5x^2+1)y'' + 2xy' - 2y = 12x^2; \quad \{x,\, 1/(x-1),\, 1/(x+1)\}$

Show answer

10. $x(1-x)y''' + (x^2-3x+3)y'' + xy' - y = 2(x-1)^2; \quad \{x,\, 1/x, e^x/x\}$

Show answer

11. $x^3y''' + x^2y'' - 2xy' + 2y = x^2; \quad \{x,\, x^2,\, 1/x\}$

Show answer

12. $xy''' - y'' - xy' + y = x^2; \quad \{x,\, e^x,\, e^{-x}\}$

Show answer

13. $xy^{(4)}+4y'''=6\ln|x|$; $\{1,\,x,\,x^2,\,1/x\}$

Show answer

14. $16x^4y^{(4)}+96x^3y'''+72x^2y''-24xy'+9y=96x^{5/2}$; $\{\sqrt{x},\,1/\sqrt{x},\,x^{3/2},\,x^{-3/2}\}$

Show answer

15. $x(x^2-6)y^{(4)}+2(x^2-12)y'''+x(6-x^2)y''+2(12-x^2)y'=2(x^2-6)^2$;

$\{1,\,1/x,\,e^x,\,e^{-x}\}$

Show answer

16. $x^4y^{(4)}-4x^3y'''+12x^2y''-24xy'+24y=x^4$; $\{x,\,x^2,\,x^3,\,x^4\}$

Show answer

17. $x^4y^{(4)}-4x^3y'''+2x^2(6-x^2)y''+4x(x^2-6)y'+(x^4-4x^2+24)y=4x^5e^x$;

$\{xe^x,\,x^2e^x,\,xe^{-x},\,x^2e^{-x}\}$

Show answer

18. $x^4y^{(4)}+6x^3y'''+2x^2y''-4xy'+4y=12x^2$; $\{x,x^2,1/x,1/x^2\}$

Show answer

19. $xy^{(4)}+4y'''-2xy''-4y'+xy=4e^x$; $\{e^x,\,e^{-x},\,e^x/x,\,e^{-x}/x\}$

Show answer

20. $xy^{(4)}+(4-6x)y'''+(13x-18)y''+(26-12x)y'+(4x-12)y=3e^x$; $\{e^x,\,e^{2x},\,e^x/x,\,e^{2x}/x\}$

Show answer

21. $x^4y^{(4)}-4x^3y'''+x^2(12-x^2)y''+2x(x^2-12)y'+2(12-x^2)y=2x^5$; $\{x,\,x^2,\,xe^x,\,xe^{-x}\}$

Show answer

In Exercises 22–33 solve the initial value problem, given the fundamental set of solutions of the complementary equation. Where indicated by **C/G** *, graph the solution.*

22. C/G $x^3y''' - 2x^2y'' + 3xy' - 3y = 4x, \quad y(1) = 4, \quad y'(1) = 4, \quad y''(1) = 2; \qquad \{x,\, x^3,\, x\ln x\}$

Show answer

23. $x^3y''' - 5x^2y'' + 14xy' - 18y = x^3, \quad y(1) = 0, \quad y'(1) = 1, \quad y''(1) = 7; \qquad \{x^2,\, x^3,\, x^3\ln x\}$

Show answer

24. \dst$(5-6x)y''' + (12x-4)y'' + (6x-23)y' + (22-12x)y = -(6x-5)^2e^x$

\dst$y(0) = -4, \quad y'(0) = -\frac{3}{2}, \quad y''(0) = -19; \qquad \{e^x,\, e^{2x},\, xe^{-x}\}$

Show answer

25. $x^3y''' - 6x^2y'' + 16xy' - 16y = 9x^4, \quad y(1) = 2, \quad y'(1) = 1, \quad y''(1) = 5;$

$\{x,\, x^4,\, x^4\ln|x|\}$

Show answer

26. C/G $(x^2-2x+2)y''' - x^2y'' + 2xy' - 2y = (x^2-2x+2)^2, \quad y(0) = 0, \quad y'(0) = 5,$

$y''(0) = 0; \qquad \{x,\, x^2,\, e^x\}$

Show answer

27. $x^3y''' + x^2y'' - 2xy' + 2y = x(x+1), \quad y(-1) = -6, \quad y'(-1) =$ \dst$\frac{43}{6}, \quad y''(-1) = -$\dst$\frac{5}{2};$

$\{x,\, x^2,\, 1/x\}$

Show answer

28. $(3x-1)y''' - (12x-1)y'' + 9(x+1)y' - 9y = 2e^x(3x-1)^2, \quad y(0) =$ \dst$\frac{3}{4},$

$y'(0) =$ \dst$\frac{5}{4}, \quad y''(0) =$ \dst$\frac{1}{4}; \qquad \{x+1,\, e^x,\, e^{3x}\}$

Show answer

29. C/G $(x^2-2)y''' - 2xy'' + (2-x^2)y' + 2xy = 2(x^2-2)^2, \quad y(0) = 1, \quad y'(0) = -5,$

$y''(0) = 5; \qquad \{x^2,\, e^x,\, e^{-x}\}$

Show answer

30. C/G $x^4y^{(4)} + 3x^3y''' - x^2y'' + 2xy' - 2y = 9x^2, \quad y(1) = -7, \quad y'(1) = -11, \quad y''(1) = -5, \; y'''(1) = 6; \quad \{x,\, x^2,\, 1/x,\, x\ln x\}$

Show answer

31. $(2x-1)y^{(4)}-4xy'''+(5-2x)y''+4xy'-4y=6(2x-1)^2, \quad y(0)=\dst\frac{55}{4}, \quad y'(0)=0,$

$y''(0)=13, \quad y'''(0)=1; \quad \{x,\, e^x,\, e^{-x},\, e^{2x}\}$

Show answer

32. $4x^4y^{(4)}+24x^3y'''+23x^2y''-xy'+y=6x, \quad y(1)=2, \quad y'(1)=0, \quad y''(1)=4, \quad y'''(1)=-\dst\frac{37}{4};$
$\{x,\sqrt{x},1/x,1/\sqrt{x}\}$

Show answer

33. $x^4y^{(4)}+5x^3y'''-3x^2y''-6xy'+6y=40x^3, \quad y(-1)=-1,\ y'(-1)=-7,$

$y''(-1)=-1, \quad y'''(-1)=-31; \quad \{x,\, x^3,\, 1/x,\, 1/x^2\}$

Show answer

34. Suppose the equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x) \tag{A}$$

is normal on an interval (a, b). Let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions of its complementary equation on (a, b), let W be the Wronskian of $\{y_1, y_2, \ldots, y_n\}$, and let W_j be the determinant obtained by deleting the last row and the j-th column of W. Suppose x_0 is in (a, b), let

$$u_j(x) = (-1)^{(n-j)} \int_{x_0}^{x} \frac{F(t)W_j(t)}{P_0(t)W(t)}\, dt, \quad 1 \le j \le n,$$

and define

$$y_p = u_1y_1 + u_2y_2 + \cdots + u_ny_n.$$

a. Show that y_p is a solution of (A) and that

$$y_p^{(r)} = u_1 y_1^{(r)} + u_2 y_2^{(r)} \cdots + u_n y_n^{(r)}, \quad 1 \le r \le n-1,$$

and

$$y_p^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + \frac{F}{P_0}.$$

Hint: See the derivation of the method of variation of parameters at the beginning of the section.

b. Show that y_p is the solution of the initial value problem

Missing dimension or its units for \hspace

c. Show that y_p can be written as

$$y_p(x) = \int_{x_0}^{x} G(x,t) F(t)\, dt,$$

where

$$G(x,t) = \frac{1}{P_0(t)W(t)} \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \end{vmatrix},$$

which is called the Green's function (http://www-history.mcs.st-and.ac.uk/Mathematicians/Green.html) for (A).

d. Show that

$$\frac{\partial^j G(x,t)}{\partial x^j} = \frac{1}{P_0(t)W(t)} \left| \begin{array}{cccc} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1^{(j)}(x) & y_2^{(j)}(x) & \cdots & y_n^{(j)}(x) \end{array} \right|, \quad 0 \le j \le n.$$

e. Show that if $a < t < b$ then

$$\left.\frac{\partial^j G(x,t)}{\partial x^j}\right|_{x=t} = \left\{ \begin{array}{cc} 0, & 1 \le j \le n-2, \\ \text{\dst}\frac{1}{P_0(t)}, & j = n-1. \end{array} \right.$$

f. Show that

$$y_p^{(j)}(x) = \left\{ \begin{array}{cc} \text{\dst}\int_{x_0}^{x} \frac{\partial^j G(x,t)}{\partial x^j} F(t)\,dt, & 1 \le j \le n-1, \\ \text{\dst}\frac{F(x)}{P_0(x)} + \int_{x_0}^{x} \frac{\partial^{(n)} G(x,t)}{\partial x^n} F(t)\,dt, & j = n. \end{array} \right.$$

In Exercises 35–42 use the method suggested by Exercise 34 to find a particular solution in the form $y_p = \int_{x_0}^{x} G(x,t)F(t)\,dt$, given the indicated fundamental set of solutions. Assume that x and x_0 are in an interval on which the equation is normal.

35. $y''' + 2y' - y' - 2y = F(x); \quad \{e^x,\, e^{-x}, e^{-2x}\}$

Show answer

36. $x^3y''' + x^2y'' - 2xy' + 2y = F(x); \quad \{x,\, x^2,\, 1/x\}$

Show answer

37. $x^3y''' - x^2(x+3)y'' + 2x(x+3)y' - 2(x+3)y = F(x); \{x, x^2, xe^x\}$

Show answer

38. $x(1-x)y''' + (x^2 - 3x + 3)y'' + xy' - y = F(x); \quad \{x,\, 1/x,\, e^x/x\}$

Show answer

39. $y^{(4)} - 5y'' + 4y = F(x); \quad \{e^x,\, e^{-x},\, e^{2x},\, e^{-2x}\}$

Show answer

40. $xy^{(4)} + 4y''' = F(x); \quad \{1,\, x,\, x^2,\, 1/x\}$

Show answer

41. $x^4y^{(4)} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = F(x); \quad \{x, x^2, 1/x, 1/x^2\}$

Show answer

42. $xy^{(4)} - y''' - 4xy' + 4y' = F(x); \quad \{1,\, x^2,\, e^{2x}, e^{-2x}\}$

Show answer

Chapter 9 Linear Higher Order Equations

9.1 Introduction to Linear Higher Order Equations

2 $y = \dst 2x^2 - 3x^3 + \frac{1}{x}$

3 $y = 2e^x + 3e^{-x} - e^{2x} + e^{-3x}$

4 $\dst y_i = \frac{(x-x_0)^{i-1}}{(i-1)!},\ 1 \le i \le n$

5 **(b)** $\dst y_1 = -\frac{1}{2}x^3 + x^2 + \frac{1}{2x}, \quad y_2 = \frac{1}{3}x^2 - \frac{1}{3x}, \quad y_3 = \frac{1}{4}x^3 - \frac{1}{3}x^2 + \frac{1}{12x}$

(c)

y
$= k_0 y_1$
$+ k_1 y_2$
$+ k_2 y_3$

7 $2e^{-x^2}$

8 $\sqrt{2} K \cos x$

9 **(a)** $W(x) = 2e^{3x}$ **(d)** $y = e^x(c_1 + c_2 x + c_3 x^2)$

10 **(a)** 2 **(b)** $-e^{3x}$ **(c)** 4 **(d)** $4/x^2$ **(e)** 1 **(f)** $2x$ **(g)** $2/x^2$ **(h)** $e^x(x^2 - 2x + 2)$

(i)
$-240/x^5$
(j)
$6e^{2x}(2x$
$- 1)$ **(l)**
$-128x$

24 **(a)** $y''' = 0$ **(b)** $xy''' - y'' - xy' + y = 0$ **(c)** $(2x-3)y''' - 2y'' - (2x-5)y' = 0$

(d) $(x^2 - 2x + 2)y''' - x^2y'' + 2xy' - 2y = 0$ **(e)** $x^3y''' + x^2y'' - 2xy' + 2y = 0$

(f) $(3x-1)y''' - (12x-1)y'' + 9(x+1)y' - 9y = 0$

(g) $x^4y^{(4)} + 5x^3y''' - 3x^2y'' - 6xy' + 6y = 0$

(h) $x^4y^{(4)} + 3x^2y''' - x^2y'' + 2xy' - 2y = 0$

(i) $(2x - 1)y^{(4)} - 4xy''' + (5 - 2x)y'' + 4xy' - 4y = 0$

(j) $xy^{(4)} - y''' - 4xy'' + 4y' = 0$

9.2 Higher Order Constant Coefficient Homogeneous Equations

1 $y = e^x(c_1 + c_2x + c_3x^2)$

2 $y = c_1e^x + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x$

3 $y = c_1e^x + c_2\cos 4x + c_3\sin 4x$

4 $y = c_1e^x + c_2e^{-x} + c_3e^{-3x/2}$

5 $y = c_1e^{-x} + e^{-2x}(c_1\cos x + c_2\sin x)$

6 $y = c_1e^x + e^{x/2}(c_2 + c_3x)$

7 $y = e^{-x/3}(c_1 + c_2x + c_3x^2)$

8 $y = c_1 + c_2x + c_3\cos x + c_4\sin x$

9 $y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$

10 $y = (c_1 + c_2x)\cos\sqrt{6}x + (c_3 + c_4x)\sin\sqrt{6}x$

11 $y = e^{3x/2}(c_1 + c_2x) + e^{-3x/2}(c_3 + c_4x)$

12 $y = c_1e^{-x/2} + c_2e^{-x/3} + c_3\cos x + c_4\sin x$

13 $y = c_1e^x + c_2e^{-2x} + c_3e^{-x/2} + c_4e^{-3x/2}$

14 $y = e^x(c_1 + c_2x + c_3\cos x + c_4\sin x)$

15 $y = \cos 2x - 2\sin 2x + e^{2x}$

16 $y = 2e^x + 3e^{-x} - 5e^{-3x}$

17 $y = 2e^x + 3xe^x - 4e^{-x}$

18 $y = 2e^{-x}\cos x - 3e^{-x}\sin x + 4e^{2x}$

19 $y = \backslash\mathrm{dst}\frac{9}{5}e^{-5x/3} + e^x(1 + 2x)$

20 $y = e^{2x}(1 - 3x + 2x^2)$

21 $y = e^{3x}(2-x) + 4e^{-x/2}$

22 $y = e^{x/2}(1-2x) + 3e^{-x/2}$

23 $y =$ \dst$\frac{1}{8}(5e^{2x} + e^{-2x} + 10\cos 2x + 4\sin 2x)$

24 $y = -4e^{x} + e^{2x} - e^{4x} + 2e^{-x}$

25 $y = 2e^{x} - e^{-x}$

26 $y = e^{2x} + e^{-2x} + e^{-x}(3\cos x + \sin x)$

27 $y = 2e^{-x/2} + \cos 2x - \sin 2x$

28 **(a)** $\{e^x,\ xe^x,\ e^{2x}\}:\ 1$ **(b)** $\{\cos 2x,\ \sin 2x,\ e^{3x}\}:\ 26$

(c) $\{e^{-x}\cos x,$
$e^{-x}\sin x,$
$e^{x}\}:\ 5$
(d) $\{1,$
$x,\ x^2,$
$e^x\}\ 2e^x$

(e) $\{e^x,\ e^{-x},\ \cos x,\ \sin x\}$\-$8$ **(f)** $\{\cos x,\ \sin x,\ e^x\cos x,\ e^x\sin x\}:\ 5$

29 $\{e^{-3x}\cos 2x,\ e^{-3x}\sin 2x,\ e^{2x},\ xe^{2x},\ 1,\ x,\ x^2\}$

30 $\{e^x,\ xe^x,\ e^{x/2},\ xe^{x/2},\ x^2e^{x/2},\ \cos x,\ \sin x\}$

31 $\{\cos 3x,\ x\cos 3x,\ x^2\cos 3x,\ \sin 3x,\ x\sin 3x,\ x^2\sin 3x,\ 1,\ x\}$

32 $\{e^{2x},\ xe^{2x},\ x^2e^{2x},\ e^{-x},\ xe^{-x},\ 1\}$

33 $\{\cos x,\ \sin x,\ \cos 3x,\ x\cos 3x,\ \sin 3x,\ x\sin 3x,\ e^{2x}\}$

34 $\{e^{2x},\ xe^{2x},\ e^{-2x},\ xe^{-2x},\ \cos 2x,\ x\cos 2x,\ \sin 2x,\ x\sin 2x\}$

35 $\{e^{-x/2}\cos 2x,\ xe^{-x/2}\cos 2x,\ x^2e^{-x/2}\cos 2x,\ e^{-x/2}\sin 2x,\ xe^{-x/2}\sin 2x,$

$x^2e^{-x/2}\sin 2x\}$

36 $\{1,\ x,\ x^2,\ e^{2x},\ xe^{2x},\ \cos 2x,\ x\cos 2x,\ \sin 2x,\ x\sin 2x\}$

<u>37</u> $\{\cos(x/2),\, x\cos(x/2),\, \sin(x/2),\, x\sin(x/2),\, \cos 2x/3\; x\cos(2x/3),$

$x^2\cos(2x/3),$
$\sin(2x/3),$
$x\sin(2x/3),$
$x^2\sin(2x/3)\}$

<u>38</u> $\{e^{-x},\, e^{3x},\, e^{x}\cos 2x,\, e^{x}\sin 2x\}$

<u>39</u> **(b)** \dst$e^{(a_1+a_2+\cdots+a_n)x}\prod_{1\le i<j\le n}(a_j-a_i)$

<u>43</u> **(a)** \dst $\left\{e^{x},\, e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$ **(b)** \dst $\left\{e^{-x},\, e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$

(c)
$\{e^{2x}\cos 2x,$
$e^{2x}\sin 2x,$
$e^{-2x}\cos 2x,$
$e^{-2x}\sin 2x\}$

(d) \dst $\left\{e^{x},\, e^{-x},\, e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\, e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$

(e)
$\{\cos 2x,$
$\sin 2x,$
$e^{-\sqrt{3}x}\cos x,$
$e^{-\sqrt{3}x}\sin x,$
$e^{\sqrt{3}x}\cos x,$
$e^{\sqrt{3}x}\sin x\}$

(f) \dst $\left\{1,\, e^{2x},\, e^{3x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{3x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\, e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$

(g)
\dst $\left\{e^{-x},\, e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\, e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\, e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$

45 $y = c_1x^{r_1} + c_2x^{r_2} + c_3x^{r_3}$ $(r_1, r_2, r_3$ distinct); $y = c_1x^{r_1} + (c_2 + c_3 \ln x)x^{r_2}$ $(r_1, r_2$

distinct);

$y = [c_1$

$+ c_2 \ln x$

$+ c_3(\ln x)^2]x^{r_1}$;

y

$= c_1x^{r_1}$

$+ x^{\lambda}[c_2 \cos(\omega \ln x)$

$+ c_3 \sin(\omega \ln x)]$

9.3 Undetermined Coefficients for Higher Order Equations

1 $y_p = e^{-x}(2 + x - x^2)$

2 $y_p = -\dst\frac{e^{-3x}}{4}(3 - x + x^2)$

3 $y_p = e^{x}(1 + x - x^2)$

4 $y_p = e^{-2x}(1 - 5x + x^2)$.

5 $y_p = -\dst\frac{xe^{x}}{2}(1 - x + x^2 - x^3)$

6 $y_p = x^2e^{x}(1 + x)$

7 $y_p = \dst\frac{xe^{-2x}}{2}(2 + x)$

8 $y_p = \dst\frac{x^2e^{x}}{2}(2 + x)$

9 $y_p = \dst\frac{x^2e^{2x}}{2}(1 + 2x)$

10 $y_p = x^2e^{3x}(2 + x - x^2)$

11 $y_p = x^2e^{4x}(2 + x)$

12 $y_p = \dst\frac{x^3e^{x/2}}{48}(1 + x)$

13 $y_p = e^{-x}(1 - 2x + x^2)$

14 $y_p = e^{2x}(1 - x)$

15 $y_p = e^{-2x}(1 + x + x^2 - x^3)$

16 $y_p = \dst\frac{e^{x}}{3}(1 - x)$

17 $y_p = e^{x}(1 + x)^2$

18 $y_p = xe^{x}(1 + x^3)$

19 $y_p = xe^{x}(2 + x)$

20 $y_p = \dst\frac{xe^{2x}}{6}(1 - x^2)$

21 $y_p = 4xe^{-x/2}(1+x)$

22 $y_p = \backslash\mathrm{dst}\frac{xe^x}{6}(1+x^2)$

23 $y_p = \backslash\mathrm{dst}\frac{x^2e^{2x}}{6}(1+x+x^2)$

24 $y_p = \backslash\mathrm{dst}\frac{x^2e^{2x}}{6}(3+x+x^2)$

25 $y_p = \backslash\mathrm{dst}\frac{x^3e^x}{48}(2+x)$

26 $y_p = \backslash\mathrm{dst}\frac{x^3e^x}{6}(1+x)$

27 $y_p = -\backslash\mathrm{dst}\frac{x^3e^{-x}}{6}(1-x+x^2)$

28 $y_p = \backslash\mathrm{dst}\frac{x^3e^{2x}}{12}(2+x-x^2)$

29 $y_p = e^{-x}\left[(1+x)\cos x + (2-x)\sin x\right]$

30 $y_p = e^{-x}\left[(1-x)\cos 2x + (1+x)\sin 2x\right]$

31 $y_p = e^{2x}[(1+x-x^2)\cos x + (1+2x)\sin x]$

32 $y_p = \backslash\mathrm{dst}\frac{e^x}{2}[(1+x)\cos 2x + (1-x+x^2)\sin 2x]$

33 $y_p = \backslash\mathrm{dst}\frac{x}{13}(8\cos 2x + 14\sin 2x)$

34 $y_p = xe^x[(1+x)\cos x + (3+x)\sin x]$

35 $y_p = \backslash\mathrm{dst}\frac{xe^{2x}}{2}[(3-x)\cos 2x + \sin 2x]$

36 $y_p = -\backslash\mathrm{dst}\frac{xe^{3x}}{12}(x\cos 3x + \sin 3x)$

37 $y_p = -\backslash\mathrm{dst}\frac{e^x}{10}(\cos x + 7\sin x)$

38 $y_p = \backslash\mathrm{dst}\frac{e^x}{12}(\cos 2x - \sin 2x)$

39 $y_p = xe^{2x}\cos 2x$

40 $y_p = -\backslash\mathrm{dst}\frac{e^{-x}}{2}[(1+x)\cos x + (2-x)\sin x]$

41 $y_p = \backslash\mathrm{dst}\frac{xe^{-x}}{10}(\cos x + 2\sin x)$

42 $y_p = \backslash\mathrm{dst}\frac{xe^x}{40}(3\cos 2x - \sin 2x)$

43 $y_p = \backslash\mathrm{dst}\frac{xe^{-2x}}{8}[(1-x)\cos 3x + (1+x)\sin 3x]$

44 $y_p = -\backslash\mathrm{dst}\frac{xe^x}{4}(1+x)\sin 2x$

45 $y_p = \backslash\mathrm{dst}\frac{x^2e^{-x}}{4}(\cos x - 2\sin x)$

46 $y_p = -\backslash\mathrm{dst}\frac{x^2e^{2x}}{32}(\cos 2x - \sin 2x)$

47 $y_p = \backslash\mathrm{dst}\frac{x^2e^{2x}}{8}(1+x)\sin x$

48 $y_p = 2x^2e^x + xe^{2x} - \cos x$

49 $y_p = e^{2x} + xe^x + 2x\cos x$

50 $y_p = 2x + x^2 + 2xe^x - 3xe^{-x} + 4e^{3x}$

51 $y_p = xe^x(\cos 2x - 2\sin 2x) + 2xe^{2x} + 1$

52 $y_p = x^2e^{-2x}(1+2x) - \cos 2x + \sin 2x$

53 $y_p = 2x^2(1+x)e^{-x} + x\cos x - 2\sin x$

54 $y_p = 2xe^x + xe^{-x} + \cos x$

55 $y_p = \backslash\mathrm{dst}\frac{xe^x}{6}(\cos x + \sin 2x)$

56 $y_p = \backslash\mathrm{dst}\frac{x^2}{54}[(2+2x)e^x + 3e^{-2x}]$

57 $y_p = \backslash\mathrm{dst}\frac{x}{8}\sinh x \sin x$

58 $y_p = x^3(1+x)e^{-x} + xe^{-2x}$

59 $y_p = xe^x(2x^2 + \cos x + \sin x)$

60 $y = e^{2x}(1+x) + c_1e^{-x} + e^x(c_2 + c_3x)$

61 $y = e^{3x}\backslash\mathrm{dst}\left(1 - x - \frac{x^2}{2}\right) + c_1e^x + e^{-x}(c_2\cos x + c_3\sin x)$

62 $y = xe^{2x}(1+x)^2 + c_1e^x + c_2e^{2x} + c_3e^{3x}$

63 $y = x^2 e^{-x}(1-x)^2 + c_1 + e^{-x}(c_2 + c_3 x)$

64 $y = \backslash\mathrm{dst}\frac{x^3 e^x}{24}(4+x) + e^x(c_1 + c_2 x + c_3 x^2)$

65 $y = \backslash\mathrm{dst}\frac{x^2 e^{-x}}{16}(1 + 2x - x^2) + e^x(c_1 + c_2 x) + e^{-x}(c_3 + c_4 x)$

66 $y = e^{-2x}\backslash\mathrm{dst}\left[(1 + \frac{x}{2})\cos x + \left(\frac{3}{2} - 2x\right)\sin x\right] + c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$

67 $y = -xe^x \sin 2x + c_1 + c_2 e^x + e^x(c_3 \cos x + c_4 \sin x)$

68 $y = -\backslash\mathrm{dst}\frac{x^2 e^x}{16}(1+x)\cos 2x + e^x\left[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\right]$

69 $y = (x^2 + 2)e^x - e^{-2x} + e^{3x}$

70 $y = e^{-x}(1 + x + x^2) + (1-x)e^x$

71 $y = \backslash\mathrm{dst}\left(\frac{x^2}{12} + 16\right) xe^{-x/2} - e^x$

72 $y = (2-x)(x^2+1)e^{-x} + \cos x - \sin x$

73 $y = (2-x)\cos x - (1-7x)\sin x + e^{-2x}$

74 $2 + e^x\left[(1+x)\cos x - \sin x - 1\right]$

9.4 Variation of Parameters for Higher Order Equations

1 $y_p = 2x^3$

2 $y_p = \displaystyle\frac{8}{105}x^{7/2}e^{-x^2}$

3 $y_p = x\ln|x|$

4 $y_p = \displaystyle-\frac{2(x^2+2)}{x}$

5 $y_p = \displaystyle-\frac{xe^{-3x}}{64}$

6 $y_p = -\displaystyle\frac{2x^2}{3}$

7 $y_p = -\displaystyle\frac{e^{-x}(x+1)}{x}$

8 $y_p = 2x^2\ln|x|$

9 $y_p = x^2+1$

10 $y_p = \displaystyle\frac{2x^2+6}{3}$

11 $y_p = \displaystyle\frac{x^2\ln|x|}{3}$

12 $y_p = -x^2-2$

13 $\displaystyle\frac{1}{4}x^3\ln|x| - \frac{25}{48}x^3$

14 $y_p = \displaystyle\frac{x^{5/2}}{4}$

15 $y_p = \displaystyle\frac{x(12-x^2)}{6}$

16 $y_p = \displaystyle\frac{x^4\ln|x|}{6}$

17 $y_p = \displaystyle\frac{x^3e^x}{2}$

18 $y_p = x^2\ln|x|$

19 $y_p = \displaystyle\frac{xe^x}{2}$

20 $y_p = \displaystyle\frac{3xe^x}{2}$

21 $y_p = -x^3$

22 $y = -x(\ln x)^2 + 3x + x^3 - 2x\ln x$

23 $y = \backslash\mathrm{dst}\frac{x^3}{2}(\ln|x|)^2 + x^2 - x^3 + 2x^3\ln|x|$

24 $y = \backslash\mathrm{dst}-\frac{1}{2}(3x+1)xe^x - 3e^x - e^{2x} + 4xe^{-x}$

25 $y = \backslash\mathrm{dst}\frac{3}{2}x^4(\ln x)^2 + 3x - x^4 + 2x^4\ln x$

26 $y = \backslash\mathrm{dst}-\frac{x^4+12}{6} + 3x - x^2 + 2e^x$

27 $y = \backslash\mathrm{dst}\left(\frac{x^2}{3} - \frac{x}{2}\right)\ln|x| + 4x - 2x^2$

28 $y = \backslash\mathrm{dst}-\frac{xe^x(1+3x)}{2} + \frac{x+1}{2} - \frac{e^x}{4} + \frac{e^{3x}}{2}$

29 $y = -8x + 2x^2 - 2x^3 + 2e^x - e^{-x}$

30 $y = 3x^2\ln x - 7x^2$

31 $y = \backslash\mathrm{dst}\frac{3(4x^2+9)}{2} + \frac{x}{2} - \frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}$

32 $y = x\ln x + x - \sqrt{x} + \backslash\mathrm{dst}\frac{1}{x} + \frac{1}{\sqrt{x}}.$

33 $y = \backslash\mathrm{dst}x^3\ln|x| + x - 2x^3 + \frac{1}{x} - \frac{1}{x^2}$

35 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{e^{(x-t)} - 3e^{-(x-t)} + 2e^{-2(x-t)}}{6}F(t)\,dt$

36 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{(x-t)^2(2x+t)}{6xt^3}F(t)\,dt$

37 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{xe^{(x-t)} - x^2 + x(t-1)}{t^4}F(t)\,dt$

38 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{x^2 - t(t-2) - 2te^{(x-t)}}{2x(t-1)^2}F(t)\,dt$

39 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{e^{2(x-t)} - 2e^{(x-t)} + 2e^{-(x-t)} - e^{-2(x-t)}}{12}F(t)\,dt$

40 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{(x-t)^3}{6x}F(t)\,dt$

41 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{(x+t)(x-t)^3}{12x^2t^3}F(t)\,dt$

42 $y_p = \backslash\mathrm{dst}\int_{x_0}^{x}\frac{e^{2(x-t)}(1+2t) + e^{-2(x-t)}(1-2t) - 4x^2 + 4t^2 - 2}{32t^2}F(t)\,dt$