7 Series Solutions of Linear Second Order Equations

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IN THIS CHAPTER we study a class of second order differential equations that occur in many applications, but can't be solved in closed form in terms of elementary functions. Here are some examples:

(1) Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

which occurs in problems displaying cylindrical symmetry, such as diffraction of light through a circular aperture, propagation of electromagnetic radiation through a coaxial cable, and vibrations of a circular drum head.

(2) Airy's equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Airy.html),

$$y'' - xy = 0,$$

which occurs in astronomy and quantum physics.

(3) Legendre's equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

which occurs in problems displaying spherical symmetry, particularly in electromagnetism.

These equations and others considered in this chapter can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{A}$$

where P_0, P_1, and P_2 are polynomials with no common factor. For most equations that occur in applications, these polynomials are of degree two or less. We'll impose this restriction, although the methods that we'll develop can be extended to the case where the coefficient functions are polynomials of arbitrary degree, or even power series that converge in some circle around the origin in the complex plane.

Since (A) does not in general have closed form solutions, we seek series representations for solutions. We'll see that if $P_0(0) \neq 0$ then solutions of (A) can be written as power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

that converge in an open interval centered at $x = 0$.

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SECTION 7.1 reviews the properties of power series.

SECTIONS 7.2 AND 7.3 are devoted to finding power series solutions of (A) in the case where $P_0(0) \neq 0$. The situation is more complicated if $P_0(0) = 0$; however, if P_1 and P_2 satisfy assumptions that apply to most equations of interest, then we're able to use a modified series method to obtain solutions of (A).

SECTION 7.4 introduces the appropriate assumptions on P_1 and P_2 in the case where $P_0(0) = 0$, and deals with Euler's equation

$$ax^2y'' + bxy' + cy = 0,$$

where a, b, and c are constants. This is the simplest equation that satisfies these assumptions.

SECTIONS 7.5 –7.7 deal with three distinct cases satisfying the assumptions introduced in Section 7.4. In all three cases, (A) has at least one solution of the form

$$y_1 = x^r \sum_{n=0}^{\infty} a_n x^n,$$

where r need not be an integer. The problem is that there are three possibilities – each requiring a different approach – for the form of a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental pair of solutions of (A).

Answers to selected exercises in this chapter

7.1 Review of Power Series

Many applications give rise to differential equations with solutions that can't be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. The solutions of some of the most important of these equations can be expressed in terms of power series. We'll study such equations in this chapter. In this section we review relevant properties of power series. We'll omit proofs, which can be found in any standard calculus text.

Definition 7.1.1

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \tag{7.1.1}$$

where x_0 and a_0, a_1, …, a_n, … are constants, is called a power series in $x-x_0$. We say that the power series (7.1.1) converges for a given x if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n$$

exists; otherwise, we say that the power series diverges for the given x.

A power series in $x-x_0$ must converge if $x=x_0$, since the positive powers of $x-x_0$ are all zero in this case. This may be the only value of x for which the power series converges. However, the next theorem shows that if the power series converges for some $x \neq x_0$ then the set of all values of x for which it converges forms an interval.

Theorem 7.1.2

For any power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

exactly one of the these statements is true:

i. The power series converges only for $x=x_0$.
ii. The power series converges for all values of x.
iii. There's a positive number R such that the power series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

In case (iii) we say that R is the radius of convergence of the power series. For convenience, we include the other two cases in this definition by defining $R=0$ in case (i) and $R=\infty$ in case (ii). We define the open interval of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ to be

$$(x_0 - R, x_0 + R) \quad \text{if} \quad 0 < R < \infty, \quad \text{or} \quad (-\infty, \infty) \quad \text{if} \quad R = \infty.$$

If R is finite, no general statement can be made concerning convergence at the endpoints $x = x_0 \pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

Recall from calculus that a series of constants $\sum_{n=0}^{\infty} \alpha_n$ is said to converge absolutely if the series of absolute values $\sum_{n=0}^{\infty} |\alpha_n|$ converges. It can be shown that a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with a positive radius of convergence R converges absolutely in its open interval of convergence; that is, the series

$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

of absolute values converges if $|x - x_0| < R$. However, if $R < \infty$, the series may fail to converge absolutely at an endpoint $x_0 \pm R$, even if it converges there.

The next theorem provides a useful method for determining the radius of convergence of a power series. It's derived in calculus by applying the ratio test to the corresponding series of absolute values. For related theorems see Exercises 2 and 4.

Theorem 7.1.3

Suppose there's an integer N such that $a_n \neq 0$ if $n \geq N$ and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = 1/L$, which should be interpreted to mean that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$.

Example 7.1.1

Find the radius of convergence of the series:

$$\textbf{(a)} \quad \sum_{n=0}^{\infty} n!x^n \quad \textbf{(b)} \quad \sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!} \quad \textbf{(c)} \quad \sum_{n=0}^{\infty} 2^n n^2 (x-1)^n.$$

Solution (a) Here $a_n = n!$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty.$$

Hence, $R = 0$.

Solution (b) Here $a_n = (1)^n/n!$ for $n \ge N = 10$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

Solution (c) Here $a_n = 2^n n^2$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}(n+1)^2}{2^n n^2} = 2\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 2.$$

Hence, $R = 1/2$.

Taylor Series

If a function f has derivatives of all orders at a point $x = x_0$, then the Taylor series of f about (http://www-history.mcs.st-and.ac.uk/Mathematicians/Taylor.html) x_0 is defined by

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the Maclaurin series of (http://www-history.mcs.st-and.ac.uk/Mathematicians/Maclaurin.html) f.

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you are probably familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty, \tag{7.1.2}$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty, \tag{7.1.3}$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty, \tag{7.1.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n, \quad -1 < x < 1. \tag{7.1.5}$$

Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

on its open interval of convergence. We say that the series represents f on the open interval of convergence. A function f represented by a power series may be a familiar elementary function as in (7.1.2)–(7.1.5); however, it often happens that f isn't a familiar function, so the series actually defines f.

The next theorem shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series, and provides power series representations of the derivatives.

Theorem 7.1.4

A power series

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x)=\backslash\mathrm{dst}\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}, \tag{7.1.6}$$

$$f''(x)=\backslash\mathrm{dst}\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}, \tag{7.1.7}$$

$$\vdots$$

$$f^{(k)}(x)=\backslash\mathrm{dst}\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}. \tag{7.1.8}$$

Moreover, all of these series have the same radius of convergence R.

Example 7.1.2

Let $f(x)=\sin x$. From (7.1.3),

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From (7.1.6),

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series (7.1.4) for $\cos x$.

Uniqueness of Power Series

The next theorem shows that if f is defined by a power series in $x - x_0$ with a positive radius of convergence, then the power series is the Taylor series of f about x_0.

Theorem 7.1.5

If the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has a positive radius of convergence, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}; \tag{7.1.9}$$

that is, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is the Taylor series of f about x_0.

This result can be obtained by setting $x = x_0$ in (7.1.8), which yields

$$f^{(k)}(x_0) = k(k-1)\cdots 1 \cdot a_k = k! a_k.$$

This implies that

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Except for notation, this is the same as (7.1.9).

The next theorem lists two important properties of power series that follow from Theorem 7.1.5.

Theorem 7.1.6

a. If

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

for all x in an open interval that contains x_0, then $a_n = b_n$ for $n = 0,\, 1,\, 2,\, \dots$.

b. If

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$$

for all x in an open interval that contains x_0, then $a_n = 0$ for $n = 0,\, 1,\, 2,\, \dots$.

To obtain **(a)** we observe that the two series represent the same function f on the open interval; hence, Theorem 7.1.5 implies that

$$a_n = b_n = \frac{f^{(n)}(x_0)}{n!}, \quad n = 0, 1, 2, \dots .$$

(b) can be obtained from **(a)** by taking $b_n = 0$ for $n = 0,\, 1,\, 2,\, \dots$.

Taylor Polynomials

If f has N derivatives at a point x_0, we say that

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is the N-th Taylor polynomial of f about x_0 (http://www-history.mcs.st-and.ac.uk/Mathematicians/Taylor.html). This definition and Theorem 7.1.5 imply that if

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

where the power series has a positive radius of convergence, then the Taylor polynomials of f about x_0 are given by

$$T_N(x) = \sum_{n=0}^{N} a_n(x-x_0)^n.$$

In numerical applications, we use the Taylor polynomials to approximate f on subintervals of the open interval of convergence of the power series. For example, (7.1.2) implies that the Taylor polynomial T_N of $f(x) = e^x$ is

$$T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}.$$

The solid curve in Figure 7.1.1 is the graph of $y = e^x$ on the interval $[0, 5]$. The dotted curves in Figure 7.1.1 are the graphs of the Taylor polynomials T_1, …, T_6 of $y = e^x$ about $x_0 = 0$. From this figure, we conclude that the accuracy of the approximation of $y = e^x$ by its Taylor polynomial T_N improves as N increases.

Figure 7.1.1. *Approximation of $y = e^x$ by Taylor polynomials about $x = 0$*

Shifting the Summation Index

In Definition 7.1.1 of a power series in $x - x_0$, the n-th term is a constant multiple of $(x - x_0)^n$. This isn't true in (7.1.6), (7.1.7), and (7.1.8), where the general terms are constant multiples of $(x - x_0)^{n-1}$, $(x - x_0)^{n-2}$, and $(x - x_0)^{n-k}$, respectively. However, these series can all be rewritten so that their n-th terms are constant multiples of $(x - x_0)^n$. For example, letting $n = k + 1$ in the series in (7.1.6) yields

$$f'(x) = \sum_{k=0}^{\infty} (k+1)a_{k+1}(x-x_0)^k, \tag{7.1.10}$$

where we start the new summation index k from zero so that the first term in (7.1.10) (obtained by setting $k = 0$) is the same as the first term in (7.1.6) (obtained by setting $n = 1$). However, the sum of a series is independent of the symbol used to denote the summation index, just as the value of a definite integral is independent of the symbol used to denote the variable of integration. Therefore we can replace k by n in (7.1.10) to obtain

$$f'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-x_0)^n, \tag{7.1.11}$$

where the general term is a constant multiple of $(x-x_0)^n$.

It isn't really necessary to introduce the intermediate summation index k. We can obtain (7.1.11) directly from (7.1.6) by replacing n by $n+1$ in the general term of (7.1.6) and subtracting 1 from the lower limit of (7.1.6). More generally, we use the following procedure for shifting indices.

Shifting the Summation Index in a Power Series

For any integer k, the power series

$$\sum_{n=n_0}^{\infty} b_n (x-x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k} (x-x_0)^{n};$$

that is, replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

Example 7.1.3

Rewrite the following power series from (7.1.7) and (7.1.8) so that the general term in each is a constant multiple of $(x-x_0)^n$:

$$\textbf{(a)}\ \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} \qquad \textbf{(b)}\ \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}.$$

Solution (a) Replacing n by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n.$$

Solution (b) Replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k} = \sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n.$$

Example 7.1.4

Given that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n.

Solution From Theorem 7.1.4 with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x) = \sum_{n=0}^{\infty} (n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. (We'll see later that sometimes it's useful to include zero terms at the beginning of a series.)

Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} ca_n(x-x_0)^n.$$

Two power series

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{and} \quad g(x)=\sum_{n=0}^{\infty} b_n(x-x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x-x_0|<R_1$ and the second converges for $|x-x_0| < R_2$, then

$$f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$$

for $|x-x_0|<R$, where R is the smaller of R_1 and R_2. More generally, linear combinations of power series can be formed term by term; for example,

$$c_1f(x)+c_2f(x)=\sum_{n=0}^{\infty}(c_1a_n+c_2b_n)(x-x_0)^n.$$

Example 7.1.5

Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x}.

Solution By definition,

$$\cosh x = \frac{1}{2}e^x+\frac{1}{2}e^{-x}.$$

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{2}[1 + (-1)^n] \frac{x^n}{n!}. \tag{7.1.12}$$

Since

$$\frac{1}{2}[1 + (-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m+1, \text{ an odd integer,} \end{cases}$$

we can rewrite (7.1.12) more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for e^x and e^{-x}.

Example 7.1.6

Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

a. Express

$$(2-x)y''+2y$$

as a power series in x on I.

b. Use the result of **(a)** to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2-x)y''+2y=0 \tag{7.1.13}$$

on I.

Solution (a) From (7.1.7) with $x_0=0$,

$$y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}.$$

Therefore

$$\begin{array}{rcl}(2-x)y''+2y &=& 2y''-xy'+2y\\ {[3pt]} &=& \backslash\mathrm{dst}\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=0}^{\infty}2a_nx^n.\end{array} \tag{7.1.14}$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of x^n; thus,

$$\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n \tag{7.1.15}$$

and

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}=\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n=\sum_{n=0}^{\infty}(n+1)na_{n+1}x^n, \tag{7.1.16}$$

where we added a zero term in the last series so that when we substitute from (7.1.15) and (7.1.16) into (7.1.14) all three series will start with $n=0$; thus,

$$(2-x)y''+2y=\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n]x^n. \tag{7.1.17}$$

Solution (b) From (7.1.17) we see that y satisfies (7.1.13) on I if

$$2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n=0, \quad n=0,1,2,\dots. \tag{7.1.18}$$

Conversely, Theorem 7.1.6 **(b)** implies that if $y=\sum_{n=0}^{\infty}a_nx^n$ satisfies (7.1.13) on I, then (7.1.18) holds.

Example 7.1.7

Suppose

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

on an open interval I that contains $x_0=1$. Express the function

$$(1+x)y''+2(x-1)^2y'+3y \tag{7.1.19}$$

as a power series in $x-1$ on I.

Solution Since we want a power series in $x-1$, we rewrite the coefficient of y'' in (7.1.19) as $1+x=2+(x-1)$, so (7.1.19) becomes

$$2y''+(x-1)y''+2(x-1)^2y'+3y.$$

From (7.1.6) and (7.1.7) with $x_0=1$,

$$y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}.$$

Therefore

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2}, \\ (x-1)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}, \\ 2(x-1)^2y' &= \sum_{n=1}^{\infty} 2na_n(x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty} 3a_n(x-1)^n. \end{aligned}$$

Before adding these four series we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. This yields

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n, \tag{7.1.20}$$

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n, \tag{7.1.21}$$

$$2(x-1)^2y' = \sum_{n=1}^{\infty} 2(n-1)a_{n-1}(x-1)^n, \tag{7.1.22}$$

$$3y = \sum_{n=0}^{\infty} 3a_n(x-1)^n, \tag{7.1.23}$$

where we added initial zero terms to the series in (7.1.21) and (7.1.22). Adding (7.1.20)–(7.1.23) yields

$$\begin{aligned} (1+x)y'' + 2(x-1)^2y' + 3y &= 2y'' + (x-1)y'' + 2(x-1)^2y' + 3y \\ &= \sum_{n=0}^{\infty} b_n(x-1)^n, \end{aligned}$$

where

$$b_0 = 4a_2 + 3a_0, \tag{7.1.24}$$

$$b_n = 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2(n-1)a_{n-1} + 3a_n,\ n \geq 1. \tag{7.1.25}$$

The formula (7.1.24) for b_0 can't be obtained by setting $n = 0$ in (7.1.25), since the summation in (7.1.22) begins with $n = 1$, while those in (7.1.20), (7.1.21), and (7.1.23) begin with $n = 0$.

7.1 Exercises

1. For each power series use Theorem 7.1.3 to find the radius of convergence R. If $R>0$, find the open interval of convergence.

(a) \dst$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n}(x-1)^n$	**(b)** \dst$\sum_{n=0}^{\infty}2^n n(x-2)^n$
(c) \dst$\sum_{n=0}^{\infty}\frac{n!}{9^n}x^n$	**(d)** \dst$\sum_{n=0}^{\infty}\frac{n(n+1)}{16^n}(x-2)^n$
(e) \dst$\sum_{n=0}^{\infty}(-1)^n\frac{7^n}{n!}x^n$	**(f)** \dst$\sum_{n=0}^{\infty}\frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$

Show answer

2. Suppose there's an integer M such that $b_m\neq 0$ for $m\geq M$, and

$$\lim_{m\to\infty}\left|\frac{b_{m+1}}{b_m}\right|=L,$$

where $0\leq L\leq\infty$. Show that the radius of convergence of

$$\sum_{m=0}^{\infty}b_m(x-x_0)^{2m}$$

is $R=1/\sqrt{L}$, which is interpreted to mean that $R=0$ if $L=\infty$ or $R=\infty$ if $L=0$. *Hint: Apply Theorem* ??? *to the series* $\sum_{m=0}^{\infty}b_m z^m$ *and then let* $z=(x-x_0)^2$*.*

3. For each power series, use the result of Exercise 2 to find the radius of convergence R. If $R>0$, find the open interval of convergence.

(a) \dst$\sum_{m=0}^{\infty}(-1)^m(3m+1)(x-1)^{2m+1}$	**(b)** \dst$\sum_{m=0}^{\infty}(-1)^m\frac{m(2m+1)}{2^m}(x+2)^{2m}$
(c) \dst$\sum_{m=0}^{\infty}\frac{m!}{(2m)!}(x-1)^{2m}$	**(d)** \dst$\sum_{m=0}^{\infty}(-1)^m\frac{m!}{9^m}(x+8)^{2m}$
(e) \dst$\sum_{m=0}^{\infty}(-1)^m\frac{(2m-1)}{3^m}x^{2m+1}$	**(f)** \dst$\sum_{m=0}^{\infty}(x-1)^{2m}$

Show answer

4. Suppose there's an integer M such that $b_m \ne 0$ for $m \ge M$, and

$$\lim_{m\to\infty}\left|\frac{b_{m+1}}{b_m}\right| = L,$$

where $0 \le L \le \infty$. Let k be a positive integer. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m (x-x_0)^{km}$$

is $R = 1/\sqrt[k]{L}$, which is interpreted to mean that $R=0$ if $L=\infty$ or $R=\infty$ if $L=0$. *Hint: Apply Theorem* $???$ *to the series* $\sum_{m=0}^{\infty} b_m z^m$ *and then let* $z=(x-x_0)^k$*.*

5. For each power series use the result of Exercise 4 to find the radius of convergence R. If $R>0$, find the open interval of convergence.

(a) \dst$\sum_{m=0}^{\infty} \frac{(-1)^m}{(27)^m}(x-3)^{3m+2}$	**(b)** \dst$\sum_{m=0}^{\infty} \frac{x^{7m+6}}{m}$
(c) \dst$\sum_{m=0}^{\infty} \frac{9^m(m+1)}{(m+2)}(x-3)^{4m+2}$	**(d)** \dst$\sum_{m=0}^{\infty} (-1)^m \frac{2^m}{m!} x^{4m+3}$
(e) \dst$\sum_{m=0}^{\infty} \frac{m!}{(26)^m}(x+1)^{4m+3}$	**(f)** \dst$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m m(m+1)}(x-1)^{3m+1}$

Show answer

6. L Graph $y=\sin x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1,\ 2,\ 3,\ \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs.

7. L Graph $y=\cos x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n}}{(2n)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1,\ 2,\ 3,\ \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs.

8. L Graph $y = 1/(1-x)$ and the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} x^n$$

on the interval $[0, .95]$ for $N = 1$, 2, 3, …, until you find a value of N for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $0 \le y \le 20$.

9. L Graph $y = \cosh x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^{M} \frac{x^{2n}}{(2n)!}$$

on the interval $(-5, 5)$ for $M = 1$, 2, 3, …, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $0 \le y \le 75$.

10. L Graph $y = \sinh x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^{M} \frac{x^{2n+1}}{(2n+1)!}$$

on the interval $(-5, 5)$ for $M = 0$, 1, 2, …, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $-75 \le y \le 75$.

In Exercises 11–15 find a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

11. $\vspace*{-5pt}(2+x)y''+xy'+3y$

Show answer

12. $(1+3x^2)y''+3x^2y'-2y$

Show answer

13. $(1+2x^2)y''+(2-3x)y'+4y$

Show answer

14. $(1+x^2)y''+(2-x)y'+3y$

Show answer

15. $(1+3x^2)y''-2xy'+4y$

Show answer

16. Suppose $y(x)=\sum_{n=0}^{\infty}a_n(x+1)^n$ on an open interval that contains $x_0 = -1$. Find a power series in $x+1$ for

$$xy''+(4+2x)y'+(2+x)y.$$

Show answer

17. Suppose $y(x)=\sum_{n=0}^{\infty}a_n(x-2)^n$ on an open interval that contains $x_0 = 2$. Find a power series in $x-2$ for

$$x^2y''+2xy'-3xy.$$

Show answer

18. L Do the following experiment for various choices of real numbers a_0 and a_1.

a. Use differential equations software to solve the initial value problem

$$(2-x)y''+2y=0, \quad y(0)=a_0, \quad y'(0)=a_1,$$

numerically on $(-1.95, 1.95)$. Choose the most accurate method your software package provides. (See Section 10.1 for a brief discussion of one such method.)

b. For $N=2$, 3, 4, …, compute a_2, …, a_N from Eqn.(7.1.18) and graph

$$T_N(x)=\sum_{n=0}^{N} a_n x^n$$

and the solution obtained in **(a)** on the same axes. Continue increasing N until it's obvious that there's no point in continuing. (This sounds vague, but you'll know when to stop.)

19. L Follow the directions of Exercise 18 for the initial value problem

$$(1+x)y''+2(x-1)^2y'+3y=0, \quad y(1)=a_0, \quad y'(1)=a_1,$$

on the interval $(0,2)$. Use Eqns. (7.1.24) and (7.1.25) to compute $\{a_n\}$.

20. Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges on an open interval $(-R,R)$, let r be an arbitrary real number, and define

$$y(x)=x^r\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} a_n x^{n+r}$$

on $(0,R)$. Use Theorem 7.1.4 and the rule for differentiating the product of two functions to show that

$$\begin{aligned}
y'(x) &= \backslash\mathrm{dst}\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1},\\
y''(x) &= \backslash\mathrm{dst}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2},\\
&\ \vdots\\
y^{(k)}(x) &= \backslash\mathrm{dst}\sum_{n=0}^{\infty}(n+r)(n+r-1)\cdots(n+r-k)a_n x^{n+r-k}
\end{aligned}$$

on $(0,R)$

In Exercises 21–26 let y be as defined in Exercise 20, and write the given expression in the form $x^r\sum_{n=0}^{\infty} b_n x^n$.

21. $x^2(1-x)y''+x(4+x)y'+(2-x)y$

Show answer

22. $x^2(1+x)y''+x(1+2x)y'-(4+6x)y$

Show answer

23. $x^2(1+x)y''-x(1-6x-x^2)y'+(1+6x+x^2)y$

Show answer

24. $x^2(1+3x)y''+x(2+12x+x^2)y'+2x(3+x)y$

Show answer

25. $x^2(1+2x^2)y''+x(4+2x^2)y'+2(1-x^2)y$

Show answer

26. $x^2(2+x^2)y''+2x(5+x^2)y'+2(3-x^2)y$

Show answer

7.2 Series Solutions Near an Ordinary Point I

Many physical applications give rise to second order homogeneous linear differential equations of the form

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0, \tag{7.2.1}$$

where P_0, P_1, and P_2 are polynomials. Usually the solutions of these equations can't be expressed in terms of familiar elementary functions. Therefore we'll consider the problem of representing solutions of (7.2.1) with series.

We assume throughout that P_0, P_1 and P_2 have no common factors. Then we say that x_0 is an ordinary point of (7.2.1) if $P_0(x_0)\neq 0$, or a singular point if $P_0(x_0)=0$. For Legendre's equation,

$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0, \tag{7.2.2}$$

$x_0=1$ and $x_0=-1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2y''+xy'+(x^2-\nu^2)y=0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If P_0 is a nonzero constant as in Airy's equation,

$$y'' - xy = 0, \tag{7.2.3}$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, P_1/P_0 and P_2/P_0 are continuous at any point x_0 that isn't a zero of P_0. Therefore, if x_0 is an ordinary point of (7.2.1) and a_0 and a_1 are arbitrary real numbers, then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{7.2.4}$$

has a unique solution on the largest open interval that contains x_0 and does not contain any zeros of P_0. To see this, we rewrite the differential equation in (7.2.4) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$. In this section and the next we consider the problem of representing solutions of (7.2.1) by power series that converge for values of x near an ordinary point x_0.

We state the next theorem without proof.

Theorem 7.2.1

Suppose P_0, P_1, and P_2 are polynomials with no common factor and P_0 isn't identically zero. Let x_0 be a point such that $P_0(x_0) \neq 0$, and let ρ be the distance from x_0 to the nearest zero of P_0 in the complex plane. (If P_0 is constant, then $\rho = \infty$.) Then every solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{7.2.5}$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{7.2.6}$$

> that converges at least on the open interval $(x_0-\rho, x_0+\rho)$. (If P_0 is nonconstant, so that ρ is necessarily finite, then the open interval of convergence of $(???)$ may be larger than $(x_0-\rho, x_0+\rho)$. If P_0 is constant then $\rho=\infty$ and $(x_0-\rho, x_0+\rho)=(-\infty,\infty)$.)

We call (7.2.6) a power series solution in $x-x_0$ of (7.2.5). We'll now develop a method for finding power series solutions of (7.2.5). For this purpose we write (7.2.5) as $Ly=0$, where

$$Ly=P_0y''+P_1y'+P_2y. \tag{7.2.7}$$

Theorem 7.2.1 implies that every solution of $Ly=0$ on $(x_0-\rho, x_0+\rho)$ can be written as

$$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Setting $x=x_0$ in this series and in the series

$$y'=\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}$$

shows that $y(x_0)=a_0$ and $y'(x_0)=a_1$. Since every initial value problem (7.2.4) has a unique solution, this means that a_0 and a_1 can be chosen arbitrarily, and a_2, a_3, … are uniquely determined by them.

To find a_2, a_3, …, we write P_0, P_1, and P_2 in powers of $x-x_0$, substitute

$$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

$$y'=\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1},$$

$$y''=\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$$

into (7.2.7), and collect the coefficients of like powers of $x-x_0$. This yields

$$Ly=\sum_{n=0}^{\infty} b_n(x-x_0)^n, \tag{7.2.8}$$

where $\{b_0, b_1, \dots, b_n, \dots\}$ are expressed in terms of $\{a_0, a_1, \dots, a_n, \dots\}$ and the coefficients of P_0, P_1, and P_2, written in powers of $x - x_0$. Since (7.2.8) and **(a)** of Theorem 7.1.6 imply that $Ly = 0$ if and only if $b_n = 0$ for $n \geq 0$, all power series solutions in $x - x_0$ of $Ly = 0$ can be obtained by choosing a_0 and a_1 arbitrarily and computing a_2, a_3, …, successively so that $b_n = 0$ for $n \geq 0$. For simplicity, we call the power series obtained this way the power series in $x - x_0$ for the general solution of $Ly = 0$, without explicitly identifying the open interval of convergence of the series.

Example 7.2.1

Let x_0 be an arbitrary real number. Find the power series in $x - x_0$ for the general solution of

$$y'' + y = 0. \tag{7.2.9}$$

Solution Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} + \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n + \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + a_n.$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \tag{7.2.10}$$

where a_0 and a_1 are arbitrary. Since the indices on the left and right sides of (7.2.10) differ by two, we write (7.2.10) separately for n even $(n = 2m)$ and n odd $(n = 2m+1)$. This yields

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Computing the coefficients of the even powers of $x - x_0$ from (7.2.11) yields

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 1} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = -\frac{1}{4 \cdot 3}\left(-\frac{a_0}{2 \cdot 1}\right) = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \\ a_6 &= -\frac{a_4}{6 \cdot 5} = -\frac{1}{6 \cdot 5}\left(\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}\right) = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, \end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \quad m \geq 0. \tag{7.2.13}$$

Computing the coefficients of the odd powers of $x - x_0$ from (7.2.12) yields

$$\begin{aligned} a_3 &= -\frac{a_1}{3 \cdot 2} \\ a_5 &= -\frac{a_3}{5 \cdot 4} = -\frac{1}{5 \cdot 4}\left(-\frac{a_1}{3 \cdot 2}\right) = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \\ a_7 &= -\frac{a_5}{7 \cdot 6} = -\frac{1}{7 \cdot 6}\left(\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}\right) = -\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}, \end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \quad m \geq 0. \tag{7.2.14}$$

Thus, the general solution of (7.2.9) can be written as

$$y=\sum_{m=0}^{\infty}a_{2m}(x-x_0)^{2m}+\sum_{m=0}^{\infty}a_{2m+1}(x-x_0)^{2m+1},$$

or, from (7.2.13) and (7.2.14), as

$$y=a_0\sum_{m=0}^{\infty}(-1)^m\frac{(x-x_0)^{2m}}{(2m)!}+a_1\sum_{m=0}^{\infty}(-1)^m\frac{(x-x_0)^{2m+1}}{(2m+1)!}. \tag{7.2.15}$$

If we recall from calculus that

$$\sum_{m=0}^{\infty}(-1)^m\frac{(x-x_0)^{2m}}{(2m)!}=\cos(x-x_0) \quad \text{and} \quad \sum_{m=0}^{\infty}(-1)^m\frac{(x-x_0)^{2m+1}}{(2m+1)!}=\sin(x-x_0),$$

then (7.2.15) becomes

$$y=a_0\cos(x-x_0)+a_1\sin(x-x_0),$$

which should look familiar. ∎

Equations like (7.2.10), (7.2.11), and (7.2.12), which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called recurrence relations. When we use a recurrence relation to compute terms of a sequence we're computing recursively.

In the remainder of this section we consider the problem of finding power series solutions in $x-x_0$ for equations of the form

$$\left(1+\alpha(x-x_0)^2\right)y''+\beta(x-x_0)y'+\gamma y=0. \tag{7.2.16}$$

Many important equations that arise in applications are of this form with $x_0=0$, including Legendre's equation (7.2.2), Airy's equation (7.2.3), Chebyshev's equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Chebyshev.html),

$$(1-x^2)y''-xy'+\alpha^2y=0,$$

and Hermite's equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Hermite.html),

$$y''-2xy'+2\alpha y=0.$$

Since

$$P_0(x) = 1 + \alpha(x - x_0)^2$$

in (7.2.16), the point x_0 is an ordinary point of (7.2.16), and Theorem 7.2.1 implies that the solutions of (7.2.16) can be written as power series in $x - x_0$ that converge on the interval $(x_0 - 1/\sqrt{|\alpha|}, x_0 + 1/\sqrt{|\alpha|})$ if $\alpha \neq 0$, or on $(-\infty, \infty)$ if $\alpha = 0$. We'll see that the coefficients in these power series can be obtained by methods similar to the one used in Example 7.2.1.

To simplify finding the coefficients, we introduce some notation for products:

$$\prod_{j=r}^{s} b_j = b_r b_{r+1} \cdots b_s \quad \text{if} \quad s \geq r.$$

Thus,

$$\prod_{j=2}^{7} b_j = b_2 b_3 b_4 b_5 b_6 b_7,$$

$$\prod_{j=0}^{4} (2j+1) = (1)(3)(5)(7)(9) = 945,$$

and

$$\prod_{j=2}^{2} j^2 = 2^2 = 4.$$

We define

$$\prod_{j=r}^{s} b_j = 1 \quad \text{if} \quad s < r,$$

no matter what the form of b_j.

Example 7.2.2

Find the power series in x for the general solution of

$$(1+2x^2)y''+6xy'+2y=0. \tag{7.2.17}$$

Solution Here

$$Ly=(1+2x^2)y''+6xy'+2y.$$

If

$$y=\sum_{n=0}^{\infty}a_nx^n$$

then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

so

$$\begin{aligned}
Ly &= (1+2x^2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+6x\sum_{n=1}^{\infty}na_nx^{n-1}+2\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}\left[2n(n-1)+6n+2\right]a_nx^n \\
&= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n.
\end{aligned}$$

To collect coefficients of x^n, we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n=\sum_{n=0}^{\infty}b_nx^n,$$

with

$$b_n=(n+2)(n+1)a_{n+2}+2(n+1)^2a_n, \quad n\ge 0.$$

To obtain solutions of (7.2.17), we set $b_n=0$ for $n\ge 0$. This is equivalent to the recurrence relation

$$a_{n+2} = -2\frac{n+1}{n+2}a_n, \quad n \geq 0. \tag{7.2.18}$$

Since the indices on the left and right differ by two, we write (7.2.18) separately for $n = 2m$ and $n = 2m + 1$, as in Example 7.2.1. This yields

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Computing the coefficients of even powers of x from (7.2.19) yields

$$\begin{aligned}
a_2 &= -\frac{1}{1}a_0, \\
a_4 &= -\frac{3}{2}a_2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0 = \frac{1\cdot 3}{1\cdot 2}a_0, \\
a_6 &= -\frac{5}{3}a_4 = -\frac{5}{3}\left(\frac{1\cdot 3}{1\cdot 2}\right)a_0 = -\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}a_0, \\
a_8 &= -\frac{7}{4}a_6 = -\frac{7}{4}\left(-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\right)a_0 = \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4}a_0.
\end{aligned}$$

In general,

$$a_{2m} = (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!}a_0, \quad m \geq 0. \tag{7.2.21}$$

(Note that (7.2.21) is correct for $m = 0$ because we defined $\prod_{j=1}^{0} b_j = 1$ for any b_j.)

Computing the coefficients of odd powers of x from (7.2.20) yields

$$\begin{aligned}
a_3 &= -4\,\frac{1}{3}a_1, \\
a_5 &= -4\,\frac{2}{5}a_3 = -4\,\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\frac{1\cdot 2}{3\cdot 5}a_1, \\
a_7 &= -4\,\frac{3}{7}a_5 = -4\,\frac{3}{7}\left(4^2\frac{1\cdot 2}{3\cdot 5}\right)a_1 = -4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}a_1, \\
a_9 &= -4\,\frac{4}{9}a_7 = -4\,\frac{4}{9}\left(4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)a_1 = 4^4\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}a_1.
\end{aligned}$$

In general,

$$a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^{m}(2j+1)}a_1, \quad m \geq 0. \tag{7.2.22}$$

From (7.2.21) and (7.2.22),

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!} x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{4^m m!}{\prod_{j=1}^{m}(2j+1)} x^{2m+1}.$$

is the power series in x for the general solution of (7.2.17). Since $P_0(x) = 1 + 2x^2$ has no real zeros, Theorem 5.1.1 implies that every solution of (7.2.17) is defined on $(-\infty, \infty)$. However, since $P_0(\pm i/\sqrt{2})$ $= 0$, Theorem 7.2.1 implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of a_0 and a_1.

The results in Examples 7.2.1 and 7.2.2 are consequences of the following general theorem.

Theorem 7.2.2

The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ of

$$\left(1 + \alpha(x - x_0)^2\right) y'' + \beta(x - x_0) y' + \gamma y = 0 \tag{7.2.23}$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)} a_n, \quad n \geq 0, \tag{7.2.24}$$

where

$$p(n) = \alpha n(n-1) + \beta n + \gamma. \tag{7.2.25}$$

Moreover, the coefficients of the even and odd powers of $x - x_0$ can be computed separately as

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where a_0 and a_1 are arbitrary.

Proof Here

$$Ly = (1 + \alpha(x - x_0)^2) y'' + \beta(x - x_0) y' + \gamma y.$$

If

$$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

then

$$y'=\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}.$$

Hence,

$$\begin{aligned} Ly &= \backslash\mathrm{dst}\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty}\left[\alpha n(n-1)+\beta n+\gamma\right]a_n(x-x_0)^n \\ &= \backslash\mathrm{dst}\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty} p(n)a_n(x-x_0)^n, \end{aligned}$$

from (7.2.25). To collect coefficients of powers of $x-x_0$, we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+p(n)a_n\right](x-x_0)^n.$$

Thus, $Ly=0$ if and only if

$$(n+2)(n+1)a_{n+2}+p(n)a_n=0, \quad n\ge 0,$$

which is equivalent to (7.2.24). Writing (7.2.24) separately for the cases where $n=2m$ and $n=2m+1$ yields (7.2.26) and (7.2.27).

Example 7.2.3

Find the power series in $x-1$ for the general solution of

$$(2+4x-2x^2)y''-12(x-1)y'-12y=0. \tag{7.2.28}$$

Solution We must first write the coefficient $P_0(x)=2+4x-x^2$ in powers of $x-1$. To do this, we write $x=(x-1)+1$ in $P_0(x)$ and then expand the terms, collecting powers of $x-1$; thus,

$$\begin{aligned} 2+4x-2x^2 &= 2+4[(x-1)+1]-2[(x-1)+1]^2 \\ &= 4-2(x-1)^2. \end{aligned}$$

Therefore we can rewrite (7.2.28) as

$$\left(4-2(x-1)^2\right)y''-12(x-1)y'-12y=0,$$

or, equivalently,

$$\left(1-\frac{1}{2}(x-1)^2\right)y''-3(x-1)y'-3y=0.$$

This is of the form (7.2.23) with $\alpha=-1/2$, $\beta=-3$, and $\gamma=-3$. Therefore, from (7.2.25)

$$p(n)=-\frac{n(n-1)}{2}-3n-3=-\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem 7.2.2 implies that

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We leave it to you to show that

$$a_{2m}=\frac{2m+1}{2^m}a_0 \quad \text{and} \quad a_{2m+1}=\frac{m+1}{2^m}a_1, \quad m\geq 0,$$

which implies that the power series in $x-1$ for the general solution of (7.2.28) is

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In the examples considered so far we were able to obtain closed formulas for coefficients in the power series solutions. In some cases this is impossible, and we must settle for computing a finite number of terms in the series. The next example illustrates this with an initial value problem.

Example 7.2.4

Compute a_0, a_1, …, a_7 in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem

$$(1+2x^2)y''+10xy'+8y=0, \quad y(0)=2, \quad y'(0)=-3. \tag{7.2.29}$$

Solution Since $\alpha=2$, $\beta=10$, and $\gamma=8$ in (7.2.29),

$$p(n) = 2n(n-1) + 10n + 8 = 2(n+2)^2.$$

Therefore

$$a_{n+2} = -2\frac{(n+2)^2}{(n+2)(n+1)}a_n = -2\frac{n+2}{n+1}a_n, \quad n \geq 0.$$

Writing this equation separately for $n = 2m$ and $n = 2m+1$ yields

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Starting with $a_0 = y(0) = 2$, we compute a_2, a_4, and a_6 from (7.2.30):

$$\begin{aligned} a_2 &= -4\frac{1}{1}2 = -8, \\ a_4 &= -4\frac{2}{3}(-8) = \frac{64}{3}, \\ a_6 &= -4\frac{3}{5}\left(\frac{64}{3}\right) = -\frac{256}{5}. \end{aligned}$$

Starting with $a_1 = y'(0) = -3$, we compute a_3, a_5 and a_7 from (7.2.31):

$$\begin{aligned} a_3 &= -\frac{3}{1}(-3) = 9, \\ a_5 &= -\frac{5}{2}9 = -\frac{45}{2}, \\ a_7 &= -\frac{7}{3}\left(-\frac{45}{2}\right) = \frac{105}{2}. \end{aligned}$$

Therefore the solution of (7.2.29) is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots .$$

Using Technology

Computing coefficients recursively as in Example 7.2.4 is tedious. We recommend that you do this kind of computation by writing a short program to implement the appropriate recurrence relation on a calculator or computer. You may wish to do this in verifying examples and doing exercises (identified by the symbol C) in this chapter that call for numerical computation of the coefficients in series solutions.

We obtained the answers to these exercises by using software that can produce answers in the form of rational numbers. However, it's perfectly acceptable - and more practical - to get your answers in decimal form. You can always check them by converting our fractions to decimals.

If you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficents is essentially irrelevant. For computational purposes it's usually more efficient to start with the given coefficients $a_0 = y(x_0)$ and $a_1 = y'(x_0)$, compute a_2, …, a_N recursively, and then compute approximate values of the solution from the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} a_n(x - x_0)^n.$$

The trick is to decide how to choose N so the approximation $y(x) \approx T_N(x)$ is sufficiently accurate on the subinterval of the interval of convergence that you're interested in. In the computational exercises in this and the next two sections, you will often be asked to obtain the solution of a given problem by numerical integration with software of your choice (see Section 10.1 for a brief discussion of one such method), and to compare the solution obtained in this way with the approximations obtained with T_N for various values of N. This is a typical textbook kind of exercise, designed to give you insight into how the accuracy of the approximation $y(x) \approx T_N(x)$ behaves as a function of N and the interval that you're working on. In real life, you would choose one or the other of the two methods (numerical integration or series solution). If you choose the method of series solution, then a practical procedure for determining a suitable value of N is to continue increasing N until the maximum of $|T_N - T_{N-1}|$ on the interval of interest is within the margin of error that you're willing to accept.

In doing computational problems that call for numerical solution of differential equations you should choose the most accurate numerical integration procedure your software supports, and experiment with the step size until you're confident that the numerical results are sufficiently accurate for the problem at hand.

7.2 Exercises

In Exercises 1 –8 find the power series in x for the general solution.

1. $(1+x^2)y''+6xy'+6y=0$

Show answer

2. $(1+x^2)y''+2xy'-2y=0$

Show answer

3. $(1+x^2)y''-8xy'+20y=0$

Show answer

4. $(1-x^2)y''-8xy'-12y=0$

Show answer

5. $(1+2x^2)y''+7xy'+2y=0$

Show answer

6. \dst$(1+x^2)y''+2xy'+\frac{1}{4}y=0$

Show answer

7. $(1-x^2)y''-5xy'-4y=0$

Show answer

8. $(1+x^2)y''-10xy'+28y=0$

Show answer

9. L

a. Find the power series in x for the general solution of $y'' + xy' + 2y = 0$.

b. For several choices of a_0 and a_1, use differential equations software to solve the initial value problem

$$y'' + xy' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{A}$$

numerically on $(-5, 5)$.

c. For fixed r in $\{1, 2, 3, 4, 5\}$ graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in **(a)** on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

Show answer

10. L Follow the directions of Exercise 9 for the differential equation

$$y'' + 2xy' + 3y = 0.$$

Show answer

In Exercises 11 –13 find a_0, …, a_N for N at least 7 in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

11. C $(1+x^2)y''+xy'+y=0, \quad y(0)=2, \quad y'(0)=-1$

Show answer

12. C $(1+2x^2)y''-9xy'-6y=0, \quad y(0)=1, \quad y'(0)=-1$

Show answer

13. C $(1+8x^2)y''+2y=0, \quad y(0)=2, \quad y'(0)=-1$

Show answer

14. L Do the next experiment for various choices of real numbers a_0, a_1, and r, with $0<r<1/\sqrt{2}$.

a. Use differential equations software to solve the initial value problem

$$(1-2x^2)y''-xy'+3y=0, \quad y(0)=a_0, \quad y'(0)=a_1, \tag{A}$$

numerically on $(-r,r)$.

b. For $N=2$, 3, 4, …, compute a_2, …, a_N in the power series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of (A), and graph

$$T_N(x)=\sum_{n=0}^{N}a_nx^n$$

and the solution obtained in **(a)** on $(-r,r)$. Continue increasing N until there's no perceptible difference between the two graphs.

15. L Do **(a)** and **(b)** for several values of r in $(0,1)$:

a. Use differential equations software to solve the initial value problem

$$(1+x^2)y''+10xy'+14y=0, \quad y(0)=5, \quad y'(0)=1, \tag{A}$$

numerically on $(-r,r)$.

b. For $N=2$, 3, 4, …, compute a_2, …, a_N in the power series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of (A) , and graph

$$T_N(x)=\sum_{n=0}^{N} a_n x^n$$

and the solution obtained in **(a)** on $(-r,r)$. Continue increasing N until there's no perceptible difference between the two graphs. What happens to the required N as $r \to 1$?

c. Try **(a)** and **(b)** with $r=1.2$. Explain your results.

In Exercises 16 –20 find the power series in $x-x_0$ for the general solution.

16. $y''-y=0; \quad x_0=3$

Show answer

17. $y''-(x-3)y'-y=0; \quad x_0=3$

Show answer

18. $(1-4x+2x^2)y''+10(x-1)y'+6y=0; \quad x_0=1$

Show answer

19. $(11-8x+2x^2)y''-16(x-2)y'+36y=0; \quad x_0=2$

Show answer

20. $(5+6x+3x^2)y''+9(x+1)y'+3y=0; \quad x_0=-1$

Show answer

In Exercises 21 –26 find a_0, …, a_N for N at least 7 in the power series $y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ for the solution of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

21. C $(x^2-4)y''-xy'-3y=0, \quad y(0)=-1, \quad y'(0)=2$

Show answer

22. C $y''+(x-3)y'+3y=0, \quad y(3)=-2, \quad y'(3)=3$

Show answer

23. C $(5-6x+3x^2)y''+(x-1)y'+12y=0, \quad y(1)=-1, \quad y'(1)=1$

Show answer

24. C $(4x^2-24x+37)y''+y=0, \quad y(3)=4, \quad y'(3)=-6$

Show answer

25. C $(x^2-8x+14)y''-8(x-4)y'+20y=0, \quad y(4)=3, \quad y'(4)=-4$

Show answer

26. C $(2x^2+4x+5)y''-20(x+1)y'+60y=0, \quad y(-1)=3, \quad y'(-1)=-3$

Show answer

27. a. Find a power series in x for the general solution of

$$(1+x^2)y''+4xy'+2y=0. \qquad \text{(A)}$$

b. Use (a) and the formula

$$\frac{1}{1-r}=1+r+r^2+\cdots+r^n+\cdots \quad (-1<r<1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (A) on $(-1,1)$.

c. Show that the expression obtained in **(b)** is actually the general solution of of (A) on $(-\infty,\infty)$.

Show answer

28. Use Theorem 7.2.2 to show that the power series in x for the general solution of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

is

$$y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j)\right]\frac{x^{2m}}{(2m)!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j+1)\right]\frac{x^{2m+1}}{(2m+1)!}.$$

29. Use Exercise 28 to show that all solutions of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

are polynomials if and only if

$$\alpha n(n-1)+\beta n+\gamma=\alpha(n-2r)(n-2s-1),$$

where r and s are nonnegative integers.

30. a. Use Exercise 28 to show that the power series in x for the general solution of

$$(1-x^2)y'' - 2bxy' + \alpha(\alpha+2b-1)y = 0$$

is $y = a_0y_1 + a_1y_2$, where

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b. Suppose $2b$ isn't a negative odd integer and k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x) = y_1(x)$ if $\alpha = 2k$, while y_2 is a polynomial of degree $2k+1$ such that $y_2(-x) = -y_2(-x)$ if $\alpha = 2k+1$. Conclude that if n is a nonnegative integer, then there's a polynomial P_n of degree n such that $P_n(-x) = (-1)^nP_n(x)$ and

$$(1-x^2)P_n'' - 2bxP_n' + n(n+2b-1)P_n = 0. \tag{A}$$

c. Show that (A) implies that

$$[(1-x^2)^bP_n']' = -n(n+2b-1)(1-x^2)^{b-1}P_n,$$

and use this to show that if m and n are nonnegative integers, then

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d. Now suppose $b > 0$. Use (B) and integration by parts to show that if $m \neq n$, then

$$\int_{-1}^{1}(1-x^2)^{b-1}P_m(x)P_n(x)\,dx = 0.$$

(We say that P_m and P_n are orthogonal on $(-1,1)$ with respect to the weighting function $(1-x^2)^{b-1}$.)

31. a. Use Exercise 28 to show that the power series in x for the general solution of Hermite's equation

$$y'' - 2xy' + 2\alpha y = 0$$

is $y = a_0 y_1 + a_1 y_1$, where

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b. Suppose k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x) = y_1(x)$ if $\alpha = 2k$, while y_2 is a polynomial of degree $2k+1$ such that $y_2(-x) = -y_2(-x)$ if $\alpha = 2k+1$. Conclude that if n is a nonnegative integer then there's a polynomial P_n of degree n such that $P_n(-x) = (-1)^n P_n(x)$ and

$$P_n'' - 2xP_n' + 2nP_n = 0. \tag{A}$$

c. Show that (A) implies that

$$[e^{-x^2} P_n']' = -2ne^{-x^2} P_n,$$

and use this to show that if m and n are nonnegative integers, then

$$[e^{-x^2} P_n']' P_m - [e^{-x^2} P_m']' P_n = 2(m-n)e^{-x^2} P_m P_n. \tag{B}$$

d. Use (B) and integration by parts to show that if $m \neq n$, then

$$\int_{-\infty}^{\infty} e^{-x^2} P_m(x) P_n(x)\, dx = 0.$$

(We say that P_m and P_n are orthogonal on $(-\infty, \infty)$ with respect to the weighting function e^{-x^2}.)

32. Consider the equation

$$\left(1+\alpha x^3\right) y'' + \beta x^2 y' + \gamma x y = 0, \tag{A}$$

and let $p(n)=\alpha n(n-1)+\beta n+\gamma$. (The special case $y''-xy=0$ of (A) is Airy's equation.)

a. Modify the argument used to prove Theorem 7.2.2 to show that

$$y=\sum_{n=0}^{\infty} a_n x^n$$

is a solution of (A) if and only if $a_2=0$ and

$$a_{n+3}=-\frac{p(n)}{(n+3)(n+2)}a_n, \quad n\ge 0.$$

b. Show from (a) that $a_n=0$ unless $n=3m$ or $n=3m+1$ for some nonnegative integer m, and that

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where a_0 and a_1 may be specified arbitrarily.

c. Conclude from **(b)** that the power series in x for the general solution of (A) is

$$y = \backslash\mathrm{dst} a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p(3j)}{3j+2}\right] \frac{x^{3m}}{3^m m!}$$
$$\backslash\mathrm{dst} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p(3j+1)}{3j+4}\right] \frac{x^{3m+1}}{3^m m!}.$$

In Exercises 33 –37 use the method of Exercise 32 to find the power series in x for the general solution.

33. $y'' - xy = 0$

Show answer

34. $(1 - 2x^3)y'' - 10x^2y' - 8xy = 0$

Show answer

35. $(1 + x^3)y'' + 7x^2y' + 9xy = 0$

Show answer

36. $(1 - 2x^3)y'' + 6x^2y' + 24xy = 0$

Show answer

37. $(1 - x^3)y'' + 15x^2y' - 63xy = 0$

Show answer

38. Consider the equation

$$\left(1+\alpha x^{k+2}\right)y''+\beta x^{k+1}y'+\gamma x^k y=0, \tag{A}$$

where k is a positive integer, and let $p(n)=\alpha n(n-1)+\beta n+\gamma$.

a. Modify the argument used to prove Theorem 7.2.2 to show that

$$y=\sum_{n=0}^{\infty}a_nx^n$$

is a solution of (A) if and only if $a_n=0$ for $2\le n\le k+1$ and

$$a_{n+k+2}=-\frac{p(n)}{(n+k+2)(n+k+1)}a_n, \quad n\ge 0.$$

b. Show from **(a)** that $a_n=0$ unless $n=(k+2)m$ or $n=(k+2)m+1$ for some nonnegative integer m, and that

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where a_0 and a_1 may be specified arbitrarily.

c. Conclude from **(b)** that the power series in x for the general solution of (A) is

$$\begin{aligned} &y=a_0\backslash\mathrm{dst}\textstyle\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{p((k+2)j)}{(k+2)(j+1)-1}\right]\frac{x^{(k+2)m}}{(k+2)^m m!}\\ &[15pt] \qquad +a_1\backslash\mathrm{dst}\textstyle\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{p((k+2)j+1)}{(k+2)(j+1)+1}\right]\frac{x^{(k+2)m+1}}{(k+2)^m m!}.\end{aligned}$$

In Exercises 39 –44 use the method of Exercise 38 to find the power series in x for the general solution.

39. $(1+2x^5)y''+14x^4y'+10x^3y=0$

Show answer

40. $y''+x^2y=0$

Show answer

41. $y''+x^6y'+7x^5y=0$

Show answer

42. $(1+x^8)y''-16x^7y'+72x^6y=0$

Show answer

43. $(1-x^6)y''-12x^5y'-30x^4y=0$

Show answer

44. $y''+x^5y'+6x^4y=0$

Show answer

7.3 Series Solutions Near an Ordinary Point II

In this section we continue to find series solutions

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

of initial value problems

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0, \quad y(x_0)=a_0, \quad y'(x_0)=a_1, \tag{7.3.1}$$

where P_0, P_1, and P_2 are polynomials and $P_0(x_0)\neq 0$, so x_0 is an ordinary point of (7.3.1). However, here we consider cases where the differential equation in (7.3.1) is not of the form

$$\left(1+\alpha(x-x_0)^2\right)y''+\beta(x-x_0)y'+\gamma y=0,$$

so Theorem 7.2.2 does not apply, and the computation of the coefficients $\{a_n\}$ is more complicated. For the equations considered here it's difficult or impossible to obtain an explicit formula for a_n in terms of n. Nevertheless, we can calculate as many coefficients as we wish. The next three examples illustrate this.

EXAMPLE 7.3.1

Find the coefficients a_0, …, a_7 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0, \quad y(0) = -1, \quad y'(0) = -2. \tag{7.3.2}$$

SOLUTION Here

$$Ly = (1 + x + 2x^2)y'' + (1 + 7x)y' + 2y.$$

The zeros $(-1 \pm i\sqrt{7})/4$ of $P_0(x) = 1 + x + 2x^2$ have absolute value $1/\sqrt{2}$, so Theorem 7.2.2 implies that the series solution converges to the solution of (7.3.2) on $(-1/\sqrt{2}, 1/\sqrt{2})$. Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

$$\begin{aligned} Ly = & \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + 2\sum_{n=2}^{\infty} n(n-1)a_n x^n \\ & + \sum_{n=1}^{\infty} n a_n x^{n-1} + 7\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting indices so the general term in each series is a constant multiple of x^n yields

$$\begin{aligned} Ly = & \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1)n a_{n+1} x^n + 2\sum_{n=0}^{\infty} n(n-1)a_n x^n \\ & + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 7\sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n, \end{aligned}$$

where

$$b_n = (n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + (n+2)(2n+1)a_n.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_{n+2} = -\frac{n+1}{n+2}\, a_{n+1} - \frac{2n+1}{n+1}\, a_n, \; n \geq 0. \tag{7.3.3}$$

From the initial conditions in (7.3.2), $a_0 = y(0) = -1$ and $a_1 = y'(0) = -2$. Setting $n = 0$ in (7.3.3) yields

$$a_2 = -\frac{1}{2}a_1 - a_0 = -\frac{1}{2}(-2) - (-1) = 2.$$

Setting $n = 1$ in (7.3.3) yields

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = -\frac{2}{3}(2) - \frac{3}{2}(-2) = \frac{5}{3}.$$

We leave it to you to compute a_4, a_5, a_6, a_7 from (7.3.3) and show that

$$y = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 - \frac{443}{56}x^7 + \cdots.$$

We also leave it to you (Exercise 13) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^{N} a_n x^n$ converge to the solution of (7.3.2) on $(-1/\sqrt{2}, 1/\sqrt{2})$.

Example 7.3.2

Find the coefficients a_0, …, a_5 in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n$$

of the initial value problem

$$(3+x)y'' + (1+2x)y' - (2-x)y = 0, \quad y(-1) = 2, \quad y'(-1) = -3. \tag{7.3.4}$$

Solution Since the desired series is in powers of $x+1$ we rewrite the differential equation in (7.3.4) as $Ly = 0$, with

$$Ly = (2 + (x+1))\, y'' - (1 - 2(x+1))\, y' - (3 - (x+1))\, y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n, \quad y' = \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2},$$

$$\begin{aligned} Ly = {} & 2\sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-1} \\ & -\sum_{n=1}^{\infty} na_n(x+1)^{n-1} + 2\sum_{n=1}^{\infty} na_n(x+1)^n \\ & -3\sum_{n=0}^{\infty} a_n(x+1)^n + \sum_{n=0}^{\infty} a_n(x+1)^{n+1}. \end{aligned}$$

Shifting indices so that the general term in each series is a constant multiple of $(x+1)^n$ yields

$$\begin{aligned} Ly &= 2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x+1)^n + \sum_{n=0}^{\infty}(n+1)na_{n+1}(x+1)^n \\ & \quad -\sum_{n=0}^{\infty}(n+1)a_{n+1}(x+1)^n + \sum_{n=0}^{\infty}(2n-3)a_n(x+1)^n + \sum_{n=1}^{\infty} a_{n-1}(x+1)^n \\ &= \sum_{n=0}^{\infty} b_n(x+1)^n, \end{aligned}$$

where

$$b_0 = 4a_2 - a_1 - 3a_0$$

and

$$b_n = 2(n+2)(n+1)a_{n+2} + (n^2-1)a_{n+1} + (2n-3)a_n + a_{n-1}, \quad n \ge 1.$$

Therefore $y = \sum_{n=0}^{\infty} a_n(x+1)^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = \frac{1}{4}(a_1 + 3a_0) \tag{7.3.5}$$

and

$$a_{n+2} = -\frac{1}{2(n+2)(n+1)}\left[(n^2-1)a_{n+1} + (2n-3)a_n + a_{n-1}\right], \quad n \ge 1. \tag{7.3.6}$$

From the initial conditions in (7.3.4), $a_0 = y(-1) = 2$ and $a_1 = y'(-1) = -3$. We leave it to you to compute a_2, …, a_5 with (7.3.5) and (7.3.6) and show that the solution of (7.3.4) is

$$y = -2 - 3(x+1) + \frac{3}{4}(x+1)^2 - \frac{5}{12}(x+1)^3 + \frac{7}{48}(x+1)^4 - \frac{1}{60}(x+1)^5 + \cdots.$$

We also leave it to you (Exercise 14) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^{N} a_n x^n$ converge to the solution of (7.3.4) on the interval of convergence of the power series solution.

Example 7.3.3

Find the coefficients a_0, …, a_5 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -3. \tag{7.3.7}$$

Solution Here

$$Ly = y'' + 3xy' + (4 + 2x^2)y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

$$Ly = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n + 2\sum_{n=0}^{\infty} a_n x^{n+2}.$$

Shifting indices so that the general term in each series is a constant multiple of x^n yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (3n+4) a_n x^n + 2\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=0}^{\infty} b_n x^n$$

where

$$b_0 = 2a_2 + 4a_0, \quad b_1 = 6a_3 + 7a_1,$$

and

$$b_n = (n+2)(n+1) a_{n+2} + (3n+4) a_n + 2a_{n-2}, \quad n \ge 2.$$

Therefore $y=\sum_{n=0}^{\infty}a_nx^n$ is a solution of $Ly=0$ if and only if

$$a_2=-2a_0,\quad a_3=-\frac{7}{6}a_1, \tag{7.3.8}$$

and

$$a_{n+2}=-\frac{1}{(n+2)(n+1)}\left[(3n+4)a_n+2a_{n-2}\right],\quad n\geq 2. \tag{7.3.9}$$

From the initial conditions in (7.3.7), $a_0=y(0)=2$ and $a_1=y'(0)=-3$. We leave it to you to compute a_2, …, a_5 with (7.3.8) and (7.3.9) and show that the solution of (7.3.7) is

$$y=2-3x-4x^2+\frac{7}{2}x^3+3x^4-\frac{79}{40}x^5+\cdots.$$

We also leave it to you (Exercise 15) to verify numerically that the Taylor polynomials $T_N(x)=\sum_{n=0}^{N}a_nx^n$ converge to the solution of (7.3.9) on the interval of convergence of the power series solution.

7.3 Exercises

In Exercises 1–12 find the coefficients a_0,…, a_N for N at least 7 in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem.

1. C $(1+3x)y''+xy'+2y=0, \quad y(0)=2, \quad y'(0)=-3$

   Show answer

2. C $(1+x+2x^2)y''+(2+8x)y'+4y=0, \quad y(0)=-1, \quad y'(0)=2$

   Show answer

3. C $(1-2x^2)y''+(2-6x)y'-2y=0, \quad y(0)=1, \quad y'(0)=0$

   Show answer

4. C $(1+x+3x^2)y''+(2+15x)y'+12y=0, \quad y(0)=0, \quad y'(0)=1$

   Show answer

5. C $(2+x)y''+(1+x)y'+3y=0, \quad y(0)=4, \quad y'(0)=3$

   Show answer

6. C $(3+3x+x^2)y''+(6+4x)y'+2y=0, \quad y(0)=7, \quad y'(0)=3$

   Show answer

7. C $(4+x)y''+(2+x)y'+2y=0, \quad y(0)=2, \quad y'(0)=5$

   Show answer

8. C $(2-3x+2x^2)y''-(4-6x)y'+2y=0, \quad y(1)=1, \quad y'(1)=-1$

   Show answer

9. C $(3x+2x^2)y''+10(1+x)y'+8y=0, \quad y(-1)=1, \quad y'(-1)=-1$

   Show answer

10. C $(1-x+x^2)y''-(1-4x)y'+2y=0, \quad y(1)=2, \quad y'(1)=-1$

    Show answer

11. C $(2+x)y''+(2+x)y'+y=0, \quad y(-1)=-2, \quad y'(-1)=3$

    Show answer

12. C $x^2y''-(6-7x)y'+8y=0, \quad y(1)=1, \quad y'(1)=-2$

    Show answer

13. L Do the following experiment for various choices of real numbers a_0, a_1, and r, with $0 < r < 1/\sqrt{2}$.

a. Use differential equations software to solve the initial value problem

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{A}$$

numerically on $(-r, r)$. (See Example 7.3.1.)

b. For $N = 2$, 3, 4, …, compute a_2, …, a_N in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in **(a)** on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

14. L Do the following experiment for various choices of real numbers a_0, a_1, and r, with $0 < r < 2$.

a. Use differential equations software to solve the initial value problem

$$(3 + x)y'' + (1 + 2x)y' - (2 - x)y = 0, \quad y(-1) = a_0, \quad y'(-1) = a_1, \tag{A}$$

numerically on $(-1 - r, -1 + r)$. (See Example 7.3.2. Why this interval?)

b. For $N = 2$, 3, 4, …, compute $a_2, \dots, a_N$ in the power series solution

$$y = \sum_{n=0}^{\infty} a_n (x + 1)^n$$

of (A), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n (x + 1)^n$$

and the solution obtained in **(a)** on $(-1 - r, -1 + r)$. Continue increasing N until there's no perceptible difference between the two graphs.

15. L Do the following experiment for several choices of a_0, a_1, and r, with $r > 0$.

a. Use differential equations software to solve the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{A}$$

numerically on $(-r, r)$. (See Example 7.3.3.)

b. Find the coefficients a_0, a_1, …, a_N in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in **(a)** on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

16. L Do the following experiment for several choices of a_0 and a_1.

a. Use differential equations software to solve the initial value problem

$$(1 - x)y'' - (2 - x)y' + y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{A}$$

numerically on $(-r, r)$.

b. Find the coefficients a_0, a_1, …, a_N in the power series solution $y = \sum_{n=0}^{N} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in **(a)** on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs. What happens as you let $r \to 1$?

17. L Follow the directions of Exercise 16 for the initial value problem

$$(1 + x)y'' + 3y' + 32y = 0, \quad y(0) = a_0, \quad y'(0) = a_1.$$

18. L Follow the directions of Exercise 16 for the initial value problem

$$(1 + x^2)y'' + y' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1.$$

In Exercises 19–28 find the coefficients a_0, …, a_N for N at least 7 in the series solution

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

19. C $(2+4x)y''-4y'-(6+4x)y=0, \quad y(0)=2, \quad y'(0)=-7$

Show answer

20. C $(1+2x)y''-(1-2x)y'-(3-2x)y=0, \quad y(1)=1, \quad y'(1)=-2$

Show answer

21. C $(5+2x)y''-y'+(5+x)y=0, \quad y(-2)=2, \quad y'(-2)=-1$

Show answer

22. C $(4+x)y''-(4+2x)y'+(6+x)y=0, \quad y(-3)=2, \quad y'(-3)=-2$

Show answer

23. C $(2+3x)y''-xy'+2xy=0, \quad y(0)=-1, \quad y'(0)=2$

Show answer

24. C $(3+2x)y''+3y'-xy=0, \quad y(-1)=2, \quad y'(-1)=-3$

Show answer

25. C $(3+2x)y''-3y'-(2+x)y=0, \quad y(-2)=-2, \quad y'(-2)=3$

Show answer

26. C $(10-2x)y''+(1+x)y=0, \quad y(2)=2, \quad y'(2)=-4$

Show answer

27. C $(7+x)y''+(8+2x)y'+(5+x)y=0, \quad y(-4)=1, \quad y'(-4)=2$

Show answer

28. C $(6+4x)y''+(1+2x)y=0, \quad y(-1)=-1, \quad y'(-1)=2$

Show answer

29. Show that the coefficients in the power series in x for the general solution of

$$(1+\alpha x+\beta x^2)y''+(\gamma+\delta x)y'+\epsilon y=0$$

satisfy the recurrrence relation

$$a_{n+2}=-\frac{\gamma+\alpha n}{n+2}\,a_{n+1}-\frac{\beta n(n-1)+\delta n+\epsilon}{(n+2)(n+1)}\,a_n.$$

30. a. Let α and β be constants, with $\beta \neq 0$. Show that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of

$$(1+\alpha x+\beta x^2)y''+(2\alpha+4\beta x)y'+2\beta y=0 \tag{A}$$

if and only if

$$a_{n+2}+\alpha a_{n+1}+\beta a_n=0, \quad n \geq 0. \tag{B}$$

An equation of this form is called a second order homogeneous linear difference equation. The polynomial $p(r)=r^2+\alpha r+\beta$ is called the characteristic polynomial of (B). If r_1 and r_2 are the zeros of p, then $1/r_1$ and $1/r_2$ are the zeros of

$$P_0(x)=1+\alpha x+\beta x^2.$$

b. Suppose $p(r)=(r-r_1)(r-r_2)$ where r_1 and r_2 are real and distinct, and let ρ be the smaller of the two numbers $\{1/|r_1|, 1/|r_2|\}$. Show that if c_1 and c_2 are constants then the sequence

$$a_n=c_1r_1^n+c_2r_2^n, \quad n \geq 0$$

satisfies (B). Conclude from this that any function of the form

$$y=\sum_{n=0}^{\infty}(c_1r_1^n+c_2r_2^n)x^n$$

is a solution of (A) on $(-\rho, \rho)$.

c. Use **(b)** and the formula for the sum of a geometric series to show that the functions

$$y_1=\frac{1}{1-r_1x} \quad \text{and} \quad y_2=\frac{1}{1-r_2x}$$

form a fundamental set of solutions of (A) on $(-\rho, \rho)$.

d. Show that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on any interval that does'nt contain either $1/r_1$ or $1/r_2$.

e. Suppose $p(r) = (r - r_1)^2$, and let $\rho = 1/|r_1|$. Show that if c_1 and c_2 are constants then the sequence

$$a_n = (c_1 + c_2 n) r_1^n, \quad n \geq 0$$

satisfies (B). Conclude from this that any function of the form

$$y = \sum_{n=0}^{\infty} (c_1 + c_2 n) r_1^n x^n$$

is a solution of (A) on $(-\rho, \rho)$.

f. Use (e) and the formula for the sum of a geometric series to show that the functions

$$y_1 = \frac{1}{1 - r_1 x} \quad \text{and} \quad y_2 = \frac{x}{(1 - r_1 x)^2}$$

form a fundamental set of solutions of (A) on $(-\rho, \rho)$.

g. Show that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on any interval that does not contain $1/r_1$.

31. Use the results of Exercise 30 to find the general solution of the given equation on any interval on which polynomial multiplying y'' has no zeros.

(a) $(1 + 3x + 2x^2)y'' + (6 + 8x)y' + 4y = 0$

(b) $(1 - 5x + 6x^2)y'' - (10 - 24x)y' + 12y = 0$

(c) $(1 - 4x + 4x^2)y'' - (8 - 16x)y' + 8y = 0$

(d) $(4 + 4x + x^2)y'' + (8 + 4x)y' + 2y = 0$

(e) $(4 + 8x + 3x^2)y'' + (16 + 12x)y' + 6y = 0$

Show answer

In Exercises 32–38 find the coefficients a_0, …, a_N for N at least 7 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

32. C $y'' + 2xy' + (3 + 2x^2)y = 0, \quad y(0) = 1, \quad y'(0) = -2$

Show answer

33. C $y'' - 3xy' + (5 + 2x^2)y = 0, \quad y(0) = 1, \quad y'(0) = -2$

Show answer

34. C $y'' + 5xy' - (3 - x^2)y = 0, \quad y(0) = 6, \quad y'(0) = -2$

Show answer

35. C $y'' - 2xy' - (2 + 3x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -5$

Show answer

36. C $y'' - 3xy' + (2 + 4x^2)y = 0, \quad y(0) = 3, \quad y'(0) = 6$

Show answer

37. C $2y'' + 5xy' + (4 + 2x^2)y = 0, \quad y(0) = 3, \quad y'(0) = -2$

Show answer

38. C $3y'' + 2xy' + (4 - x^2)y = 0, \quad y(0) = -2, \quad y'(0) = 3$

Show answer

39. Find power series in x for the solutions y_1 and y_2 of

$$y'' + 4xy' + (2 + 4x^2)y = 0$$

such that $y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 1$, and identify y_1 and y_2 in terms of familiar elementary functions.

Show answer

In Exercises 40–49 find the coefficients a_0, …, a_N for N at least 7 in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

40. C $(1+x)y''+x^2y'+(1+2x)y=0, \quad y(0)-2, \quad y'(0)=3$

Show answer

41. C $y''+(1+2x+x^2)y'+2y=0, \quad y(0)=2, \quad y'(0)=3$

Show answer

42. C $(1+x^2)y''+(2+x^2)y'+xy=0, \quad y(0)=-3, \quad y'(0)=5$

Show answer

43. C $(1+x)y''+(1-3x+2x^2)y'-(x-4)y=0, \quad y(1)=-2, \quad y'(1)=3$

Show answer

44. C $y''+(13+12x+3x^2)y'+(5+2x), \quad y(-2)=2, \quad y'(-2)=-3$

Show answer

45. C $(1+2x+3x^2)y''+(2-x^2)y'+(1+x)y=0, \quad y(0)=1, \quad y'(0)=-2$

Show answer

46. C $(3+4x+x^2)y''-(5+4x-x^2)y'-(2+x)y=0, \quad y(-2)=2, \quad y'(-2)=-1$

Show answer

47. C $(1+2x+x^2)y''+(1-x)y=0, \quad y(0)=2, \quad y'(0)=-1$

Show answer

48. C $(x-2x^2)y''+(1+3x-x^2)y'+(2+x)y=0, \quad y(1)=1, \quad y'(1)=0$

Show answer

49. C $(16-11x+2x^2)y''+(10-6x+x^2)y'-(2-x)y, \quad y(3)=1, \quad y'(3)=-2$

Show answer

7.4 Regular Singular Points: Euler Equations

This section sets the stage for Sections 1.5, 1.6, and 1.7. If you're not interested in those sections, but wish to learn about Euler equations, omit the introductory paragraphs and start reading at Definition 7.4.2.

In the next three sections we'll continue to study equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{7.4.1}$$

where P_0, P_1, and P_2 are polynomials, but the emphasis will be different from that of Sections 7.2 and 7.3, where we obtained solutions of (7.4.1) near an ordinary point x_0 in the form of power series in $x - x_0$. If x_0 is a singular point of (7.4.1) (that is, if $P(x_0) = 0$), the solutions can't in general be represented by power series in $x - x_0$. Nevertheless, it's often necessary in physical applications to study the behavior of solutions of (7.4.1) near a singular point. Although this can be difficult in the absence of some sort of assumption on the nature of the singular point, equations that satisfy the requirements of the next definition can be solved by series methods discussed in the next three sections. Fortunately, many equations arising in applications satisfy these requirements.

Definition 7.4.1

Let P_0, P_1, and P_2 be polynomials with no common factor and suppose $P_0(x_0) = 0$. Then x_0 is a regular singular point of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{7.4.2}$$

if $(???)$ can be written as

$$(x - x_0)^2 A(x)y'' + (x - x_0)B(x)y' + C(x)y = 0 \tag{7.4.3}$$

where A, B, and C are polynomials and $A(x_0) \neq 0$; otherwise, x_0 is an irregular singular point of (7.4.2).

Example 7.4.1

Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{7.4.4}$$

has the singular point $x_0 = 0$. Since this equation is of the form (7.4.3) with $x_0 = 0$, $A(x) = 1$, $B(x) = 1$, and $C(x) = x^2 - \nu^2$, it follows that $x_0 = 0$ is a regular singular point of (7.4.4).

Example 7.4.2

Legendre's equation,

$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0, \tag{7.4.5}$$

has the singular points $x_0=\pm 1$. Mutiplying through by $1-x$ yields

$$(x-1)^2(x+1)y''+2x(x-1)y'-\alpha(\alpha+1)(x-1)y=0,$$

which is of the form (7.4.3) with $x_0=1$, $A(x)=x+1$, $B(x)=2x$, and $C(x)=-\alpha(\alpha+1)(x-1)$. Therefore $x_0=1$ is a regular singular point of (7.4.5). We leave it to you to show that $x_0=-1$ is also a regular singular point of (7.4.5).

Example 7.4.3

The equation

$$x^3y''+xy'+y=0$$

has an irregular singular point at $x_0=0$. (Verify.)

For convenience we restrict our attention to the case where $x_0=0$ is a regular singular point of (7.4.2). This isn't really a restriction, since if $x_0\neq 0$ is a regular singular point of (7.4.2) then introducing the new independent variable $t=x-x_0$ and the new unknown $Y(t)=y(t+x_0)$ leads to a differential equation with polynomial coefficients that has a regular singular point at $t_0=0$. This is illustrated in Exercise 22 for Legendre's equation, and in Exercise 23 for the general case.

Euler Equations

The simplest kind of equation with a regular singular point at $x_0=0$ is the Euler equation, defined as follows.

Definition 7.4.2

An Euler equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html) is an equation that can be written in the form

$$ax^2y''+bxy'+cy=0, \tag{7.4.6}$$

> where a, b, and c are real constants and $a \neq 0$.

Theorem 5.1.1 implies that (7.4.6) has solutions defined on $(0, \infty)$ and $(-\infty, 0)$, since (7.4.6) can be rewritten as

$$ay'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0.$$

For convenience we'll restrict our attention to the interval $(0, \infty)$. (Exercise 19 deals with solutions of (7.4.6) on $(-\infty, 0)$.) The key to finding solutions on $(0, \infty)$ is that if $x > 0$ then x^r is defined as a real-valued function on $(0, \infty)$ for all values of r, and substituting $y = x^r$ into (7.4.6) produces

$$\begin{aligned} ax^2(x^r)'' + bx(x^r)' + cx^r &= ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r \\ &= [ar(r-1) + br + c]x^r. \end{aligned} \tag{7.4.7}$$

The polynomial

$$p(r) = ar(r-1) + br + c$$

is called the indicial polynomial of (7.4.6), and $p(r) = 0$ is its indicial equation. From (7.4.7) we can see that $y = x^r$ is a solution of (7.4.6) on $(0, \infty)$ if and only if $p(r) = 0$. Therefore, if the indicial equation has distinct real roots r_1 and r_2 then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ form a fundamental set of solutions of (7.4.6) on $(0, \infty)$, since $y_2/y_1 = x^{r_2 - r_1}$ is nonconstant. In this case

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

is the general solution of (7.4.6) on $(0, \infty)$.

> **Example 7.4.4**
>
> Find the general solution of
>
> $$x^2y'' - xy' - 8y = 0 \tag{7.4.8}$$
>
> on $(0, \infty)$.

Solution The indicial polynomial of (7.4.8) is

$$p(r) = r(r-1) - r - 8 = (r-4)(r+2).$$

Therefore $y_1 = x^4$ and $y_2 = x^{-2}$ are solutions of (7.4.8) on $(0, \infty)$, and its general solution on $(0, \infty)$ is

$$y = c_1 x^4 + \frac{c_2}{x^2}.$$

Example 7.4.5

Find the general solution of

$$6x^2 y'' + 5xy' - y = 0 \tag{7.4.9}$$

on $(0, \infty)$.

Solution The indicial polynomial of (7.4.9) is

$$p(r) = 6r(r-1) + 5r - 1 = (2r-1)(3r+1).$$

Therefore the general solution of (7.4.9) on $(0, \infty)$ is

$$y = c_1 x^{1/2} + c_2 x^{-1/3}.\text{\bbox}$$

If the indicial equation has a repeated root r_1, then $y_1 = x^{r_1}$ is a solution of

$$ax^2 y'' + bxy' + cy = 0, \tag{7.4.10}$$

on $(0, \infty)$, but (7.4.10) has no other solution of the form $y = x^r$. If the indicial equation has complex conjugate zeros then (7.4.10) has no real–valued solutions of the form $y = x^r$. Fortunately we can use the results of Section 5.2 for constant coefficient equations to solve (7.4.10) in any case.

Theorem 7.4.3

Suppose the roots of the indicial equation

$$ar(r-1) + br + c = 0 \tag{7.4.11}$$

are r_1 and r_2. Then the general solution of the Euler equation

$$ax^2y'' + bxy' + cy = 0 \tag{7.4.12}$$

on $(0, \infty)$ is

$$\begin{aligned}
&y = c_1x^{r_1} + c_2x^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers ;}\\
&y = x^{r_1}(c_1 + c_2\ln x) \text{ if } r_1 = r_2 \text{ ;}\\
&y = x^{\lambda}\left[c_1\cos(\omega\ln x) + c_2\sin(\omega\ln x)\right] \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega > 0.
\end{aligned}$$

Proof We first show that $y = y(x)$ satisfies (7.4.12) on $(0, \infty)$ if and only if $Y(t) = y(e^t)$ satisfies the constant coefficient equation

$$a\frac{d^2Y}{dt^2} + (b-a)\frac{dY}{dt} + cY = 0 \tag{7.4.13}$$

on $(-\infty, \infty)$. To do this, it's convenient to write $x = e^t$, or, equivalently, $t = \ln x$; thus, $Y(t) = y(x)$, where $x = e^t$. From the chain rule,

$$\frac{dY}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and, since

$$\frac{dx}{dt} = e^t = x,$$

it follows that

$$\frac{dY}{dt} = x\frac{dy}{dx}. \tag{7.4.14}$$

Differentiating this with respect to t and using the chain rule again yields

$$\begin{aligned}
\frac{d^2Y}{dt^2} &= \frac{d}{dt}\left(\frac{dY}{dt}\right) = \frac{d}{dt}\left(x\frac{dy}{dx}\right)\\
&= \frac{dx}{dt}\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{dt}\\
&= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} \quad \left(\text{since } \frac{dx}{dt} = x\right).
\end{aligned}$$

From this and (7.4.14),

$$x^2\frac{d^2y}{dx^2}=\frac{d^2Y}{dt^2}-\frac{dY}{dt}.$$

Substituting this and (7.4.14) into (7.4.12) yields (7.4.13). Since (7.4.11) is the characteristic equation of (7.4.13), Theorem 5.2.1 implies that the general solution of (7.4.13) on $(-\infty,\infty)$ is

$$\begin{aligned}&Y(t)=c_1e^{r_1t}+c_2e^{r_2t} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;}\\&Y(t)=e^{r_1t}(c_1+c_2t) \text{ if } r_1=r_2;\\&Y(t)=e^{\lambda t}\left(c_1\cos\omega t+c_2\sin\omega t\right) \text{ if } r_1,r_2=\lambda\pm i\omega \text{ with } \omega\neq 0.\end{aligned}$$

Since $Y(t)=y(e^t)$, substituting $t=\ln x$ in the last three equations shows that the general solution of (7.4.12) on $(0,\infty)$ has the form stated in the theorem.

Example 7.4.6

Find the general solution of

$$x^2y''-5xy'+9y=0 \tag{7.4.15}$$

on $(0,\infty)$.

Solution The indicial polynomial of (7.4.15) is

$$p(r)=r(r-1)-5r+9=(r-3)^2.$$

Therefore the general solution of (7.4.15) on $(0,\infty)$ is

$$y=x^3(c_1+c_2\ln x).$$

Example 7.4.7

Find the general solution of

$$x^2y''+3xy'+2y=0 \tag{7.4.16}$$

on $(0, \infty)$.

Solution The indicial polynomial of (7.4.16) is

$$p(r) = r(r-1) + 3r + 2 = (r+1)^2 + 1.$$

The roots of the indicial equation are $r = -1 \pm i$ and the general solution of (7.4.16) on $(0, \infty)$ is

$$y = \frac{1}{x}\left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right].$$

7.4 Exercises

In Exercises 1–18 find the general solution of the given Euler equation on $(0, \infty)$.

1. $x^2y'' + 7xy' + 8y = 0$

Show answer

2. $x^2y'' - 7xy' + 7y = 0$

Show answer

3. $x^2y'' - xy' + y = 0$

Show answer

4. $x^2y'' + 5xy' + 4y = 0$

Show answer

5. $x^2y'' + xy' + y = 0$

Show answer

6. $x^2y'' - 3xy' + 13y = 0$

Show answer

7. $x^2y'' + 3xy' - 3y = 0$

Show answer

8. $12x^2y'' - 5xy'' + 6y = 0$

Show answer

9. $4x^2y'' + 8xy' + y = 0$

Show answer

10. $3x^2y'' - xy' + y = 0$

Show answer

11. $2x^2y'' - 3xy' + 2y = 0$

Show answer

12. $x^2y'' + 3xy' + 5y = 0$

Show answer

13. $9x^2y'' + 15xy' + y = 0$

Show answer

14. $x^2y'' - xy' + 10y = 0$

Show answer

15. $x^2y'' - 6y = 0$

Show answer

16. $2x^2y'' + 3xy' - y = 0$

Show answer

17. $x^2y'' - 3xy' + 4y = 0$

Show answer

18. $2x^2y'' + 10xy' + 9y = 0$

Show answer

19. a. Adapt the proof of Theorem 7.4.3 to show that $y = y(x)$ satisfies the Euler equation

$$ax^2y'' + bxy' + cy = 0 \tag{7.4.1}$$

on $(-\infty, 0)$ if and only if $Y(t) = y(-e^t)$

$$a\frac{d^2Y}{dt^2} + (b-a)\frac{dY}{dt} + cY = 0.$$

on $(-\infty, \infty)$.

b. Use **(a)** to show that the general solution of (7.4.1) on $(-\infty, 0)$ is

$$\begin{aligned}
&y = c_1|x|^{r_1} + c_2|x|^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;}\\
&y = |x|^{r_1}(c_1 + c_2\ln|x|) \text{ if } r_1 = r_2;\\
&y = |x|^{\lambda}\left[c_1\cos(\omega\ln|x|) + c_2\sin(\omega\ln|x|)\right] \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega > 0.
\end{aligned}$$

20. Use reduction of order to show that if

$$ar(r-1)+br+c=0$$

has a repeated root r_1 then $y=x^{r_1}(c_1+c_2\ln x)$ is the general solution of

$$ax^2y''+bxy'+cy=0$$

on $(0,\infty)$.

21. A nontrivial solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0$$

is said to be oscillatory on an interval (a,b) if it has infinitely many zeros on (a,b). Otherwise y is said to be nonoscillatory on (a,b). Show that the equation

$$x^2y''+ky=0 \quad (k=\text{ constant})$$

has oscillatory solutions on $(0,\infty)$ if and only if $k>1/4$.

22. In Example 7.4.2 we saw that $x_0 = 1$ and $x_0 = -1$ are regular singular points of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0. \tag{A}$$

a. Introduce the new variables $t = x - 1$ and $Y(t) = y(t + 1)$, and show that y is a solution of (A) if and only if Y is a solution of

$$t(2 + t)\frac{d^2Y}{dt^2} + 2(1 + t)\frac{dY}{dt} - \alpha(\alpha + 1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

b. Introduce the new variables $t = x + 1$ and $Y(t) = y(t - 1)$, and show that y is a solution of (A) if and only if Y is a solution of

$$t(2 - t)\frac{d^2Y}{dt^2} + 2(1 - t)\frac{dY}{dt} + \alpha(\alpha + 1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

23. Let P_0, P_1, and P_2 be polynomials with no common factor, and suppose $x_0 \neq 0$ is a singular point of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{A}$$

Let $t = x - x_0$ and $Y(t) = y(t + x_0)$.

a. Show that y is a solution of (A) if and only if Y is a solution of

$$R_0(t)\frac{d^2Y}{dt^2} + R_1(t)\frac{dY}{dt} + R_2(t)Y = 0. \tag{B}$$

where

$$R_i(t) = P_i(t + x_0), \quad i = 0, 1, 2.$$

b. Show that R_0, R_1, and R_2 are polynomials in t with no common factors, and $R_0(0) = 0$; thus, $t_0 = 0$ is a singular point of (B).

7.5 The Method of Frobenius I

In this section we begin to study series solutions of a homogeneous linear second order differential equation with a regular singular point at $x_0 = 0$, so it can be written as

$$x^2A(x)y'' + xB(x)y' + C(x)y = 0, \tag{7.5.1}$$

where A, B, C are polynomials and $A(0) \neq 0$.

We'll see that (7.5.1) always has at least one solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

where $a_0 \neq 0$ and r is a suitably chosen number. The method we will use to find solutions of this form and other forms that we'll encounter in the next two sections is called the method of Frobenius (http://www-history.mcs.st-and.ac.uk/Mathematicians/Frobenius.html), and we'll call them Frobenius solutions (http://www-history.mcs.st-and.ac.uk/Mathematicians/Frobenius.html).

It can be shown that the power series $\sum_{n=0}^{\infty} a_n x^n$ in a Frobenius solution of (7.5.1) converges on some open interval $(-\rho, \rho)$, where $0 < \rho \leq \infty$. However, since x^r may be complex for negative x or undefined if $x = 0$, we'll consider solutions defined for positive values of x. Easy modifications of our results yield solutions defined for negative values of x. (Exercise 54).

We'll restrict our attention to the case where A, B, and C are polynomials of degree not greater than two, so (7.5.1) becomes

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0, \tag{7.5.2}$$

where α_i, β_i, and γ_i are real constants and $\alpha_0 \neq 0$. Most equations that arise in applications can be written this way. Some examples are

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where we would multiply the last equation through by x to put it in the form (7.5.2). However, the method of Frobenius can be extended to the case where A, B, and C are functions that can be represented by power series in x on some interval that contains zero, and $A_0(0) \neq 0$ (Exercises 57 and 58).

The next two theorems will enable us to develop systematic methods for finding Frobenius solutions of (7.5.2).

Theorem 7.5.1

Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y,$$

and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose the series

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{7.5.3}$$

converges on $(0, \rho)$. Then

$$Ly = \sum_{n=0}^{\infty} b_n x^{n+r} \tag{7.5.4}$$

on $(0, \rho)$, where

$$\begin{array}{rcl} b_0 &=& p_0(r)a_0, \\ b_1 &=& p_0(r+1)a_1 + p_1(r)a_0, \\ b_n &=& p_0(n+r)a_n + p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}, \quad n \geq 2. \end{array} \tag{7.5.5}$$

Proof We begin by showing that if y is given by (7.5.3) and α, β, and γ are constants, then

$$\alpha x^2 y'' + \beta x y' + \gamma y = \sum_{n=0}^{\infty} p(n+r) a_n x^{n+r}, \tag{7.5.6}$$

where

$$p(r) = \alpha r(r-1) + \beta r + \gamma.$$

Differentiating (3) twice yields

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \tag{7.5.7}$$

and

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}. \tag{7.5.8}$$

Multiplying (7.5.7) by x and (7.5.8) by x^2 yields

$$xy' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

and

$$x^2y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}.$$

Therefore

$$\begin{aligned} \alpha x^2y'' + \beta xy' + \gamma y &= \sum_{n=0}^{\infty}\left[\alpha(n+r)(n+r-1) + \beta(n+r) + \gamma\right] a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r}, \end{aligned}$$

which proves (7.5.6).

Multiplying (7.5.6) by x yields

$$x(\alpha x^2y'' + \beta xy' + \gamma y) = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r+1} = \sum_{n=1}^{\infty} p(n+r-1)a_{n-1}x^{n+r}. \tag{7.5.9}$$

Multiplying (7.5.6) by x^2 yields

$$x^2(\alpha x^2y'' + \beta xy' + \gamma y) = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r+2} = \sum_{n=2}^{\infty} p(n+r-2)a_{n-2}x^{n+r}. \tag{7.5.10}$$

To use these results, we rewrite

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y$$

as

$$\begin{aligned} Ly &= \left(\alpha_0 x^2 y'' + \beta_0 xy' + \gamma_0 y\right) + x\left(\alpha_1 x^2 y'' + \beta_1 xy' + \gamma_1 y\right) \\ &\quad + x^2\left(\alpha_2 x^2 y'' + \beta_2 xy' + \gamma_2 y\right). \end{aligned} \tag{7.5.11}$$

From (7.5.6) with $p = p_0$,

$$\alpha_0 x^2 y'' + \beta_0 xy' + \gamma_0 y = \sum_{n=0}^{\infty} p_0(n+r)a_n x^{n+r}.$$

From (7.5.9) with $p = p_1$,

$$x\left(\alpha_1 x^2 y'' + \beta_1 xy' + \gamma_1 y\right) = \sum_{n=1}^{\infty} p_1(n+r-1)a_{n-1} x^{n+r}.$$

From (7.5.10) with $p = p_2$,

$$x^2\left(\alpha_2 x^2 y'' + \beta_2 xy' + \gamma_2 y\right) = \sum_{n=2}^{\infty} p_2(n+r-2)a_{n-2} x^{n+r}.$$

Therefore we can rewrite (7.5.11) as

$$\begin{aligned} Ly = \sum_{n=0}^{\infty} p_0(n+r)a_n x^{n+r} &+ \sum_{n=1}^{\infty} p_1(n+r-1)a_{n-1} x^{n+r} \\ &+ \sum_{n=2}^{\infty} p_2(n+r-2)a_{n-2} x^{n+r}, \end{aligned}$$

or

$$\begin{aligned} Ly &= p_0(r)a_0 x^r + \left[p_0(r+1)a_1 + p_1(r)a_2\right] x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} \left[p_0(n+r)a_n + p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}\right] x^{n+r}, \end{aligned}$$

which implies (7.5.4) with $\{b_n\}$ defined as in (7.5.5).

Theorem 7.5.2

Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n. Define

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\dst \frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &= -\dst \frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \end{aligned} \tag{7.5.12}$$

Then the Frobenius series

$$y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n \tag{7.5.13}$$

converges and satisfies

$$Ly(x,r) = p_0(r) x^r \tag{7.5.14}$$

on the interval $(0,\rho)$, where ρ is the distance from the origin to the nearest zero of $A(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ in the complex plane. (If A is constant, then $\rho = \infty$.)

If $\{a_n(r)\}$ is determined by the recurrence relation (7.5.12) then substituting $a_n = a_n(r)$ into (7.5.5) yields $b_0 = p_0(r)$ and $b_n = 0$ for $n \geq 1$, so (7.5.4) reduces to (7.5.14). We omit the proof that the series (7.5.13) converges on $(0,\rho)$. ∎

If $\alpha_i = \beta_i = \gamma_i = 0$ for $i = 1, 2$, then $Ly = 0$ reduces to the Euler equation

$$\alpha_0 x^2 y'' + \beta_0 x y' + \gamma_0 y = 0.$$

Theorem 7.4.3 shows that the solutions of this equation are determined by the zeros of the indicial polynomial

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Since (7.5.14) implies that this is also true for the solutions of $Ly = 0$, we'll also say that p_0 is the indicial polynomial of (7.5.2), and that $p_0(r) = 0$ is the indicial equation of $Ly = 0$. We'll consider only cases where the indicial equation has real roots r_1 and r_2, with $r_1 \geq r_2$.

Theorem 7.5.3

Let L and $\{a_n(r)\}$ be as in Theorem ???, and suppose the indicial equation $p_0(r) = 0$ of $Ly = 0$ has real roots r_1 and r_2, where $r_1 \geq r_2$. Then

$$y_1(x) = y(x, r_1) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

is a Frobenius solution of $Ly = 0$. Moreover, if $r_1 - r_2$ isn't an integer then

$$y_2(x) = y(x, r_2) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n$$

is also a Frobenius solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Proof Since r_1 and r_2 are roots of $p_0(r) = 0$, the indicial polynomial can be factored as

$$p_0(r) = \alpha_0 (r - r_1)(r - r_2). \tag{7.5.15}$$

Therefore

$$p_0(n + r_1) = n\alpha_0 (n + r_1 - r_2),$$

which is nonzero if $n > 0$, since $r_1 - r_2 \geq 0$. Therefore the assumptions of Theorem 7.5.2 hold with $r = r_1$, and (7.5.14) implies that $Ly_1 = p_0(r_1) x^{r_1} = 0$.

Now suppose $r_1 - r_2$ isn't an integer. From (7.5.15),

$$p_0(n + r_2) = n\alpha_0 (n - r_1 + r_2) \neq 0 \quad \text{if} \quad n = 1, 2, \cdots.$$

Hence, the assumptions of Theorem 7.5.2 hold with $r = r_2$, and (7.5.14) implies that $Ly_2 = p_0(r_2)x^{r_2} = 0$. We leave the proof that $\{y_1, y_2\}$ is a fundamental set of solutions as an exercise (Exercise 52). ∎

It isn't always possible to obtain explicit formulas for the coefficients in Frobenius solutions. However, we can always set up the recurrence relations and use them to compute as many coefficients as we want. The next example illustrates this.

Example 7.5.1

Find a fundamental set of Frobenius solutions of

$$2x^2(1+x+x^2)y'' + x(9+11x+11x^2)y' + (6+10x+7x^2)y = 0. \tag{7.5.16}$$

Compute just the first six coefficients a_0,…, a_5 in each solution.

Solution For the given equation, the polynomials defined in Theorem 7.5.2 are

$$\begin{array}{rcccl} p_0(r) & = & 2r(r-1)+9r+6 & = & (2r+3)(r+2), \\ p_1(r) & = & 2r(r-1)+11r+10 & = & (2r+5)(r+2), \\ [3pt]p_2(r) & = & 2r(r-1)+11r+7 & = & (2r+7)(r+1). \end{array}$$

The zeros of the indicial polynomial p_0 are $r_1 = -3/2$ and $r_2 = -2$, so $r_1 - r_2 = 1/2$. Therefore Theorem 7.5.3 implies that

$$y_1 = x^{-3/2}\sum_{n=0}^{\infty} a_n(-3/2)x^n \quad \text{and} \quad y_2 = x^{-2}\sum_{n=0}^{\infty} a_n(-2)x^n \tag{7.5.17}$$

form a fundamental set of Frobenius solutions of (7.5.16). To find the coefficients in these series, we use the recurrence relation of Theorem 7.5.2; thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\,\backslash\text{dst}\frac{p_1(r)}{p_0(r+1)} = -\backslash\text{dst}\frac{(2r+5)(r+2)}{(2r+5)(r+3)} = -\backslash\text{dst}\frac{r+2}{r+3}, \\ a_n(r) &= -\,\backslash\text{dst}\frac{p_1(n+r-1)a_{n-1}+p_2(n+r-2)a_{n-2}}{p_0(n+r)} \\ &= -\,\backslash\text{dst}\frac{(n+r+1)(2n+2r+3)a_{n-1}(r)+(n+r-1)(2n+2r+3)a_{n-2}(r)}{(n+r+2)(2n+2r+3)} \\ &= -\,\backslash\text{dst}\frac{(n+r+1)a_{n-1}(r)+(n+r-1)a_{n-2}(r)}{n+r+2}, \quad n \ge 2. \end{aligned}$$

Setting $r = -3/2$ in these equations yields

$$\begin{aligned} a_0(-3/2) &= 1, \\ a_1(-3/2) &= -1/3, \\ a_n(-3/2) &= -\backslash\text{dst}\tfrac{(2n-1)a_{n-1}(-3/2)+(2n-5)a_{n-2}(-3/2)}{2n+1}, \quad n \geq 2, \end{aligned} \tag{7.5.18}$$

and setting $r = -2$ yields

$$\begin{aligned} a_0(-2) &= 1, \\ a_1(-2) &= 0, \\ a_n(-2) &= -\backslash\text{dst}\tfrac{(n-1)a_{n-1}(-2)+(n-3)a_{n-2}(-2)}{n}, \quad n \geq 2. \end{aligned} \tag{7.5.19}$$

Calculating with (7.5.18) and (7.5.19) and substituting the results into (7.5.17) yields the fundamental set of Frobenius solutions

$$\begin{aligned} y_1 &= x^{-3/2}\left(1 - \frac{1}{3}x + \frac{2}{5}x^2 - \frac{5}{21}x^3 + \frac{7}{135}x^4 + \frac{76}{1155}x^5 + \cdots\right), \\ y_2 &= x^{-2}\left(1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \cdots\right). \end{aligned}$$

Special Cases With Two Term Recurrence Relations

For $n \geq 2$, the recurrence relation (7.5.12) of Theorem 7.5.2 involves the three coefficients $a_n(r)$, $a_{n-1}(r)$, and $a_{n-2}(r)$. We'll now consider some special cases where (7.5.12) reduces to a two term recurrence relation; that is, a relation involving only $a_n(r)$ and $a_{n-1}(r)$ or only $a_n(r)$ and $a_{n-2}(r)$. This simplification often makes it possible to obtain explicit formulas for the coefficents of Frobenius solutions.

We first consider equations of the form

$$x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y = 0$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_2 = \beta_2 = \gamma_2 = 0$, so $p_2 \equiv 0$ and the recurrence relations in Theorem 7.5.2 simplify to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\backslash\text{dst}\tfrac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \quad n \geq 1. \end{aligned} \tag{7.5.20}$$

Example 7.5.2

Find a fundamental set of Frobenius solutions of

$$x^2(3+x)y'' + 5x(1+x)y' - (1-4x)y = 0. \tag{7.5.21}$$

Give explicit formulas for the coefficients in the solutions.

Solution For this equation, the polynomials defined in Theorem 7.5.2 are

$$\begin{aligned} p_0(r) &= 3r(r-1)+5r-1 &= (3r-1)(r+1), \\ p_1(r) &= r(r-1)+5r+4 &= (r+2)^2, \\ p_2(r) &= 0. \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = 1/3$ and $r_2 = -1$, so $r_1 - r_2 = 4/3$. Therefore Theorem 7.5.3 implies that

$$y_1 = x^{1/3}\sum_{n=0}^{\infty} a_n(1/3)x^n \quad \text{and} \quad y_2 = x^{-1}\sum_{n=0}^{\infty} a_n(-1)x^n$$

form a fundamental set of Frobenius solutions of (7.5.21). To find the coefficients in these series, we use the recurrence relationss (7.5.20); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\mathrm{\backslash dst}\tfrac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ &= -\mathrm{\backslash dst}\tfrac{(n+r+1)^2}{(3n+3r-1)(n+r+1)}a_{n-1}(r) \\ &= -\mathrm{\backslash dst}\tfrac{n+r+1}{3n+3r-1}a_{n-1}(r), \quad n \ge 1. \end{aligned} \tag{7.5.22}$$

Setting $r = 1/3$ in (7.5.22) yields

$$\begin{aligned} a_0(1/3) &= 1, \\ a_n(1/3) &= -\frac{3n+4}{9n}a_{n-1}(1/3), \quad n \ge 1. \end{aligned}$$

By using the product notation introduced in Section 7.2 and proceeding as we did in the examples in that section yields

$$a_n(1/3) = \frac{(-1)^n\prod_{j=1}^{n}(3j+4)}{9^n n!}, \quad n \ge 0.$$

Therefore

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(3j+4)}{9^n n!} x^n$$

is a Frobenius solution of (7.5.21).

Setting $r=-1$ in (7.5.22) yields

$$\begin{aligned} a_0(-1) &= 1, \\ a_n(-1) &= -\frac{n}{3n-4} a_{n-1}(-1), \quad n \geq 1, \end{aligned}$$

so

$$a_n(-1) = \frac{(-1)^n n!}{\prod_{j=1}^{n}(3j-4)}.$$

Therefore

$$y_2 = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\prod_{j=1}^{n}(3j-4)} x^n$$

is a Frobenius solution of (7.5.21), and $\{y_1, y_2\}$ is a fundamental set of solutions. ∎

We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0 \tag{7.5.23}$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_1 = \beta_1 = \gamma_1 = 0$, so $p_1 \equiv 0$ and the recurrence relations in Theorem 7.5.2 simplify to

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= 0, \\ a_n(r) &= -\backslash\mathrm{dst}\frac{p_2(n+r-2)}{p_0(n+r)} a_{n-2}(r), \quad n \geq 2. \end{aligned}$$

Since $a_1(r) = 0$, the last equation implies that $a_n(r) = 0$ if n is odd, so the Frobenius solutions are of the form

$$y(x,r) = x^r \sum_{m=0}^{\infty} a_{2m}(r) x^{2m},$$

where

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\backslash\text{dst}\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \ge 1. \end{aligned} \tag{7.5.24}$$

EXAMPLE 7.5.3

Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y'' - x(3+4x^2)y' + (2-2x^2)y = 0. \tag{7.5.25}$$

Give explicit formulas for the coefficients in the solutions.

SOLUTION For this equation, the polynomials defined in Theorem 7.5.2 are

$$\begin{aligned} p_0(r) &= 2r(r-1) - 3r + 2 &&= (r-2)(2r-1), \\ p_1(r) &= 0 \\ p_2(r) &= -\left[r(r-1) + 4r + 2\right] &&= -(r+1)(r+2). \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = 2$ and $r_2 = 1/2$, so $r_1 - r_2 = 3/2$. Therefore Theorem 7.5.3 implies that

$$y_1 = x^2 \sum_{m=0}^{\infty} a_{2m}(1/3) x^{2m} \quad \text{and} \quad y_2 = x^{1/2} \sum_{m=0}^{\infty} a_{2m}(1/2) x^{2m}$$

form a fundamental set of Frobenius solutions of (7.5.25). To find the coefficients in these series, we use the recurrence relation (7.5.24); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\backslash\text{dst}\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= \backslash\text{dst}\frac{(2m+r)(2m+r-1)}{(2m+r-2)(4m+2r-1)} a_{2m-2}(r), \quad m \ge 1. \end{aligned} \tag{7.5.26}$$

Setting $r = 2$ in (7.5.26) yields

$$
\begin{aligned}
a_0(2) &= 1,\\
a_{2m}(2) &= \backslash\mathrm{dst}\frac{(m+1)(2m+1)}{m(4m+3)}a_{2m-2}(2), \quad m \ge 1,
\end{aligned}
$$

so

$$a_{2m}(2) = (m+1)\prod_{j=1}^{m}\frac{2j+1}{4j+3}.$$

Therefore

$$y_1 = x^2\sum_{m=0}^{\infty}(m+1)\left(\prod_{j=1}^{m}\frac{2j+1}{4j+3}\right)x^{2m}$$

is a Frobenius solution of (7.5.25).

Setting $r = 1/2$ in (7.5.26) yields

$$
\begin{aligned}
a_0(1/2) &= 1,\\
a_{2m}(1/2) &= \backslash\mathrm{dst}\frac{(4m-1)(4m+1)}{8m(4m-3)}a_{2m-2}(1/2), \quad m \ge 1,
\end{aligned}
$$

so

$$a_{2m}(1/2) = \frac{1}{8^m m!}\prod_{j=1}^{m}\frac{(4j-1)(4j+1)}{4j-3}.$$

Therefore

$$y_2 = x^{1/2}\sum_{m=0}^{\infty}\frac{1}{8^m m!}\left(\prod_{j=1}^{m}\frac{(4j-1)(4j+1)}{4j-3}\right)x^{2m}$$

is a Frobenius solution of (7.5.25) and $\{y_1, y_2\}$ is a fundamental set of solutions.

Remark

Thus far, we considered only the case where the indicial equation has real roots that don't differ by an integer, which allows us to apply Theorem 7.5.3. However, for equations of the form (7.5.23), the sequence $\{a_{2m}(r)\}$ in (7.5.24) is defined for $r = r_2$ if $r_1 - r_2$ isn't an even integer. It can be shown (Exercise 56) that in this case

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1) x^{2m} \quad \text{and} \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} a_{2m}(r_2) x^{2m}$$

form a fundamental set Frobenius solutions of (7.5.23)

.

Using Technology

As we said at the end of Section 7.2, if you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficents is essentially irrelevant; recursive computation is usually more efficient. Since it's also laborious, we encourage you to write short programs to implement recurrence relations on a calculator or computer, even in exercises where this is not specifically required.

In practical use of the method of Frobenius when $x_0 = 0$ is a regular singular point, we're interested in how well the functions

$$y_N(x, r_i) = x^{r_i} \sum_{n=0}^{N} a_n(r_i) x^n, \quad i = 1, 2,$$

approximate solutions to a given equation when r_i is a zero of the indicial polynomial. In dealing with the corresponding problem for the case where $x_0 = 0$ is an ordinary point, we used numerical integration to solve the differential equation subject to initial conditions $y(0) = a_0, \quad y'(0) = a_1$, and compared the result with values of the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} a_n x^n.$$

We can't do that here, since in general we can't prescribe arbitrary initial values for solutions of a differential equation at a singular point. Therefore, motivated by Theorem 7.5.2 (specifically, (7.5.14)), we suggest the following procedure.

Verification Procedure

Let L and $Y_n(x; r_i)$ be defined by

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2) y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2) y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2) y$$

and

$$y_N(x;r_i) = x^{r_i}\sum_{n=0}^{N} a_n(r_i)x^n,$$

where the coefficients $\{a_n(r_i)\}_{n=0}^{N}$ are computed as in $(???)$, Theorem $???$. Compute the error

$$E_N(x;r_i) = x^{-r_i}Ly_N(x;r_i)/\alpha_0 \tag{7.5.27}$$

for various values of N and various values of x in the interval $(0,\rho)$, with ρ as defined in Theorem $???$.

The multiplier x^{-r_i}/α_0 on the right of (7.5.27) eliminates the effects of small or large values of x^{r_i} near $x = 0$, and of multiplication by an arbitrary constant. In some exercises you will be asked to estimate the maximum value of $E_N(x;r_i)$ on an interval $(0,\delta]$ by computing $E_N(x_m;r_i)$ at the M points $x_m = m\delta/M$, $m = 1$, 2, …, M, and finding the maximum of the absolute values:

$$\sigma_N(\delta) = \max\{|E_N(x_m;r_i)|,\ m = 1,2,\dots,M\}. \tag{7.5.28}$$

(For simplicity, this notation ignores the dependence of the right side of the equation on i and M.)

To implement this procedure, you'll have to write a computer program to calculate $\{a_n(r_i)\}$ from the applicable recurrence relation, and to evaluate $E_N(x;r_i)$.

The next exercise set contains five exercises specifically identified by L that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in any of the Exercises 1–10, 14-25, and 28–51

7.5 Exercises

This set contains exercises specifically identified by L that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in Exercises 1–10, 14-25, and 28–51.

In Exercises 1–10 find a fundamental set of Frobenius solutions. Compute a_0, a_1 …, a_N for N at least 7 in each solution.

1. C $2x^2(1+x+x^2)y''+x(3+3x+5x^2)y'-y=0$

Show answer

2. C $3x^2y''+2x(1+x-2x^2)y'+(2x-8x^2)y=0$

Show answer

3. C $x^2(3+3x+x^2)y''+x(5+8x+7x^2)y'-(1-2x-9x^2)y=0$

Show answer

4. C $4x^2y''+x(7+2x+4x^2)y'-(1-4x-7x^2)y=0$

Show answer

5. C $12x^2(1+x)y''+x(11+35x+3x^2)y'-(1-10x-5x^2)y=0$

Show answer

6. C $x^2(5+x+10x^2)y''+x(4+3x+48x^2)y'+(x+36x^2)y=0$

Show answer

7. C $8x^2y''-2x(3-4x-x^2)y'+(3+6x+x^2)y=0$

Show answer

8. C $18x^2(1+x)y''+3x(5+11x+x^2)y'-(1-2x-5x^2)y=0$

Show answer

9. C $x(3+x+x^2)y''+(4+x-x^2)y'+xy=0$

Show answer

10. C $10x^2(1+x+2x^2)y''+x(13+13x+66x^2)y'-(1+4x+10x^2)y=0$

Show answer

11. L The Frobenius solutions of

$$2x^2(1+x+x^2)y''+x(9+11x+11x^2)y'+(6+10x+7x^2)y=0$$

obtained in Example 7.5.1 are defined on $(0,\rho)$, where ρ is defined in Theorem 7.5.2. Find ρ. Then do the following experiments for each Frobenius solution, with $M=20$ and $\delta=.5\rho$, $.7\rho$, and $.9\rho$ in the verification procedure described at the end of this section.

a. Compute $\sigma_N(\delta)$ (see Eqn. (7.5.28)) for $N=5$, 10, 15,…, 50.

b. Find N such that $\sigma_N(\delta)<10^{-5}$.

c. Find N such that $\sigma_N(\delta)<10^{-10}$.

12. L By Theorem 7.5.2 the Frobenius solutions of the equation in Exercise 4 are defined on $(0,\infty)$. Do experiments **(a)**, **(b)**, and **(c)** of Exercise 11 for each Frobenius solution, with $M=20$ and $\delta=1$, 2, and 3 in the verification procedure described at the end of this section.

13. L The Frobenius solutions of the equation in Exercise 6 are defined on $(0,\rho)$, where ρ is defined in Theorem 7.5.2. Find ρ and do experiments **(a)**, **(b)**, and **(c)** of Exercise 11 for each Frobenius solution, with $M=20$ and $\delta=.3\rho$, $.4\rho$, and $.5\rho$, in the verification procedure described at the end of this section.

In Exercises 14–25 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

14. $2x^2y'' + x(3+2x)y' - (1-x)y = 0$

Show answer

15. $x^2(3+x)y'' + x(5+4x)y' - (1-2x)y = 0$

Show answer

16. $2x^2y'' + x(5+x)y' - (2-3x)y = 0$

Show answer

17. $3x^2y'' + x(1+x)y' - y = 0$

Show answer

18. $2x^2y'' - xy' + (1-2x)y = 0$

Show answer

19. $9x^2y'' + 9xy' - (1+3x)y = 0$

Show answer

20. $3x^2y'' + x(1+x)y' - (1+3x)y = 0$

Show answer

21. $2x^2(3+x)y'' + x(1+5x)y' + (1+x)y = 0$

Show answer

22. $x^2(4+x)y'' - x(1-3x)y' + y = 0$

Show answer

23. $2x^2y'' + 5xy' + (1+x)y = 0$

Show answer

24. $x^2(3+4x)y'' + x(5+18x)y' - (1-12x)y = 0$

Show answer

25. $6x^2y'' + x(10-x)y' - (2+x)y = 0$

Show answer

26. L By Theorem 7.5.2 the Frobenius solutions of the equation in Exercise 17 are defined on $(0, \infty)$. Do experiments **(a)**, **(b)**, and **(c)** of Exercise 11 for each Frobenius solution, with $M = 20$ and $\delta = 3$, 6, 9, and 12 in the verification procedure described at the end of this section.

27. L The Frobenius solutions of the equation in Exercise 22 are defined on $(0, \rho)$, where ρ is defined in Theorem 7.5.2. Find ρ and do experiments **(a)**, **(b)**, and **(c)** of Exercise 11 for each Frobenius solution, with $M = 20$ and $\delta = .25\rho$, $.5\rho$, and $.75\rho$ in the verification procedure described at the end of this section.

In Exercises 28–32 find a fundamental set of Frobenius solutions. Compute coefficients a_0, …, a_N for N at least 7 in each solution.

28. C $x^2(8+x)y'' + x(2+3x)y' + (1+x)y = 0$

Show answer

29. C $x^2(3+4x)y'' + x(11+4x)y' - (3+4x)y = 0$

Show answer

30. C $2x^2(2+3x)y'' + x(4+11x)y' - (1-x)y = 0$

Show answer

31. C $x^2(2+x)y'' + 5x(1-x)y' - (2-8x)y$

Show answer

32. C $x^2(6+x)y'' + x(11+4x)y' + (1+2x)y = 0$

Show answer

In Exercises 33–46 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

33. $8x^2y'' + x(2+x^2)y' + y = 0$

Show answer

34. $8x^2(1-x^2)y'' + 2x(1-13x^2)y' + (1-9x^2)y = 0$

Show answer

35. $x^2(1+x^2)y'' - 2x(2-x^2)y' + 4y = 0$

Show answer

36. $x(3+x^2)y'' + (2-x^2)y' - 8xy = 0$

Show answer

37. $4x^2(1-x^2)y'' + x(7-19x^2)y' - (1+14x^2)y = 0$

Show answer

38. $3x^2(2-x^2)y'' + x(1-11x^2)y' + (1-5x^2)y = 0$

Show answer

39. $2x^2(2+x^2)y'' - x(12-7x^2)y' + (7+3x^2)y = 0$

Show answer

40. $2x^2(2+x^2)y'' + x(4+7x^2)y' - (1-3x^2)y = 0$

Show answer

41. $2x^2(1+2x^2)y'' + 5x(1+6x^2)y' - (2-40x^2)y = 0$

Show answer

42. $3x^2(1+x^2)y'' + 5x(1+x^2)y' - (1+5x^2)y = 0$

Show answer

43. $x(1+x^2)y'' + (4+7x^2)y' + 8xy = 0$

Show answer

44. $x^2(2+x^2)y'' + x(3+x^2)y' - y = 0$

Show answer

45. $2x^2(1+x^2)y'' + x(3+8x^2)y' - (3-4x^2)y = 0$

Show answer

46. $9x^2y'' + 3x(3 + x^2)y' - (1 - 5x^2)y = 0$

Show answer

In Exercises 47–51 find a fundamental set of Frobenius solutions. Compute the coefficients a_0, …, a_{2M} for M at least 7 in each solution.

47. C $6x^2y'' + x(1 + 6x^2)y' + (1 + 9x^2)y = 0$

Show answer

48. C $x^2(8 + x^2)y'' + 7x(2 + x^2)y' - (2 - 9x^2)y = 0$

Show answer

49. C $9x^2(1 + x^2)y'' + 3x(3 + 13x^2)y' - (1 - 25x^2)y = 0$

Show answer

50. C $4x^2(1 + x^2)y'' + 4x(1 + 6x^2)y' - (1 - 25x^2)y = 0$

Show answer

51. C $8x^2(1 + 2x^2)y'' + 2x(5 + 34x^2)y' - (1 - 30x^2)y = 0$

Show answer

52. Suppose $r_1 > r_2$, $a_0 = b_0 = 1$, and the Frobenius series

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$$

both converge on an interval $(0, \rho)$.

a. Show that y_1 and y_2 are linearly independent on $(0, \rho)$.

Hint

Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on $(0, \rho)$, then

$$c_1x^{r_1-r_2}\sum_{n=0}^{\infty} a_n x^n + c_2\sum_{n=0}^{\infty} b_n x^n = 0, \quad 0 < x < \rho.$$

Then let $x \to 0+$ to conclude that $c_2 = 0$.

b. Use the result of **(b)** to complete the proof of Theorem 7.5.3.

53. The equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{7.5.1}$$

is Bessel's equation of order ν (http://www-history.mcs.st-and.ac.uk/Mathematicians/Bessel.html). (Here ν is a parameter, and this use of "order" should not be confused with its usual use as in "the order of the equation.") The solutions of (7.5.1) are Bessel functions of order (http://www-history.mcs.st-and.ac.uk/Mathematicians/Bessel.html) ν.

a. Assuming that ν isn't an integer, find a fundamental set of Frobenius solutions of (7.5.1).

b. If $\nu = 1/2$, the solutions of (7.5.1) reduce to familiar elementary functions. Identify these functions.

Show answer

54. a. Verify that

$$\frac{d}{dx}\left(|x|^r x^n\right) = (n+r)|x|^r x^{n-1} \quad \text{and} \quad \frac{d^2}{dx^2}\left(|x|^r x^n\right) = (n+r)(n+r-1)|x|^r x^{n-2}$$

if $x \neq 0$.

b. Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0.$$

Show that if $x^r \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ on $(0, \rho)$ then $|x|^r \sum_{n=0}^{\infty} a_n x^n$ is a solution on $(-\rho, 0)$ and $(0, \rho)$.

55. a. Deduce from Eqn. (7.5.20) that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{p_1(j+r-1)}{p_0(j+r)}.$$

b. Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an integer, then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n$$

form a fundamental set of Frobenius solutions of

$$x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y = 0.$$

c. Show that if p_0 satisfies the hypotheses of **(b)** then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n}(j + r_1 - r_2)} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

and

$$y_2 = x^{r_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n}(j + r_2 - r_1)} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x)y = 0.$$

56. Let

$$Ly = x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_2(r) = \alpha_2 r(r-1) + \beta_2 r + \gamma_2.$$

a. Use Theorem 7.5.2 to show that if

$$\begin{aligned} a_0(r) &= 1, \\ p_0(2m+r)a_{2m}(r) + p_2(2m+r-2)a_{2m-2}(r) &= 0, \quad m \geq 1, \end{aligned} \tag{7.5.1}$$

then the Frobenius series $y(x,r) = x^r \sum_{m=0}^{\infty} a_{2m} x^{2m}$ satisfies $Ly(x,r) = p_0(r)x^r$.

b. Deduce from (7.5.1) that if $p_0(2m+r)$ is nonzero for every positive integer m then

$$a_{2m}(r) = (-1)^m \prod_{j=1}^{m} \frac{p_2(2j+r-2)}{p_0(2j+r)}.$$

c. Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an even integer, then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1) x^{2m} \quad \text{and} \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} a_{2m}(r_2) x^{2m}$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

d. Show that if p_0 satisfies the hypotheses of **(c)** then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m! \prod_{j=1}^{m}(2j + r_1 - r_2)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

and

$$y_2 = x^{r_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m! \prod_{j=1}^{m}(2j + r_2 - r_1)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_2 x^2) y = 0.$$

57. Let

$$Ly = x^2 q_0(x) y'' + x q_1(x) y' + q_2(x) y,$$

where

$$q_0(x) = \sum_{j=0}^{\infty} \alpha_j x^j, \quad q_1(x) = \sum_{j=0}^{\infty} \beta_j x^j, \quad q_2(x) = \sum_{j=0}^{\infty} \gamma_j x^j,$$

and define

$$p_j(r) = \alpha_j r(r-1) + \beta_j r + \gamma_j, \quad j = 0, 1, \dots .$$

Let $y = x^r \sum_{n=0}^{\infty} a_n x^n$. Show that

$$Ly = x^r \sum_{n=0}^{\infty} b_n x^n,$$

where

$$b_n = \sum_{j=0}^{n} p_j(n + r - j) a_{n-j}.$$

58. a. Let L be as in Exercise 57. Show that if

$$y(x,r)=x^r\sum_{n=0}^{\infty}a_n(r)x^n$$

where

$$\begin{aligned}a_0(r)&=1,\\ a_n(r)&=-\,\backslash\mathrm{dst}\frac{1}{p_0(n+r)}\sum_{j=1}^{n}p_j(n+r-j)a_{n-j}(r),\quad n\ge 1,\end{aligned}$$

then

$$Ly(x,r)=p_0(r)x^r.$$

b. Conclude that if

$$p_0(r)=\alpha_0(r-r_1)(r-r_2)$$

where r_1-r_2 isn't an integer then $y_1=y(x,r_1)$ and $y_2=y(x,r_2)$ are solutions of $Ly=0$.

59. Let

$$Ly = x^2(\alpha_0 + \alpha_q x^q)y'' + x(\beta_0 + \beta_q x^q)y' + (\gamma_0 + \gamma_q x^q)y$$

where q is a positive integer, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_q(r) = \alpha_q r(r-1) + \beta_q r + \gamma_q.$$

a. Show that if

$$y(x,r)=x^r\sum_{m=0}^\infty a_{qm}(r)x^{qm}$$

where

$$\begin{array}{rcl} a_0(r) & = & 1, \\ a_{qm}(r) & = & -\dst\frac{p_q(q(m-1)+r)}{p_0(qm+r)}a_{q(m-1)}(r), \quad m\ge 1, \end{array} \tag{7.5.1}$$

then

$$Ly(x,r)=p_0(r)x^r.$$

b. Deduce from (7.5.1) that

$$a_{qm}(r)=(-1)^m\prod_{j=1}^m\frac{p_q\left(q(j-1)+r\right)}{p_0(qj+r)}.$$

c. Conclude that if $p_0(r)=\alpha_0(r-r_1)(r-r_2)$ where r_1-r_2 is not an integer multiple of q, then

$$y_1=x^{r_1}\sum_{m=0}^\infty a_{qm}(r_1)x^{qm} \quad \text{and} \quad y_2=x^{r_2}\sum_{m=0}^\infty a_{qm}(r_2)x^{qm}$$

form a fundamental set of Frobenius solutions of $Ly=0$.

d. Show that if p_0 satisfies the hypotheses of **(c)** then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{q^m m! \prod_{j=1}^{m}(qj + r_1 - r_2)} \left(\frac{\gamma_q}{\alpha_0}\right)^m x^{qm}$$

and

$$y_2 = x^{r_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{q^m m! \prod_{j=1}^{m}(qj + r_2 - r_1)} \left(\frac{\gamma_q}{\alpha_0}\right)^m x^{qm}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_q x^q) y = 0.$$

60. a. Suppose α_0, α_1, and α_2 are real numbers with $\alpha_0 \neq 0$, and $\{a_n\}_{n=0}^{\infty}$ is defined by

$$\alpha_0 a_1 + \alpha_1 a_0 = 0$$

and

$$\alpha_0 a_n + \alpha_1 a_{n-1} + \alpha_2 a_{n-2} = 0, \quad n \geq 2.$$

Show that

$$(\alpha_0 + \alpha_1 x + \alpha_2 x^2) \sum_{n=0}^{\infty} a_n x^n = \alpha_0 a_0,$$

and infer that

$$\sum_{n=0}^{\infty} a_n x^n = \frac{\alpha_0 a_0}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}.$$

b. With α_0, α_1, and α_2 as in **(a)**, consider the equation

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0, \tag{7.5.1}$$

and define

$$p_j(r) = \alpha_j r(r-1) + \beta_j r + \gamma_j, \quad j = 0, 1, 2.$$

Suppose

$$\frac{p_1(r-1)}{p_0(r)} = \frac{\alpha_1}{\alpha_0}, \qquad \frac{p_2(r-2)}{p_0(r)} = \frac{\alpha_2}{\alpha_0},$$

and

$$p_0(r) = \alpha_0(r-r_1)(r-r_2),$$

where $r_1 > r_2$. Show that

$$y_1 = \frac{x^{r_1}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2} \quad \text{and} \quad y_2 = \frac{x^{r_2}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}$$

form a fundamental set of Frobenius solutions of (7.5.1) on any interval $(0, \rho)$ on which $\alpha_0 + \alpha_1 x + \alpha_2 x^2$ has no zeros.

In Exercises 61–68 use the method suggested by Exercise 60 to find the general solution on some interval $(0, \rho)$.

61. $2x^2(1+x)y'' - x(1-3x)y' + y = 0$

Show answer

62. $6x^2(1+2x^2)y'' + x(1+50x^2)y' + (1+30x^2)y = 0$

Show answer

63. $28x^2(1-3x)y'' - 7x(5+9x)y' + 7(2+9x)y = 0$

Show answer

64. $9x^2(5+x)y'' + 9x(5+3x)y' - (5-8x)y = 0$

Show answer

65. $8x^2(2-x^2)y'' + 2x(10-21x^2)y' - (2+35x^2)y = 0$

Show answer

66. $4x^2(1+3x+x^2)y'' - 4x(1-3x-3x^2)y' + 3(1-x+x^2)y = 0$

Show answer

67. $3x^2(1+x)^2y'' - x(1-10x-11x^2)y' + (1+5x^2)y = 0$

Show answer

68. $4x^2(3+2x+x^2)y'' - x(3-14x-15x^2)y' + (3+7x^2)y = 0$

Show answer

7.6 The Method of Frobenius II

In this section we discuss a method for finding two linearly independent Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated real root. As in the preceding section, we consider equations that can be written as

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0 \tag{7.6.1}$$

where $\alpha_0 \neq 0$. We assume that the indicial equation $p_0(r) = 0$ has a repeated real root r_1. In this case Theorem 7.5.3 implies that (7.6.1) has one solution of the form

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n,$$

but does not provide a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental set of solutions. The following extension of Theorem 7.5.2 provides a way to find a second solution.

Theorem 7.6.1

Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y, \tag{7.6.2}$$

where $\alpha_0 \neq 0$ and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n, and define

$$\begin{array}{rcl} a_0(r) &=& 1, \\ a_1(r) &=& -\dst \frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &=& -\dst \frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \end{array}$$

Then the Frobenius series

$$y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n \tag{7.6.3}$$

satisfies

$$Ly(x,r) = p_0(r) x^r. \tag{7.6.4}$$

Moreover,

$$\frac{\partial y}{\partial r}(x,r) = y(x,r)\ln x + x^r \sum_{n=1}^{\infty} a_n'(r) x^n, \tag{7.6.5}$$

and

$$L\left(\frac{\partial y}{\partial r}(x,r)\right)=p_0'(r)x^r+x^rp_0(r)\ln x. \tag{7.6.6}$$

Proof Theorem 7.5.2 implies (7.6.4). Differentiating formally with respect to r in (7.6.3) yields

$$\begin{aligned}\frac{\partial y}{\partial r}(x,r)&=\backslash\text{dst}\frac{\partial}{\partial r}(x^r)\sum_{n=0}^{\infty}a_n(r)x^n+x^r\sum_{n=1}^{\infty}a_n'(r)x^n\\&=\backslash\text{dst}x^r\ln x\sum_{n=0}^{\infty}a_n(r)x^n+x^r\sum_{n=1}^{\infty}a_n'(r)x^n\\&=y(x,r)\ln x+x^r\sum_{n=1}^{\infty}a_n'(r)x^n,\end{aligned}$$

which proves (7.6.5).

To prove that $\partial y(x,r)/\partial r$ satisfies (7.6.6), we view y in (7.6.2) as a function $y=y(x,r)$ of two variables, where the prime indicates partial differentiation with respect to x; thus,

$$y'=y'(x,r)=\frac{\partial y}{\partial x}(x,r)\quad\text{and}\quad y''=y''(x,r)=\frac{\partial^2 y}{\partial x^2}(x,r).$$

With this notation we can use (7.6.2) to rewrite (7.6.4) as

$$x^2q_0(x)\frac{\partial^2 y}{\partial x^2}(x,r)+xq_1(x)\frac{\partial y}{\partial x}(x,r)+q_2(x)y(x,r)=p_0(r)x^r, \tag{7.6.7}$$

where

$$\begin{aligned}q_0(x)&=\alpha_0+\alpha_1x+\alpha_2x^2,\\q_1(x)&=\beta_0+\beta_1x+\beta_2x^2,\\q_2(x)&=\gamma_0+\gamma_1x+\gamma_2x^2.\end{aligned}$$

Differentiating both sides of (7.6.7) with respect to r yields

$$x^2q_0(x)\frac{\partial^3 y}{\partial r\partial x^2}(x,r)+xq_1(x)\frac{\partial^2 y}{\partial r\partial x}(x,r)+q_2(x)\frac{\partial y}{\partial r}(x,r)=p_0'(r)x^r+p_0(r)x^r\ln x.$$

By changing the order of differentiation in the first two terms on the left we can rewrite this as

$$x^2 q_0(x)\frac{\partial^3 y}{\partial x^2 \partial r}(x,r) + xq_1(x)\frac{\partial^2 y}{\partial x \partial r}(x,r) + q_2(x)\frac{\partial y}{\partial r}(x,r) = p_0'(r)x^r + p_0(r)x^r \ln x,$$

or

$$x^2 q_0(x)\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial r}(x,r)\right) + xq_1(x)\frac{\partial}{\partial r}\left(\frac{\partial y}{\partial x}(x,r)\right) + q_2(x)\frac{\partial y}{\partial r}(x,r) = p_0'(r)x^r + p_0(r)x^r \ln x,$$

which is equivalent to (7.6.6).

Theorem 7.6.2

Let L be as in Theorem 7.6.1 and suppose the indicial equation $p_0(r) = 0$ has a repeated real root r_1. Then

$$y_1(x) = y(x, r_1) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

and

$$y_2(x) = \frac{\partial y}{\partial r}(x, r_1) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n \tag{7.6.8}$$

form a fundamental set of solutions of $Ly = 0$.

Proof Since r_1 is a repeated root of $p_0(r) = 0$, the indicial polynomial can be factored as

$$p_0(r) = \alpha_0 (r - r_1)^2,$$

so

$$p_0(n + r_1) = \alpha_0 n^2,$$

which is nonzero if $n > 0$. Therefore the assumptions of Theorem 7.6.1 hold with $r = r_1$, and (7.6.4) implies that $Ly_1 = p_0(r_1)x^{r_1} = 0$. Since

$$p_0'(r) = 2\alpha(r - r_1)$$

it follows that $p_0'(r_1)=0$, so (7.6.6) implies that

$$Ly_2 = p_0'(r_1)x^{r_1} + x^{r_1}p_0(r_1)\ln x = 0.$$

This proves that y_1 and y_2 are both solutions of $Ly=0$. We leave the proof that $\{y_1, y_2\}$ is a fundamental set as an exercise (Exercise 53).

Example 7.6.1

Find a fundamental set of solutions of

$$x^2(1-2x+x^2)y'' - x(3+x)y' + (4+x)y = 0. \tag{7.6.9}$$

Compute just the terms involving x^{n+r_1}, where $0 \le n \le 4$ and r_1 is the root of the indicial equation.

Solution For the given equation, the polynomials defined in Theorem 7.6.1 are

$$\begin{array}{rcrcl} p_0(r) &=& r(r-1)-3r+4 &=& (r-2)^2, \\ p_1(r) &=& -2r(r-1)-r+1 &=& -(r-1)(2r+1), \\ p_2(r) &=& r(r-1). && \end{array}$$

Since $r_1=2$ is a repeated root of the indicial polynomial p_0, Theorem 7.6.2 implies that

$$y_1 = x^2\sum_{n=0}^\infty a_n(2)x^n \quad \text{and} \quad y_2 = y_1\ln x + x^2\sum_{n=1}^\infty a_n'(2)x^n \tag{7.6.10}$$

form a fundamental set of Frobenius solutions of (7.6.9). To find the coefficients in these series, we use the recurrence formulas from Theorem 7.6.1:

$$\begin{array}{rcl} a_0(r) &=& 1, \\ a_1(r) &=& -\backslash\text{dst}\,\frac{p_1(r)}{p_0(r+1)} = -\backslash\text{dst}\,\frac{(r-1)(2r+1)}{(r-1)^2} = \backslash\text{dst}\,\frac{2r+1}{r-1}, \\ a_n(r) &=& -\backslash\text{dst}\,\frac{p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)} \\ &=& \backslash\text{dst}\,\frac{(n+r-2)[(2n+2r-1)a_{n-1}(r)-(n+r-3)a_{n-2}(r)]}{(n+r-2)^2} \\ &=& \backslash\text{dst}\,\frac{(2n+2r-1)}{(n+r-2)}a_{n-1}(r) - \frac{(n+r-3)}{(n+r-2)}a_{n-2}(r), n \ge 2. \end{array} \tag{7.6.11}$$

Differentiating yields

$$\begin{array}{rcl} a_1'(r) &=& -\dst\frac{3}{(r-1)^2}, \\ a_n'(r) &=& \dst\frac{2n+2r-1}{n+r-2}a_{n-1}'(r) - \frac{n+r-3}{n+r-2}a_{n-2}'(r) \\ && \dst-\frac{3}{(n+r-2)^2}a_{n-1}(r) - \frac{1}{(n+r-2)^2}a_{n-2}(r), \quad n \geq 2. \end{array} \tag{7.6.12}$$

Setting $r = 2$ in (7.6.11) and (7.6.12) yields

$$\begin{array}{rcl} a_0(2) &=& 1, \\ a_1(2) &=& 5, \\ a_n(2) &=& \dst\frac{(2n+3)}{n}a_{n-1}(2) - \frac{(n-1)}{n}a_{n-2}(2), \quad n \geq 2 \end{array} \tag{7.6.13}$$

and

$$\begin{array}{rcl} a_1'(2) &=& -3, \\ a_n'(2) &=& \dst\frac{2n+3}{n}a_{n-1}'(2) - \frac{n-1}{n}a_{n-2}'(2) - \frac{3}{n^2}a_{n-1}(2) - \frac{1}{n^2}a_{n-2}(2), \quad n \geq 2. \end{array} \tag{7.6.14}$$

Computing recursively with (7.6.13) and (7.6.14) yields

$$a_0(2) = 1,\ a_1(2) = 5,\ a_2(2) = 17,\ a_3(2) = \frac{143}{3},\ a_4(2) = \frac{355}{3},$$

and

$$a_1'(2) = -3,\ a_2'(2) = -\frac{29}{2},\ a_3'(2) = -\frac{859}{18},\ a_4'(2) = -\frac{4693}{36}.$$

Substituting these coefficients into (7.6.10) yields

$$y_1 = x^2\left(1 + 5x + 17x^2 + \frac{143}{3}x^3 + \frac{355}{3}x^4 + \cdots\right)$$

and

$$y_2 = y_1 \ln x - x^3\left(3 + \frac{29}{2}x + \frac{859}{18}x^2 + \frac{4693}{36}x^3 + \cdots\right).\bbox$$

Since the recurrence formula (7.6.11) involves three terms, it's not possible to obtain a simple explicit formula for the coefficients in the Frobenius solutions of (7.6.9). However, as we saw in the preceding sections, the recurrrence formula for $\{a_n(r)\}$ involves only two terms if either $\alpha_1 = \beta_1 = \gamma_1 = 0$ or $\alpha_2 = \beta_2 = \gamma_2 = 0$ in (7.6.1). In this case, it's often possible to find explicit formulas for the coefficients. The next two examples illustrate this.

Example 7.6.2

Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y'' + 5x^2y' + (1+x)y = 0. \tag{7.6.15}$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.6.1 are

$$\begin{array}{rcccl} p_0(r) & = & 4r(r-1)+1 & = & (2r-1)^2, \\ p_1(r) & = & 2r(r-1)+5r+1 & = & (r+1)(2r+1), \\ p_2(r) & = & 0. & & \end{array}$$

Since $r_1 = 1/2$ is a repeated zero of the indicial polynomial p_0, Theorem 7.6.2 implies that

$$y_1 = x^{1/2}\sum_{n=0}^{\infty} a_n(1/2)x^n \tag{7.6.16}$$

and

$$y_2 = y_1 \ln x + x^{1/2}\sum_{n=1}^{\infty} a_n'(1/2)x^n \tag{7.6.17}$$

form a fundamental set of Frobenius solutions of (7.6.15). Since $p_2 \equiv 0$, the recurrence formulas in Theorem 7.6.1 reduce to

$$\begin{array}{rcl} a_0(r) & = & 1, \\ a_n(r) & = & -\dst\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \\ & = & -\dst\frac{(n+r)(2n+2r-1)}{(2n+2r-1)^2}a_{n-1}(r), \\ & = & -\dst\frac{n+r}{2n+2r-1}a_{n-1}(r), \quad n \geq 0. \end{array}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r}{2j+2r-1}, \quad n \ge 0. \tag{7.6.18}$$

Setting $r = 1/2$ yields

$$\begin{aligned} a_n(1/2) &= (-1)^n \backslash\mathrm{dst} \prod_{j=1}^n \backslash\mathrm{dst} \tfrac{j+1/2}{2j} = (-1)^n \prod_{j=1}^n \backslash\mathrm{dst} \tfrac{2j+1}{4j}, \\ &= \backslash\mathrm{dst} \tfrac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!}, \quad n \ge 0. \end{aligned} \tag{7.6.19}$$

Substituting this into (7.6.16) yields

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} x^n.$$

To obtain y_2 in (7.6.17), we must compute $a_n'(1/2)$ for $n = 1$, 2,…. We'll do this by logarithmic differentiation. From (7.6.18),

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r|}{|2j+2r-1|}, \quad n \ge 1.$$

Therefore

$$\ln |a_n(r)| = \sum_{j=1}^{n} \left(\ln |j+r| - \ln |2j+2r-1| \right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Setting $r = 1/2$ here and recalling (7.6.19) yields

$$a_n'(1/2) = \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j+1/2} - \sum_{j=1}^n \frac{1}{j} \right). \tag{7.6.20}$$

Since

$$\frac{1}{j+1/2} - \frac{1}{j} = \frac{j-j-1/2}{j(j+1/2)} = -\frac{1}{j(2j+1)},$$

(7.6.20) can be rewritten as

$$a_n'(1/2) = -\frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \sum_{j=1}^n \frac{1}{j(2j+1)}.$$

Therefore, from (7.6.17),

$$y_2 = y_1 \ln x - x^{1/2} \sum_{n=1}^\infty \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j(2j+1)} \right) x^n.$$

Example 7.6.3

Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y'' - 2x(1+2x^2)y' + (2-2x^2)y = 0. \tag{7.6.21}$$

Give explicit formulas for the coefficients in the solutions.

Solution For (7.6.21), the polynomials defined in Theorem 7.6.1 are

$$\begin{array}{rcccl} p_0(r) & = & 2r(r-1) - 2r + 2 & = & 2(r-1)^2, \\ p_1(r) & = & 0, & & \\ p_2(r) & = & -r(r-1) - 4r - 2 & = & -(r+1)(r+2). \end{array}$$

As in Section 7.5, since $p_1 \equiv 0$, the recurrence formulas of Theorem 7.6.1 imply that $a_n(r) = 0$ if n is odd, and

$$\begin{array}{rcl} a_0(r) & = & 1, \\ a_{2m}(r) & = & -\dst\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ & = & \dst\frac{(2m+r-1)(2m+r)}{2(2m+r-1)^2}a_{2m-2}(r) \\ & = & \dst\frac{2m+r}{2(2m+r-1)}a_{2m-2}(r), \quad m \ge 1. \end{array}$$

Since $r_1 = 1$ is a repeated root of the indicial polynomial p_0, Theorem 7.6.2 implies that

$$y_1 = x\sum_{m=0}^{\infty} a_{2m}(1)x^{2m} \tag{7.6.22}$$

and

$$y_2 = y_1 \ln x + x\sum_{m=1}^{\infty} a'_{2m}(1)x^{2m} \tag{7.6.23}$$

form a fundamental set of Frobenius solutions of (7.6.21). We leave it to you to show that

$$a_{2m}(r) = \frac{1}{2^m}\prod_{j=1}^{m}\frac{2j+r}{2j+r-1}. \tag{7.6.24}$$

Setting $r = 1$ yields

$$a_{2m}(1) = \frac{1}{2^m}\prod_{j=1}^{m}\frac{2j+1}{2j} = \frac{\prod_{j=1}^{m}(2j+1)}{4^m m!}, \tag{7.6.25}$$

and substituting this into (7.6.22) yields

$$y_1 = x\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{m}(2j+1)}{4^m m!}x^{2m}.$$

To obtain y_2 in (7.6.23), we must compute $a'_{2m}(1)$ for $m = 1$, 2, …. Again we use logarithmic differentiation. From (7.6.24),

$$|a_{2m}(r)| = \frac{1}{2^m}\prod_{j=1}^{m}\frac{|2j+r|}{|2j+r-1|}.$$

Taking logarithms yields

$$\ln|a_{2m}(r)| = -m\ln 2 + \sum_{j=1}^{m}\left(\ln|2j+r| - \ln|2j+r-1|\right).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^{m}\left(\frac{1}{2j+r} - \frac{1}{2j+r-1}\right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r)\sum_{j=1}^{m}\left(\frac{1}{2j+r} - \frac{1}{2j+r-1}\right).$$

Setting $r=1$ and recalling (7.6.25) yields

$$a'_{2m}(1) = \backslash\mathrm{dst}\frac{\prod_{j=1}^{m}(2j+1)}{4^m m!}\sum_{j=1}^{m}\left(\frac{1}{2j+1} - \frac{1}{2j}\right). \tag{7.6.26}$$

Since

$$\frac{1}{2j+1} - \frac{1}{2j} = -\frac{1}{2j(2j+1)},$$

(7.6.26) can be rewritten as

$$a'_{2m}(1) = -\backslash\mathrm{dst}\frac{\prod_{j=1}^{m}(2j+1)}{2\cdot 4^m m!}\sum_{j=1}^{m}\frac{1}{j(2j+1)}.$$

Substituting this into (7.6.23) yields

$$y_2 = y_1\ln x - \frac{x}{2}\sum_{m=1}^{\infty}\frac{\prod_{j=1}^{m}(2j+1)}{4^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(2j+1)}\right)x^{2m}.\backslash\mathrm{bbox}$$

If the solution $y_1 = y(x, r_1)$ of $Ly=0$ reduces to a finite sum, then there's a difficulty in using logarithmic differentiation to obtain the coefficients $\{a'_n(r_1)\}$ in the second solution. The next example illustrates this difficulty and shows how to overcome it.

Example 7.6.4

Find a fundamental set of Frobenius solutions of

$$x^2y'' - x(5-x)y' + (9-4x)y = 0. \tag{7.6.27}$$

Give explicit formulas for the coefficients in the solutions.

Solution For (7.6.27) the polynomials defined in Theorem 7.6.1 are

$$\begin{array}{rcccl} p_0(r) & = & r(r-1)-5r+9 & = & (r-3)^2, \\ p_1(r) & = & r-4, & & \\ p_2(r) & = & 0. & & \end{array}$$

Since $r_1 = 3$ is a repeated zero of the indicial polynomial p_0, Theorem 7.6.2 implies that

$$y_1 = x^3 \sum_{n=0}^{\infty} a_n(3)x^n \tag{7.6.28}$$

and

$$y_2 = y_1 \ln x + x^3 \sum_{n=1}^{\infty} a_n'(3)x^n \tag{7.6.29}$$

are linearly independent Frobenius solutions of (7.6.27). To find the coefficients in (7.6.28) we use the recurrence formulas

$$\begin{array}{rcl} a_0(r) & = & 1, \\ a_n(r) & = & -\backslash\mathrm{dst}\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ & = & -\backslash\mathrm{dst}\frac{n+r-5}{(n+r-3)^2}a_{n-1}(r), \quad n \geq 1. \end{array}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r-5}{(j+r-3)^2}. \tag{7.6.30}$$

Setting $r = 3$ here yields

$$a_n(3) = (-1)^n \prod_{j=1}^{n} \frac{j-2}{j^2},$$

so $a_1(3) = 1$ and $a_n(3) = 0$ if $n \geq 2$. Substituting these coefficients into (7.6.28) yields

$$y_1 = x^3(1+x).$$

To obtain y_2 in (7.6.29) we must compute $a_n'(3)$ for $n = 1$, 2, …. Let's first try logarithmic differentiation. From (7.6.30),

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r-5|}{|j+r-3|^2}, \quad n \geq 1,$$

so

$$\ln|a_n(r)| = \sum_{j=1}^{n} \left(\ln|j+r-5| - 2\ln|j+r-3|\right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r-5} - \frac{2}{j+r-3}\right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r-5} - \frac{2}{j+r-3}\right). \tag{7.6.31}$$

However, we can't simply set $r = 3$ here if $n \geq 2$, since the bracketed expression in the sum corresponding to $j = 2$ contains the term $1/(r-3)$. In fact, since $a_n(3) = 0$ for $n \geq 2$, the formula (7.6.31) for $a_n'(r)$ is actually an indeterminate form at $r = 3$.

We overcome this difficulty as follows. From (7.6.30) with $n = 1$,

$$a_1(r) = -\frac{r-4}{(r-2)^2}.$$

Therefore

$$a_1'(r) = \frac{r-6}{(r-2)^3},$$

so

$$a_1'(3) = -3. \tag{7.6.32}$$

From (7.6.30) with $n \geq 2$,

$$a_n(r) = (-1)^n (r-4)(r-3) \frac{\prod_{j=3}^{n}(j+r-5)}{\prod_{j=1}^{n}(j+r-3)^2} = (r-3)c_n(r),$$

where

$$c_n(r) = (-1)^n (r-4) \frac{\prod_{j=3}^{n}(j+r-5)}{\prod_{j=1}^{n}(j+r-3)^2}, \quad n \geq 2.$$

Therefore

$$a_n'(r) = c_n(r) + (r-3)c_n'(r), \quad n \geq 2,$$

which implies that $a_n'(3) = c_n(3)$ if $n \geq 3$. We leave it to you to verify that

$$a_n'(3) = c_n(3) = \frac{(-1)^{n+1}}{n(n-1)n!}, \quad n \geq 2.$$

Substituting this and (7.6.32) into (7.6.29) yields

$$y_2 = x^3(1+x)\ln x - 3x^4 - x^3 \backslash\mathrm{dst} \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)n!} x^n.$$

7.6 Exercises

In Exercises 1–11 find a fundamental set of Frobenius solutions. Compute the terms involving x^{n+r_1}, where $0 \leq n \leq N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

1. C $x^2y'' - x(1-x)y' + (1-x^2)y = 0$

Show answer

2. C $x^2(1+x+2x^2)y' + x(3+6x+7x^2)y' + (1+6x-3x^2)y = 0$

Show answer

3. C $x^2(1+2x+x^2)y'' + x(1+3x+4x^2)y' - x(1-2x)y = 0$

Show answer

4. C $4x^2(1+x+x^2)y'' + 12x^2(1+x)y' + (1+3x+3x^2)y = 0$

Show answer

5. C $x^2(1+x+x^2)y'' - x(1-4x-2x^2)y' + y = 0$

Show answer

6. C $9x^2y'' + 3x(5+3x-2x^2)y' + (1+12x-14x^2)y = 0$

Show answer

7. C $x^2y'' + x(1+x+x^2)y' + x(2-x)y = 0$

Show answer

8. C $x^2(1+2x)y'' + x(5+14x+3x^2)y' + (4+18x+12x^2)y = 0$

Show answer

9. C $4x^2y'' + 2x(4+x+x^2)y' + (1+5x+3x^2)y = 0$

Show answer

10. C $16x^2y'' + 4x(6+x+2x^2)y' + (1+5x+18x^2)y = 0$

Show answer

11. C $9x^2(1+x)y'' + 3x(5+11x-x^2)y' + (1+16x-7x^2)y = 0$

Show answer

In Exercises 12–22 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

12. $4x^2y'' + (1+4x)y = 0$

Show answer

13. $36x^2(1-2x)y'' + 24x(1-9x)y' + (1-70x)y = 0$

Show answer

14. $x^2(1+x)y'' - x(3-x)y' + 4y = 0$

Show answer

15. $x^2(1-2x)y'' - x(5-4x)y' + (9-4x)y = 0$

Show answer

16. $25x^2y'' + x(15+x)y' + (1+x)y = 0$

Show answer

17. $2x^2(2+x)y'' + x^2y' + (1-x)y = 0$

Show answer

18. $x^2(9+4x)y'' + 3xy' + (1+x)y = 0$

Show answer

19. $x^2y'' - x(3-2x)y' + (4+3x)y = 0$

Show answer

20. $x^2(1-4x)y'' + 3x(1-6x)y' + (1-12x)y = 0$

Show answer

21. $x^2(1+2x)y'' + x(3+5x)y' + (1-2x)y = 0$

Show answer

22. $2x^2(1+x)y'' - x(6-x)y' + (8-x)y = 0$

Show answer

In Exercises 23–27 find a fundamental set of Frobenius solutions. Compute the terms involving x^{n+r_1}, where $0 \le n \le N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

23. C $x^2(1+2x)y''+x(5+9x)y'+(4+3x)y=0$

Show answer

24. C $x^2(1-2x)y''-x(5+4x)y'+(9+4x)y=0$

Show answer

25. C $x^2(1+4x)y''-x(1-4x)y'+(1+x)y=0$

Show answer

26. C $x^2(1+x)y''+x(1+2x)y'+xy=0$

Show answer

27. C $x^2(1-x)y''+x(7+x)y'+(9-x)y=0$

Show answer

In Exercises 28–38 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

28. $x^2y'' - x(1-x^2)y' + (1+x^2)y = 0$

Show answer

29. $x^2(1+x^2)y'' - 3x(1-x^2)y' + 4y = 0$

Show answer

30. $4x^2y'' + 2x^3y' + (1+3x^2)y = 0$

Show answer

31. $x^2(1+x^2)y'' - x(1-2x^2)y' + y = 0$

Show answer

32. $2x^2(2+x^2)y'' + 7x^3y' + (1+3x^2)y = 0$

Show answer

33. $x^2(1+x^2)y'' - x(1-4x^2)y' + (1+2x^2)y = 0$

Show answer

34. $4x^2(4+x^2)y'' + 3x(8+3x^2)y' + (1-9x^2)y = 0$

Show answer

35. $3x^2(3+x^2)y'' + x(3+11x^2)y' + (1+5x^2)y = 0$

Show answer

36. $4x^2(1+4x^2)y'' + 32x^3y' + y = 0$

Show answer

37. $9x^2y'' - 3x(7-2x^2)y' + (25+2x^2)y = 0$

Show answer

38. $x^2(1+2x^2)y'' + x(3+7x^2)y' + (1-3x^2)y = 0$

Show answer

In Exercises 39–43 find a fundamental set of Frobenius solutions. Compute the terms involving x^{2m+r_1}, where $0 \le m \le M$ (M at least 3) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $M > 3$.

39. C $x^2(1+x^2)y''+x(3+8x^2)y'+(1+12x^2)y$

Show answer

40. C $x^2y''-x(1-x^2)y'+(1+x^2)y=0$

Show answer

41. C $x^2(1-2x^2)y''+x(5-9x^2)y'+(4-3x^2)y=0$

Show answer

42. C $x^2(2+x^2)y''+x(14-x^2)y'+2(9+x^2)y=0$

Show answer

43. C $x^2(1+x^2)y''+x(3+7x^2)y'+(1+8x^2)y=0$

Show answer

In Exercises 44–52 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

44. $x^2(1-2x)y'' + 3xy' + (1+4x)y = 0$

Show answer

45. $x(1+x)y'' + (1-x)y' + y = 0$

Show answer

46. $x^2(1-x)y'' + x(3-2x)y' + (1+2x)y = 0$

Show answer

47. $4x^2(1+x)y'' - 4x^2y' + (1-5x)y = 0$

Show answer

48. $x^2(1-x)y'' - x(3-5x)y' + (4-5x)y = 0$

Show answer

49. $x^2(1+x^2)y'' - x(1+9x^2)y' + (1+25x^2)y = 0$

Show answer

50. $9x^2y'' + 3x(1-x^2)y' + (1+7x^2)y = 0$

Show answer

51. $x(1+x^2)y'' + (1-x^2)y' - 8xy = 0$

Show answer

52. $4x^2y'' + 2x(4-x^2)y' + (1+7x^2)y = 0$

Show answer

53. Under the assumptions of Theorem 7.6.2, suppose the power series

$$\sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad \sum_{n=1}^{\infty} a_n'(r_1)x^n$$

converge on $(-\rho, \rho)$.

a. Show that

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n$$

are linearly independent on $(0, \rho)$.

Hint

Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on $(0, \rho)$, then

$$(c_1 + c_2 \ln x)\sum_{n=0}^{\infty} a_n(r_1)x^n + c_2\sum_{n=1}^{\infty} a_n'(r_1)x^n = 0, \quad 0 < x < \rho.$$

Then let $x \to 0+$ to conclude that $c_2 = 0$.

b. Use the result of **(a)** to complete the proof of Theorem 7.6.2.

54. Let

$$Ly = x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_1(r) = \alpha_1 r(r-1) + \beta_1 r + \gamma_1.$$

Theorem 7.6.1 and Exercise 7.5. 55**(a)** imply that if

$$y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n$$

where

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{p_1(j+r-1)}{p_0(j+r)},$$

then

$$Ly(x,r) = p_0(r)x^r.$$

Now suppose $p_0(r) = \alpha_0(r-r_1)^2$ and $p_1(k+r_1) \neq 0$ if k is a nonnegative integer.

a. Show that $Ly = 0$ has the solution

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n,$$

where

$$a_n(r_1) = \frac{(-1)^n}{\alpha_0^n (n!)^2} \prod_{j=1}^{n} p_1(j + r_1 - 1).$$

b. Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n(r_1) J_n x^n,$$

where

$$J_n = \sum_{j=1}^{n} \frac{p_1'(j + r_1 - 1)}{p_1(j + r_1 - 1)} - 2 \sum_{j=1}^{n} \frac{1}{j}.$$

c. Conclude from **(a)** and **(b)** that if $\gamma_1 \neq 0$ then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n$$

and

$$y_2 = y_1 \ln x - 2x^{r_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0} \right)^n \left(\sum_{j=1}^{n} \frac{1}{j} \right) x^n$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x) y = 0.$$

(The conclusion is also valid if $\gamma_1 = 0$. Why?)

55. Let

$$Ly = x^2(\alpha_0 + \alpha_q x^q)y'' + x(\beta_0 + \beta_q x^q)y' + (\gamma_0 + \gamma_q x^q)y$$

where q is a positive integer, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_q(r) = \alpha_q r(r-1) + \beta_q r + \gamma_q.$$

Suppose

$$p_0(r) = \alpha_0(r - r_1)^2 \quad \text{and} \quad p_q(r) \not\equiv 0.$$

a. Recall from Exercise 7.5. 59 that $Ly = 0$ has the solution

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{qm}(r_1) x^{qm},$$

where

$$a_{qm}(r_1) = \frac{(-1)^m}{(q^2\alpha_0)^m (m!)^2} \prod_{j=1}^{m} p_q\left(q(j-1)+r_1\right).$$

b. Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1} \sum_{m=1}^{\infty} a'_{qm}(r_1) J_m x^{qm},$$

where

$$J_m = \sum_{j=1}^{m} \frac{p'_q\left(q(j-1)+r_1\right)}{p_q\left(q(j-1)+r_1\right)} - \frac{2}{q} \sum_{j=1}^{m} \frac{1}{j}.$$

c. Conclude from **(a)** and **(b)** that if $\gamma_q \neq 0$ then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2\alpha_0}\right)^m x^{qm}$$

and

$$y_2 = y_1 \ln x - \frac{2}{q} x^{r_1} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2\alpha_0}\right)^m \left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{qm}$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_q x^q) y = 0.$$

56. The equation

$$xy'' + y' + xy = 0$$

is Bessel's equation of order zero. (http://www-history.mcs.st-and.ac.uk/Mathematicians/Bessel.html) (See Exercise 53.) Find two linearly independent Frobenius solutions of this equation.

Show answer

57. Suppose the assumptions of Exercise 7.5. 53 hold, except that

$$p_0(r) = \alpha_0(r - r_1)^2.$$

Show that

$$y_1 = \frac{x^{r_1}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2} \quad \text{and} \quad y_2 = \frac{x^{r_1} \ln x}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}$$

are linearly independent Frobenius solutions of

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0$$

on any interval $(0, \rho)$ on which $\alpha_0 + \alpha_1 x + \alpha_2 x^2$ has no zeros.

In Exercises 58–65 use the method suggested by Exercise 57 to find the general solution on some interval $(0, \rho)$.

58. $4x^2(1+x)y'' + 8x^2y' + (1+x)y = 0$

Show answer

59. $9x^2(3+x)y'' + 3x(3+7x)y' + (3+4x)y = 0$

Show answer

60. $x^2(2-x^2)y'' - x(2+3x^2)y' + (2-x^2)y = 0$

Show answer

61. $16x^2(1+x^2)y'' + 8x(1+9x^2)y' + (1+49x^2)y = 0$

Show answer

62. $x^2(4+3x)y'' - x(4-3x)y' + 4y = 0$

Show answer

63. $4x^2(1+3x+x^2)y'' + 8x^2(3+2x)y' + (1+3x+9x^2)y = 0$

Show answer

64. $x^2(1-x)^2y'' - x(1+2x-3x^2)y' + (1+x^2)y = 0$

Show answer

65. $9x^2(1+x+x^2)y'' + 3x(1+7x+13x^2)y' + (1+4x+25x^2)y = 0$

Show answer

66. a. Let L and $y(x,r)$ be as in Exercises 57 and 58. Extend Theorem 7.6.1 by showing that

$$L\left(\frac{\partial y}{\partial r}(x,r)\right) = p_0'(r)x^r + x^r p_0(r)\ln x.$$

b. Show that if

$$p_0(r) = \alpha_0(r-r_1)^2$$

then

$$y_1 = y(x,r_1) \quad \text{and} \quad y_2 = \frac{\partial y}{\partial r}(x,r_1)$$

are solutions of $Ly = 0$.

7.7 The Method of Frobenius III

In Sections 7.5 and 7.6 we discussed methods for finding Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated root or distinct real roots that don't differ by an integer. In this section we consider the case where the indicial equation has distinct real roots that differ by an integer. We'll limit our discussion to equations that can be written as

$$x^2(\alpha_0+\alpha_1x)y''+x(\beta_0+\beta_1x)y'+(\gamma_0+\gamma_1x)y=0 \tag{7.7.1}$$

or

$$x^2(\alpha_0+\alpha_2x^2)y''+x(\beta_0+\beta_2x^2)y'+(\gamma_0+\gamma_2x^2)y=0,$$

where the roots of the indicial equation differ by a positive integer.

We begin with a theorem that provides a fundamental set of solutions of equations of the form (7.7.1).

Theorem 7.7.1

Let

$$Ly=x^2(\alpha_0+\alpha_1x)y''+x(\beta_0+\beta_1x)y'+(\gamma_0+\gamma_1x)y,$$

where $\alpha_0\ne0$, and define

$$\begin{aligned}p_0(r)&=\alpha_0r(r-1)+\beta_0r+\gamma_0,\\ p_1(r)&=\alpha_1r(r-1)+\beta_1r+\gamma_1.\end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n, and define

$$\begin{array}{rcl}a_0(r)&=&1,\\ a_n(r)&=&-\backslash\mathrm{dst}\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r),\quad n\ge1.\end{array} \tag{7.7.2}$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r)=0$, and suppose $r_1=r_2+k$, where k is a positive integer. Then

$$y_1=x^{r_1}\sum_{n=0}^{\infty}a_n(r_1)x^n$$

is a Frobenius solution of $Ly = 0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_n(r_2) &= -\dst \frac{p_1(n+r_2-1)}{p_0(n+r_2)} a_{n-1}(r_2), \quad 1 \le n \le k-1, \end{aligned} \tag{7.7.3}$$

and

$$C = -\frac{p_1(r_1-1)}{k\alpha_0} a_{k-1}(r_2), \tag{7.7.4}$$

then

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} a_n(r_2) x^n + C \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n \right) \tag{7.7.5}$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Proof Theorem 7.5.3 implies that $Ly_1 = 0$. We'll now show that $Ly_2 = 0$. Since L is a linear operator, this is equivalent to showing that

$$L\left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2) x^n \right) + CL \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n \right) = 0. \tag{7.7.6}$$

To verify this, we'll show that

$$L\left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2) x^n \right) = p_1(r_1 - 1) a_{k-1}(r_2) x^{r_1} \tag{7.7.7}$$

and

$$L\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n \right) = k\alpha_0 x^{r_1}. \tag{7.7.8}$$

This will imply that $Ly_2 = 0$, since substituting (7.7.7) and (7.7.8) into (7.7.6) and using (7.7.4) yields

$$\begin{aligned} Ly_2 &= \left[p_1(r_1-1) a_{k-1}(r_2) + Ck\alpha_0 \right] x^{r_1} \\ &= \left[p_1(r_1-1) a_{k-1}(r_2) - p_1(r_1-1) a_{k-1}(r_2) \right] x^{r_1} = 0. \end{aligned}$$

We'll prove (7.7.8) first. From Theorem 7.6.1,

$$L\left(y(x,r)\ln x + x^r\sum_{n=1}^{\infty} a_n'(r)x^n\right) = p_0'(r)x^r + x^r p_0(r)\ln x.$$

Setting $r = r_1$ and recalling that $p_0(r_1) = 0$ and $y_1 = y(x, r_1)$ yields

$$L\left(y_1\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = p_0'(r_1)x^{r_1}. \tag{7.7.9}$$

Since r_1 and r_2 are the roots of the indicial equation, the indicial polynomial can be written as

$$p_0(r) = \alpha_0(r-r_1)(r-r_2) = \alpha_0\left[r^2 - (r_1+r_2)r + r_1r_2\right].$$

Differentiating this yields

$$p_0'(r) = \alpha_0(2r - r_1 - r_2).$$

Therefore $p_0'(r_1) = \alpha_0(r_1 - r_2) = k\alpha_0$, so (7.7.9) implies (7.7.8).

Before proving (7.7.7), we first note $a_n(r_2)$ is well defined by (7.7.3) for $1 \le n \le k-1$, since $p_0(n+r_2) \ne 0$ for these values of n. However, we can't define $a_n(r_2)$ for $n \ge k$ with (7.7.3), since $p_0(k+r_2) = p_0(r_1) = 0$. For convenience, we define $a_n(r_2) = 0$ for $n \ge k$. Then, from Theorem 7.5.1,

$$L\left(x^{r_2}\sum_{n=0}^{k-1} a_n(r_2)x^n\right) = L\left(x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n\right) = x^{r_2}\sum_{n=0}^{\infty} b_nx^n, \tag{7.7.10}$$

where $b_0 = p_0(r_2) = 0$ and

$$b_n = p_0(n+r_2)a_n(r_2) + p_1(n+r_2-1)a_{n-1}(r_2), \quad n \ge 1.$$

If $1 \le n \le k-1$, then (7.7.3) implies that $b_n = 0$. If $n \ge k+1$, then $b_n = 0$ because $a_{n-1}(r_2) = a_n(r_2) = 0$. Therefore (7.7.10) reduces to

$$L\left(x^{r_2}\sum_{n=0}^{k-1} a_n(r_2)x^n\right) = \left[p_0(k+r_2)a_k(r_2) + p_1(k+r_2-1)a_{k-1}(r_2)\right]x^{k+r_2}.$$

Since $a_k(r_2) = 0$ and $k + r_2 = r_1$, this implies (7.7.7).

We leave the proof that $\{y_1, y_2\}$ is a fundamental set as an exercise (Exercise 41).

Example 7.7.1

Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y'' - x(4-7x)y' - (5-3x)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.1 are

$$\begin{array}{rcccl} p_0(r) & = & 4r(r-1) - 4r - 5 & = & (2r+1)(2r-5), \\ p_1(r) & = & 2r(r-1) + 7r + 3 & = & (r+1)(2r+3). \end{array}$$

The roots of the indicial equation are $r_1 = 5/2$ and $r_2 = -1/2$, so $k = r_1 - r_2 = 3$. Therefore Theorem 7.7.1 implies that

$$y_1 = x^{5/2} \sum_{n=0}^{\infty} a_n(5/2)x^n \tag{7.7.11}$$

and

$$y_2 = x^{-1/2} \sum_{n=0}^{2} a_n(-1/2) + C \left(y_1 \ln x + x^{5/2} \sum_{n=1}^{\infty} a_n'(5/2)x^n \right) \tag{7.7.12}$$

(with C as in (7.7.4)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (7.7.2) is

$$\begin{array}{rcl} a_0(r) & = & 1, \\ a_n(r) & = & -\backslash\mathrm{dst}\,\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ & = & -\backslash\mathrm{dst}\,\frac{(n+r)(2n+2r+1)}{(2n+2r+1)(2n+2r-5)} a_{n-1}(r), \\ & = & -\backslash\mathrm{dst}\,\frac{n+r}{2n+2r-5} a_{n-1}(r),\ n \geq 1, \end{array} \tag{7.7.13}$$

which implies that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r}{2j+2r-5},\ n \geq 0. \tag{7.7.14}$$

Therefore

$$a_n(5/2) = \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!}. \tag{7.7.15}$$

Substituting this into (7.7.11) yields

$$y_1 = x^{5/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} x^n.$$

To compute the coefficients $a_0(-1/2), a_1(-1/2)$ and $a_2(-1/2)$ in y_2, we set $r=-1/2$ in (7.7.13) and apply the resulting recurrence formula for $n=1,\ 2$; thus,

$$\begin{aligned} a_0(-1/2) &= 1, \\ a_n(-1/2) &= -\frac{2n-1}{4(n-3)} a_{n-1}(-1/2),\ n=1,2. \end{aligned}$$

The last formula yields

$$a_1(-1/2) = 1/8 \quad \text{and} \quad a_2(-1/2) = 3/32.$$

Substituting $r_1 = 5/2, r_2 = -1/2, k=3$, and $\alpha_0 = 4$ into (7.7.4) yields $C = -15/128$. Therefore, from (7.7.12),

$$y_2 = x^{-1/2}\left(1 + \frac{1}{8}x + \frac{3}{32}x^2\right) - \frac{15}{128}\left(y_1 \ln x + x^{5/2}\sum_{n=1}^{\infty} a_n'(5/2)x^n\right). \tag{7.7.16}$$

We use logarithmic differentiation to obtain obtain $a_n'(r)$. From (7.7.14),

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r|}{|2j+2r-5|},\ n \geq 1.$$

Therefore

$$\ln|a_n(r)| = \sum_{j=1}^{n} \left(\ln|j+r| - \ln|2j+2r-5|\right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-5} \right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-5} \right).$$

Setting $r = 5/2$ here and recalling (7.7.15) yields

$$a_n'(5/2) = \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} \sum_{j=1}^{n} \left(\frac{1}{j+5/2} - \frac{1}{j} \right). \tag{7.7.17}$$

Since

$$\frac{1}{j+5/2} - \frac{1}{j} = -\frac{5}{j(2j+5)},$$

we can rewrite (7.7.17) as

$$a_n'(5/2) = -5 \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+5)} \right).$$

Substituting this into (7.7.16) yields

$$\begin{aligned} y_2 = {} & x^{-1/2} \left(1 + \frac{1}{8}x + \frac{3}{32}x^2 \right) - \frac{15}{128} y_1 \ln x \\ & + \frac{75}{128} x^{5/2} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+5)} \right) x^n . \text{\bbox} \end{aligned}$$

If $C = 0$ in (7.7.4), there’s no need to compute

$$y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n$$

in the formula (7.7.5) for y_2. Therefore it's best to compute C before computing $\{a_n'(r_1)\}_{n=1}^{\infty}$. This is illustrated in the next example. (See also Exercises 44 and 45.)

Example 7.7.2

Find a fundamental set of Frobenius solutions of

$$x^2(1-2x)y'' + x(8-9x)y' + (6-3x)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.1 are

$$\begin{array}{rcccl} p_0(r) & = & r(r-1)+8r+6 & = & (r+1)(r+6) \\ p_1(r) & = & -2r(r-1)-9r-3 & = & -(r+3)(2r+1). \end{array}$$

The roots of the indicial equation are $r_1 = -1$ and $r_2 = -6$, so $k = r_1 - r_2 = 5$. Therefore Theorem 7.7.1 implies that

$$y_1 = x^{-1}\sum_{n=0}^{\infty} a_n(-1)x^n \tag{7.7.18}$$

and

$$y_2 = x^{-6}\sum_{n=0}^{4} a_n(-6) + C\left(y_1 \ln x + x^{-1}\sum_{n=1}^{\infty} a_n'(-1)x^n\right) \tag{7.7.19}$$

(with C as in (7.7.4)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (7.7.2) is

$$\begin{array}{rcl} a_0(r) & = & 1, \\ a_n(r) & = & -\backslash\text{dst}\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ & = & \backslash\text{dst}\frac{(n+r+2)(2n+2r-1)}{(n+r+1)(n+r+6)}a_{n-1}(r),\ n \geq 1, \end{array} \tag{7.7.20}$$

which implies that

$$\begin{aligned} a_n(r) &= \backslash\mathrm{dst}\prod_{j=1}^n \tfrac{(j+r+2)(2j+2r-1)}{(j+r+1)(j+r+6)} \\ &= \backslash\mathrm{dst}\left(\prod_{j=1}^n \tfrac{j+r+2}{j+r+1}\right)\left(\prod_{j=1}^n \tfrac{2j+2r-1}{j+r+6}\right). \end{aligned} \tag{7.7.21}$$

Since

$$\prod_{j=1}^n \frac{j+r+2}{j+r+1} = \frac{(r+3)(r+4)\cdots(n+r+2)}{(r+2)(r+3)\cdots(n+r+1)} = \frac{n+r+2}{r+2}$$

because of cancellations, (7.7.21) simplifies to

$$a_n(r) = \frac{n+r+2}{r+2}\prod_{j=1}^n \frac{2j+2r-1}{j+r+6}.$$

Therefore

$$a_n(-1) = (n+1)\prod_{j=1}^n \frac{2j-3}{j+5}.$$

Substituting this into (7.7.18) yields

$$y_1 = x^{-1}\sum_{n=0}^\infty (n+1)\left(\prod_{j=1}^n \frac{2j-3}{j+5}\right)x^n.$$

To compute the coefficients $a_0(-6), \dots, a_4(-6)$ in y_2, we set $r=-6$ in (7.7.20) and apply the resulting recurrence formula for $n=1$, 2, 3, 4; thus,

$$\begin{aligned} &a_0(-6) = 1, \\ &a_n(-6) = \backslash\mathrm{dst}\frac{(n-4)(2n-13)}{n(n-5)}a_{n-1}(-6),\ n=1,2,3,4. \end{aligned}$$

The last formula yields

$$a_1(-6) = -\frac{33}{4},\ a_2(-6) = \frac{99}{4},\ a_3(-6) = -\frac{231}{8},\ a_4(-6) = 0.$$

Since $a_4(-6)=0$, (7.7.4) implies that the constant C in (7.7.19) is zero. Therefore (7.7.19) reduces to

$$y_2 = x^{-6}\left(1 - \frac{33}{4}x + \frac{99}{4}x^2 - \frac{231}{8}x^3\right).\bbox$$

We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0,$$

where the roots of the indicial equation are real and differ by an even integer. The case where the roots are real and differ by an odd integer can be handled by the method discussed in 56.

The proof of the next theorem is similar to the proof of Theorem 7.7.1 (Exercise 43).

Theorem 7.7.2

Let

$$Ly = x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(2m+r)$ is nonzero for all positive integers m, and define

$$\begin{array}{rcl} a_0(r) &=& 1, \\ a_{2m}(r) &=& -\dst\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \geq 1. \end{array} \tag{7.7.22}$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r) = 0$, and suppose $r_1 = r_2 + 2k$, where k is a positive integer. Then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1) x^{2m}$$

is a Frobenius solution of $Ly = 0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_{2m}(r_2) &= -\dst\frac{p_2(2m+r_2-2)}{p_0(2m+r_2)} a_{2m-2}(r_2), \quad 1 \leq m \leq k-1 \end{aligned}$$

and

$$C = -\frac{p_2(r_1-2)}{2k\alpha_0} a_{2k-2}(r_2), \tag{7.7.23}$$

then

$$y_2 = x^{r_2} \sum_{m=0}^{k-1} a_{2m}(r_2) x^{2m} + C\left(y_1 \ln x + x^{r_1} \sum_{m=1}^{\infty} a'_{2m}(r_1) x^{2m}\right) \tag{7.7.24}$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Example 7.7.3

Find a fundamental set of Frobenius solutions of

$$x^2(1+x^2)y'' + x(3+10x^2)y' - (15-14x^2)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.2 are

$$\begin{aligned} p_0(r) &= r(r-1) + 3r - 15 &= (r-3)(r+5) \\ p_2(r) &= r(r-1) + 10r + 14 &= (r+2)(r+7). \end{aligned}$$

The roots of the indicial equation are $r_1 = 3$ and $r_2 = -5$, so $k = (r_1 - r_2)/2 = 4$. Therefore Theorem 7.7.2 implies that

$$y_1 = x^3 \sum_{m=0}^{\infty} a_{2m}(3) x^{2m} \tag{7.7.25}$$

and

$$y_2 = x^{-5} \sum_{m=0}^{3} a_{2m}(-5) x^{2m} + C\left(y_1 \ln x + x^3 \sum_{m=1}^{\infty} a'_{2m}(3) x^{2m}\right)$$

(with C as in (7.7.23)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (7.7.22) is

$$\begin{aligned}
a_0(r) &= 1, \\
a_{2m}(r) &= -\backslash\mathrm{dst}\,\tfrac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\
&= -\backslash\mathrm{dst}\,\tfrac{(2m+r)(2m+r+5)}{(2m+r-3)(2m+r+5)}a_{2m-2}(r) \\
&= -\backslash\mathrm{dst}\,\tfrac{2m+r}{2m+r-3}a_{2m-2}(r),\ m \geq 1,
\end{aligned} \tag{7.7.26}$$

which implies that

$$a_{2m}(r) = (-1)^m \prod_{j=1}^{m} \frac{2j+r}{2j+r-3},\ m \geq 0. \tag{7.7.27}$$

Therefore

$$a_{2m}(3) = \frac{(-1)^m \prod_{j=1}^{m}(2j+3)}{2^m m!}. \tag{7.7.28}$$

Substituting this into (7.7.25) yields

$$y_1 = x^3 \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+3)}{2^m m!} x^{2m}.$$

To compute the coefficients $a_2(-5)$, $a_4(-5)$, and $a_6(-5)$ in y_2, we set $r=-5$ in (7.7.26) and apply the resulting recurrence formula for $m=1$, 2, 3; thus,

$$a_{2m}(-5) = -\frac{2m-5}{2(m-4)}a_{2m-2}(-5),\ m=1,2,3.$$

This yields

$$a_2(-5) = -\frac{1}{2},\ a_4(-5) = \frac{1}{8},\ a_6(-5) = \frac{1}{16}.$$

Substituting $r_1=3$, $r_2=-5$, $k=4$, and $\alpha_0=1$ into (7.7.23) yields $C=-3/16$. Therefore, from (7.7.24),

$$y_2 = x^{-5}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{16}x^6\right) - \frac{3}{16}\left(y_1 \ln x + x^3 \sum_{m=1}^{\infty} a_{2m}'(3)x^{2m}\right). \tag{7.7.29}$$

To obtain $a_{2m}'(r)$ we use logarithmic differentiation. From (7.7.27),

$$|a_{2m}(r)| = \prod_{j=1}^{m} \frac{|2j+r|}{|2j+r-3|},\ m \geq 1.$$

Therefore

$$\ln|a_{2m}(r)| = \sum_{j=1}^{n} \left(\ln|2j+r| - \ln|2j+r-3|\right).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^{m} \left(\frac{1}{2j+r} - \frac{1}{2j+r-3}\right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^{n} \left(\frac{1}{2j+r} - \frac{1}{2j+r-3}\right).$$

Setting $r=3$ here and recalling (7.7.28) yields

$$a'_{2m}(3) = \frac{(-1)^m \prod_{j=1}^{m}(2j+3)}{2^m m!} \sum_{j=1}^{m} \left(\frac{1}{2j+3} - \frac{1}{2j}\right). \tag{7.7.30}$$

Since

$$\frac{1}{2j+3} - \frac{1}{2j} = -\frac{3}{2j(2j+3)},$$

we can rewrite (7.7.30) as

$$a'_{2m}(3) = -\frac{3}{2} \frac{(-1)^n \prod_{j=1}^{m}(2j+3)}{2^m m!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+3)}\right).$$

Substituting this into (7.7.29) yields

$$\begin{aligned} y_2 = x^{-5}\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6\right) - \frac{3}{16}y_1 \ln x \\ + \frac{9}{32}x^3 \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+3)}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j(2j+3)}\right) x^{2m}. \end{aligned}$$

Example 7.7.4

Find a fundamental set of Frobenius solutions of

$$x^2(1-2x^2)y'' + x(7-13x^2)y' - 14x^2y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.2 are

$$\begin{aligned} p_0(r) &= r(r-1)+7r &&= r(r+6), \\ p_2(r) &= -2r(r-1)-13r-14 &&= -(r+2)(2r+7). \end{aligned}$$

The roots of the indicial equation are $r_1 = 0$ and $r_2 = -6$, so $k = (r_1 - r_2)/2 = 3$. Therefore Theorem 7.7.2 implies that

$$y_1 = \sum_{m=0}^{\infty} a_{2m}(0)x^{2m}, \tag{7.7.31}$$

and

$$y_2 = x^{-6}\sum_{m=0}^{2} a_{2m}(-6)x^{2m} + C\left(y_1 \ln x + \sum_{m=1}^{\infty} a'_{2m}(0)x^{2m}\right) \tag{7.7.32}$$

(with C as in (7.7.23)) form a fundamental set of solutions of $Ly = 0$. The recurrence formulas (7.7.22) are

$$\begin{aligned}
a_0(r) &= 1, \\
a_{2m}(r) &= -\backslash\mathrm{dst}\,\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\
&= \backslash\mathrm{dst}\,\frac{(2m+r)(4m+2r+3)}{(2m+r)(2m+r+6)}a_{2m-2}(r) \\
&= \backslash\mathrm{dst}\,\frac{4m+2r+3}{2m+r+6}a_{2m-2}(r),\ m \ge 1,
\end{aligned} \tag{7.7.33}$$

which implies that

$$a_{2m}(r) = \prod_{j=1}^{m} \frac{4j+2r+3}{2j+r+6}.$$

Setting $r = 0$ yields

$$a_{2m}(0) = 6\frac{\prod_{j=1}^{m}(4j+3)}{2^m(m+3)!}.$$

Substituting this into (7.7.31) yields

$$y_1 = 6\sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(4j+3)}{2^m(m+3)!}x^{2m}.$$

To compute the coefficients $a_0(-6)$, $a_2(-6)$, and $a_4(-6)$ in y_2, we set $r=-6$ in (7.7.33) and apply the resulting recurrence formula for $m = 1$, 2; thus,

$$\begin{aligned}
a_0(-6) &= 1, \\
a_{2m}(-6) &= \backslash\mathrm{dst}\,\frac{4m-9}{2m}a_{2m-2}(-6),\ m = 1, 2.
\end{aligned}$$

The last formula yields

$$a_2(-6) = -\frac{5}{2} \quad \text{and} \quad a_4(-6) = \frac{5}{8}.$$

Since $p_2(-2) = 0$, the constant C in (7.7.23) is zero. Therefore (7.7.32) reduces to

$$y_2 = x^{-6}\left(1 - \frac{5}{2}x^2 + \frac{5}{8}x^4\right).$$

7.7 Exercises

In Exercises 1–40 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

1. $x^2y'' - 3xy' + (3+4x)y = 0$

Show answer

2. $xy'' + y = 0$

Show answer

3. $4x^2(1+x)y'' + 4x(1+2x)y' - (1+3x)y = 0$

Show answer

4. $xy'' + xy' + y = 0$

Show answer

5. $2x^2(2+3x)y'' + x(4+21x)y' - (1-9x)y = 0$

Show answer

6. $x^2y'' + x(2+x)y' - (2-3x)y = 0$

Show answer

7. $4x^2y'' + 4xy' - (9-x)y = 0$

Show answer

8. $x^2y'' + 10xy' + (14+x)y = 0$

Show answer

9. $4x^2(1+x)y'' + 4x(3+8x)y' - (5-49x)y = 0$

Show answer

10. $x^2(1+x)y'' - x(3+10x)y' + 30xy = 0$

Show answer

11. $x^2y'' + x(1+x)y' - 3(3+x)y = 0$

Show answer

12. $x^2y'' + x(1-2x)y' - (4+x)y = 0$

Show answer

13. $x(1+x)y'' - 4y' - 2y = 0$

Show answer

14. $x^2(1+2x)y''+x(9+13x)y'+(7+5x)y=0$

Show answer

15. $4x^2y''-2x(4-x)y'-(7+5x)y=0$

Show answer

16. $3x^2(3+x)y''-x(15+x)y'-20y=0$

Show answer

17. $x^2(1+x)y''+x(1-10x)y'-(9-10x)y=0$

Show answer

18. $x^2(1+x)y''+3x^2y'-(6-x)y=0$

Show answer

19. $x^2(1+2x)y''-2x(3+14x)y'+(6+100x)y=0$

Show answer

20. $x^2(1+x)y''-x(6+11x)y'+(6+32x)y=0$

Show answer

21. $4x^2(1+x)y''+4x(1+4x)y'-(49+27x)y=0$

Show answer

22. $x^2(1+2x)y''-x(9+8x)y'-12xy=0$

Show answer

23. $x^2(1+x^2)y''-x(7-2x^2)y'+12y=0$

Show answer

24. $x^2y''-x(7-x^2)y'+12y=0$

Show answer

25. $xy''-5y'+xy=0$

Show answer

26. $x^2y''+x(1+2x^2)y'-(1-10x^2)y=0$

Show answer

27. $x^2y'' - xy' - (3 - x^2)y = 0$

Show answer

28. $4x^2y'' + 2x(8 + x^2)y' + (5 + 3x^2)y = 0$

Show answer

29. $x^2y'' + x(1 + x^2)y' - (1 - 3x^2)y = 0$

Show answer

30. $x^2y'' + x(1 - 2x^2)y' - 4(1 + 2x^2)y = 0$

Show answer

31. $4x^2y'' + 8xy' - (35 - x^2)y = 0$

Show answer

32. $9x^2y'' - 3x(11 + 2x^2)y' + (13 + 10x^2)y = 0$

Show answer

33. $x^2y'' + x(1 - 2x^2)y' - 4(1 - x^2)y = 0$

Show answer

34. $x^2y'' + x(1 - 3x^2)y' - 4(1 - 3x^2)y = 0$

Show answer

35. $x^2(1 + x^2)y'' + x(5 + 11x^2)y' + 24x^2y = 0$

Show answer

36. $4x^2(1 + x^2)y'' + 8xy' - (35 - x^2)y = 0$

Show answer

37. $x^2(1 + x^2)y'' - x(5 - x^2)y' - (7 + 25x^2)y = 0$

Show answer

38. $x^2(1 + x^2)y'' + x(5 + 2x^2)y' - 21y = 0$

Show answer

39. $x^2(1 + 2x^2)y'' - x(3 + x^2)y' - 2x^2y = 0$

Show answer

40. $4x^2(1+x^2)y'' + 4x(2+x^2)y' - (15+x^2)y = 0$

Show answer

41. a. Under the assumptions of Theorem 7.7.1, show that

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n$$

and

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n + C\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n\right)$$

are linearly independent.

Hint

Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on an interval $(0, \rho)$, then

$$x^{-r_2}(c_1y_1(x) + c_2y_2(x)) = 0, \quad 0 < x < \rho.$$

Then let $x \to 0+$ to conclude that c_2=0.

b. Use the result of **(a)** to complete the proof of Theorem 7.7.1.

42. Find a fundamental set of Frobenius solutions of Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

in the case where ν is a positive integer.

Show answer

43. Prove Theorem 7.7.2.

44. Under the assumptions of Theorem 7.7.1, show that $C = 0$ if and only if $p_1(r_2 + \backslash\mathrm{l}) = 0$ for some integer $\backslash\mathrm{l}$ in $\{0, 1, \dots, k-1\}$.

45. Under the assumptions of Theorem 7.7.2, show that $C = 0$ if and only if $p_2(r_2 + 2\backslash\mathrm{l}) = 0$ for some integer ℓ in $\{0, 1, \dots, k-1\}$.

46. Let

$$Ly = \alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x) y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Show that if

$$p_0(r) = \alpha_0 (r - r_1)(r - r_2)$$

where $r_1 - r_2 = k$, a positive integer, then $Ly = 0$ has the solutions

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n} (j+k)} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n$$

and

$$\begin{aligned} y_2 = {} & x^{r_2} \sum_{n=0}^{k-1} \frac{(-1)^n}{n! \prod_{j=1}^{n} (j-k)} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n \\ & - \frac{1}{k!(k-1)!} \left(\frac{\gamma_1}{\alpha_0} \right)^k \left(y_1 \ln x - x^{r_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n} (j+k)} \left(\frac{\gamma_1}{\alpha_0} \right)^n \left(\sum_{j=1}^{n} \frac{2j+k}{j(j+k)} \right) x^n \right). \end{aligned}$$

47. Let

$$Ly = \alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_2 x^2)y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Show that if

$$p_0(r) = \alpha_0 (r - r_1)(r - r_2)$$

where $r_1 - r_2 = 2k$, an even positive integer, then $Ly = 0$ has the solutions

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j+k)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

and

$$\begin{aligned} y_2 = {} & x^{r_2} \sum_{m=0}^{k-1} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j-k)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m} \\ & - \frac{2}{4^k k!(k-1)!} \left(\frac{\gamma_2}{\alpha_0}\right)^k \left(y_1 \ln x - \frac{x^{r_1}}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j+k)} \left(\frac{\gamma_2}{\alpha_0}\right)^m \left(\sum_{j=1}^{m} \frac{2j+k}{j(j+k)} \right) x^{2m} \right). \end{aligned}$$

48. Let L be as in Exercises 7.5. 57 and 7.5. 58, and suppose the indicial polynomial of $Ly = 0$ is

$$p_0(r) = \alpha_0(r - r_1)(r - r_2),$$

with $k = r_1 - r_2$, where k is a positive integer. Define $a_0(r) = 1$ for all r. If r is a real number such that $p_0(n + r)$ is nonzero for all positive integers n, define

$$a_n(r) = -\frac{1}{p_0(n + r)} \sum_{j=1}^{n} p_j(n + r - j)a_{n-j}(r),\ n \geq 1,$$

and let

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n.$$

Define

$$a_n(r_2) = -\frac{1}{p_0(n + r_2)} \sum_{j=1}^{n} p_j(n + r_2 - j)a_{n-j}(r_2) \text{ if } n \geq 1 \text{ and } n \neq k,$$

and let $a_k(r_2)$ be arbitrary.

a. Conclude from Exercise 7.6. .66 that

$$L\left(y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = k\alpha_0 x^{r_1}.$$

b. Conclude from Exercise 7.5. .57 that

$$L\left(x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n\right) = Ax^{r_1},$$

where

$$A = \sum_{j=1}^{k} p_j(r_1 - j)a_{k-j}(r_2).$$

c. Show that y_1 and

$$y_2 = x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n - \frac{A}{k\alpha_0}\left(y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right)$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

d. Show that choosing the arbitrary quantity $a_k(r_2)$ to be nonzero merely adds a multiple of y_1 to y_2. Conclude that we may as well take $a_k(r_2) = 0$.

Chapter 7 Series Solutions of Linear Second Order Equations

7.1 Review of Power Series

1 **(a)** $R = 2$; $I = (-1, 3)$; **(b)** $R = 1/2$; $I = (3/2, 5/2)$ **(c)** $R = 0$; **(d)** $R = 16$;

I

$= (-14, 18)$

(e) R

$= \infty$; I

$= (-\infty, \infty)$

(f) R

$= 4/3$;

I

$= (-25/3, -17/3)$

3 **(a)** $R = 1$; $I = (0, 2)$ **(b)** $R = \sqrt{2}$; $I = (-2 - \sqrt{2}, -2 + \sqrt{2})$; **(c)** $R = \infty$;

I

$= (-\infty, \infty)$

(d) R

$= 0$ **(e)**

R

$= \sqrt{3}$; I

$= (-\sqrt{3}, \sqrt{3})$

(f) R

$= 1$ I

$= (0, 2)$

5 **(a)** $R = 3$; $I = (0, 6)$ **(b)** $R = 1$; $I = (-1, 1)$ **(c)** $R = 1/\sqrt{3}$

I $I = (-1, 3)$
$= (3$
$- 1/\sqrt{3}, 3$
$+ 1/\sqrt{3})$
(d) R
$= \infty$; I
$= (-\infty, \infty)$
(e) R
$= 0$ **(f)**
$R = 2$;

11 $b_n = 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (n+3)a_n$

12 $b_0 = 2a_2 - 2a_0$ $b_n = (n+2)(n+1)a_{n+2} + [3n(n-1) - 2]a_n + 3(n-1)a_{n-1},\ n \geq 1$

13 $b_n = (n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (2n^2 - 5n + 4)a_n$

14 $b_n = (n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (n^2 - 2n + 3)a_n$

15 $b_n = (n+2)(n+1)a_{n+2} + (3n^2 - 5n + 4)a_n$

16 $b_0 = -2a_2 + 2a_1 + a_0$,

b_n
$= -(n$
$+ 2)(n$
$+ 1)a_{n+2}$
$+ (n$
$+ 1)(n$
$+ 2)a_{n+1}$
$+ (2n$
$+ 1)a_n$
$+ a_{n-1}$,
$n \geq 2$

17 $b_0 = 8a_2 + 4a_1 - 6a_0,$

b_n
$= 4(n$
$+ 2)(n$
$+ 1)a_{n+2}$
$+ 4(n+1)^2 a_{n+1}$
$+ (n^2$
$+ n$
$- 6)a_n$
$- 3a_{n-1},$
$n \geq 1$

21 $b_0 = (r+1)(r+2)a_0,$

b_n
$= (n$
$+ r$
$+ 1)(n$
$+ r$
$+ 2)a_n$
$- (n+r-2)^2 a_{n-1},$
$n \geq 1.$

22 $b_0 = (r-2)(r+2)a_0,$

b_n
$= (n$
$+ r$
$- 2)(n$
$+ r$
$+ 2)a_n$
$+ (n$
$+ r$
$+ 2)(n$
$+ r$
$- 3)a_{n-1},$
$n \geq 14$

23 $b_0 = (r-1)^2 a_0,\ b_1 = r^2 a_1 + (r+2)(r+3)a_0,$

$$b_n = (n+r-1)^2 a_n + (n+r+1)(n+r+2)a_{n-1} + (n+r-1)a_{n-2},\quad n \ge 2$$

24 $b_0 = r(r+1)a_0,\ b_1 = (r+1)(r+2)a_1 + 3(r+1)(r+2)a_0,$

$$b_n = (n+r)(n+r+1)a_n + 3(n+r)(n+r+1)a_{n-1} + (n+r)a_{n-2},\quad n \ge 2$$

25 $b_0=(r+2)(r+1)a_0 \quad b_1=(r+3)(r+2)a_1,$

$$b_n=(n+r+2)(n+r+1)a_n+2(n+r-1)(n+r-3)a_{n-2},\quad n\ge 2$$

26 $b_0=2(r+1)(r+3)a_0,\ \ b_1=2(r+2)(r+4)a_1,$

$$b_n=2(n+r+1)(n+r+3)a_n+(n+r-3)(n+r)a_{n-2},\quad n\ge 2$$

7.2 Series Solutions Near an Ordinary Point I

1 $y =$ \dst$a_0 \sum_{m=0}^{\infty} (-1)^m (2m+1) x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m (m+1) x^{2m+1}$

2 $y =$ \dst$a_0 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{x^{2m}}{2m-1} + a_1 x$

3 $y =$ \dst$a_0 (1 - 10x^2 + 5x^4) + a_1 \left(x - 2x^3 + \frac{1}{5} x^5\right)$

4 $y =$ \dst$a_0 \sum_{m=0}^{\infty} (m+1)(2m+1) x^{2m} + \frac{a_1}{3} \sum_{m=0}^{\infty} (m+1)(2m+3) x^{2m+1}$

5 $y =$ \dst$a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{4j+1}{2j+1}\right] x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} (4j+3)\right] \frac{x^{2m+1}}{2^m m!}$

6 $y =$ \dst$a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{(4j+1)^2}{2j+1}\right] \frac{x^{2m}}{8^m m!} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{(4j+3)^2}{2j+3}\right] \frac{x^{2m+1}}{8^m m!}$

7 $y =$ \dst$a_0 \sum_{m=0}^{\infty} \frac{2^m m!}{\prod_{j=0}^{m-1} (2j+1)} x^{2m} + a_1 \sum_{m=0}^{\infty} \frac{\prod_{j=0}^{m-1} (2j+3)}{2^m m!} x^{2m+1}$

8 $y =$ \dst$a_0 \left(1 - 14x^2 + \frac{35}{3} x^4\right) + a_1 \left(x - 3x^3 + \frac{3}{5} x^5 + \frac{1}{35} x^7\right)$

9 **(a)** $y =$ \dst$a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{\prod_{j=0}^{m-1} (2j+1)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{2^m m!}$

10 **(a)** $y =$ \dst$a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{4j+3}{2j+1}\right] \frac{x^{2m}}{2^m m!} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{4j+5}{2j+3}\right] \frac{x^{2m+1}}{2^m m!}$

11 $y =$ \dst$2 - x - x^2 + \frac{1}{3} x^3 + \frac{5}{12} x^4 - \frac{1}{6} x^5 - \frac{17}{72} x^6 + \frac{13}{126} x^7 + \cdots$

12 $y =$ \dst$1 - x + 3x^2 - \frac{5}{2} x^3 + 5x^4 - \frac{21}{8} x^5 + 3x^6 - \frac{11}{16} x^7 + \cdots$

13 $y =$ \dst$2 - x - 2x^2 + \frac{1}{3} x^3 + 3x^4 - \frac{5}{6} x^5 - \frac{49}{5} x^6 + \frac{45}{14} x^7 + \cdots$

16 $y =$ \dst$a_0 \sum_{m=0}^{\infty} \frac{(x-3)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{(x-3)^{2m+1}}{(2m+1)!}$

17 $y =$ \dst$a_0 \sum_{m=0}^{\infty} \frac{(x-3)^{2m}}{2^m m!} + a_1 \sum_{m=0}^{\infty} \frac{(x-3)^{2m+1}}{\prod_{j=0}^{m-1} (2j+3)}$

18 $y =$ \dst$a_0 \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j+3)\right] \frac{(x-1)^{2m}}{m!} + a_1 \sum_{m=0}^{\infty} \frac{4^m (m+1)!}{\prod_{j=0}^{m-1} (2j+3)} (x-1)^{2m+1}$

19 $y =$ \dst$a_0 \left(1 - 6(x-2)^2 + \frac{4}{3} (x-2)^4 + \frac{8}{135} (x-2)^6\right) + a_1 \left((x-2) - \frac{10}{9} (x-2)^3\right)$

20 $y =$ \dst$a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} (2j+1)\right] \frac{3^m}{4^m m!} (x+1)^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{3^m m!}{\prod_{j=0}^{m-1} (2j+3)} (x+1)^{2m+1}$

21 $y =$ \dst$-1 + 2x + \frac{3}{8} x^2 - \frac{1}{3} x^3 - \frac{3}{128} x^4 - \frac{1}{1024} x^6 + \cdots$

22 $y =$ \dst$-2+3(x-3)+3(x-3)^2-2(x-3)^3-\frac{5}{4}(x-3)^4+\frac{3}{5}(x-3)^5+\frac{7}{24}(x-3)^6-\frac{4}{35}(x-3)^7+\cdots$

23 $y =$ \dst$-1+(x-1)+3(x-1)^2-\frac{5}{2}(x-1)^3-\frac{27}{4}(x-1)^4+\frac{21}{4}(x-1)^5+\frac{27}{2}(x-1)^6-\frac{81}{8}(x-1)^7+\cdots$

24 $y =$ \dst$4-6(x-3)-2(x-3)^2+(x-3)^3+\frac{3}{2}(x-3)^4-\frac{5}{4}(x-3)^5-\frac{49}{20}(x-3)^6+\frac{135}{56}(x-3)^7+\cdots$

25 $y =$ \dst$3-4(x-4)+15(x-4)^2-4(x-4)^3+\frac{15}{4}(x-4)^4-\frac{1}{5}(x-4)^5$

26 $y =$ \dst$3-3(x+1)-30(x+1)^2+\frac{20}{3}(x+1)^3+20(x+1)^4-\frac{4}{3}(x+1)^5-\frac{8}{9}(x+1)^6$

27 **(a)**$y =$ \dst$a_0\sum_{m=0}^\infty(-1)^m x^{2m}+a_1\sum_{m=0}^\infty(-1)^m x^{2m+1}$ **(b)**$y =$ \dst$\frac{a_0+a_1x}{1+x^2}$

33 $y =$ \dst$a_0\sum_{m=0}^\infty \frac{x^{3m}}{3^m m!\prod_{j=0}^{m-1}(3j+2)}+a_1\sum_{m=0}^\infty \frac{x^{3m+1}}{3^m m!\prod_{j=0}^{m-1}(3j+4)}$

34 $y =$ \dst$a_0\sum_{m=0}^\infty\left(\frac{2}{3}\right)^m\left[\prod_{j=0}^{m-1}(3j+2)\right]\frac{x^{3m}}{m!}+a_1\sum_{m=0}^\infty\frac{6^m m!}{\prod_{j=0}^{m-1}(3j+4)}x^{3m+1}$

35 $y =$ \dst$a_0\sum_{m=0}^\infty(-1)^m\frac{3^m m!}{\prod_{j=0}^{m-1}(3j+2)}x^{3m}+a_1\sum_{m=0}^\infty(-1)^m\left[\prod_{j=0}^{m-1}(3j+4)\right]\frac{x^{3m+1}}{3^m m!}$

36 $y =$ \dst$a_0(1-4x^3+4x^6)+a_1\sum_{m=0}^\infty 2^m\left[\prod_{j=0}^{m-1}\frac{3j-5}{3j+4}\right]x^{3m+1}$

37 $y =$ \dst$a_0\left(1+\frac{21}{2}x^3+\frac{42}{5}x^6+\frac{7}{20}x^9\right)+a_1\left(x+4x^4+\frac{10}{7}x^7\right)$

39 $y =$ \dst$a_0\sum_{m=0}^\infty(-2)^m\left[\prod_{j=0}^{m-1}\frac{5j+1}{5j+4}\right]x^{5m}+a_1\sum_{m=0}^\infty\left(-\frac{2}{5}\right)^m\left[\prod_{j=0}^{m-1}(5j+2)\right]\frac{x^{5m+1}}{m!}$

40 $y =$ \dst$a_0\sum_{m=0}^\infty(-1)^m\frac{x^{4m}}{4^m m!\prod_{j=0}^{m-1}(4j+3)}+a_1\sum_{m=0}^\infty(-1)^m\frac{x^{4m+1}}{4^m m!\prod_{j=0}^{m-1}(4j+5)}$

41 $y =$ \dst$a_0\sum_{m=0}^\infty(-1)^m\frac{x^{7m}}{\prod_{j=0}^{m-1}(7j+6)}+a_1\sum_{m=0}^\infty(-1)^m\frac{x^{7m+1}}{7^m m!}$

42 $y =$ \dst$a_0\left(1-\frac{9}{7}x^8\right)+a_1\left(x-\frac{7}{9}x^9\right)$

43 $y =$ \dst$a_0\sum_{m=0}^\infty x^{6m}+a_1\sum_{m=0}^\infty x^{6m+1}$

44 $y =$ \dst$a_0\sum_{m=0}^\infty(-1)^m\frac{x^{6m}}{\prod_{j=0}^{m-1}(6j+5)}+a_1\sum_{m=0}^\infty(-1)^m\frac{x^{6m+1}}{6^m m!}$

7.3 Series Solutions Near an Ordinary Point II

1 $y = \backslash\text{dst}2-3x-2x^2+\frac{7}{2}x^3-\frac{55}{12}x^4+\frac{59}{8}x^5-\frac{83}{6}x^6+\frac{9547}{336}x^7+\cdots$

2 $y = \backslash\text{dst}-1+2x-4x^3+4x^4+4x^5-12x^6+4x^7+\cdots$

3 $y = \backslash\text{dst}1+x^2-\frac{2}{3}x^3+\frac{11}{6}x^4-\frac{9}{5}x^5+\frac{329}{90}x^6-\frac{1301}{315}x^7+\cdots$

4 $y = \backslash\text{dst}x-x^2-\frac{7}{2}x^3+\frac{15}{2}x^4+\frac{45}{8}x^5-\frac{261}{8}x^6+\frac{207}{16}x^7+\cdots$

5 $y = \backslash\text{dst}4+3x-\frac{15}{4}x^2+\frac{1}{4}x^3+\frac{11}{16}x^4-\frac{5}{16}x^5+\frac{1}{20}x^6+\frac{1}{120}x^7+\cdots$

6 $y = \backslash\text{dst}7+3x-\frac{16}{3}x^2+\frac{13}{3}x^3-\frac{23}{9}x^4+\frac{10}{9}x^5-\frac{7}{27}x^6-\frac{1}{9}x^7+\cdots$

7 $y = \backslash\text{dst}2+5x-\frac{7}{4}x^2-\frac{3}{16}x^3+\frac{37}{192}x^4-\frac{7}{192}x^5-\frac{1}{1920}x^6+\frac{19}{11520}x^7+\cdots$

8 $y = \backslash\text{dst}1-(x-1)+\frac{4}{3}(x-1)^3-\frac{4}{3}(x-1)^4-\frac{4}{5}(x-1)^5+\frac{136}{45}(x-1)^6-\frac{104}{63}(x-1)^7+\cdots$

9 y
$= \backslash\text{dst}1-(x+1)+4(x+1)^2-\frac{13}{3}(x+1)^3+\frac{77}{6}(x+1)^4-\frac{278}{15}(x+1)^5+\frac{1942}{45}(x+1)^6-\frac{23332}{315}(x+1)^7+\cdots$

10 $y = \backslash\text{dst}2-(x-1)-\frac{1}{2}(x-1)^2+\frac{5}{3}(x-1)^3-\frac{19}{12}(x-1)^4+\frac{7}{30}(x-1)^5+\frac{59}{45}(x-1)^6-\frac{1091}{630}(x-1)^7+\cdots$

11 $y = \backslash\text{dst}-2+3(x+1)-\frac{1}{2}(x+1)^2-\frac{2}{3}(x+1)^3+\frac{5}{8}(x+1)^4-\frac{11}{30}(x+1)^5+\frac{29}{144}(x+1)^6-\frac{101}{840}(x+1)^7+\cdots$

12 $y = \backslash\text{dst}1-2(x-1)-3(x-1)^2+8(x-1)^3-4(x-1)^4-\frac{42}{5}(x-1)^5+19(x-1)^6-\frac{604}{35}(x-1)^7+\cdots$

19 $y = \backslash\text{dst}2-7x-4x^2-\frac{17}{6}x^3-\frac{3}{4}x^4-\frac{9}{40}x^5+\cdots$

20 $y = \backslash\text{dst}1-2(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3+\frac{5}{36}(x-1)^4-\frac{73}{1080}(x-1)^5+\cdots$

21 $y = \backslash\text{dst}2-(x+2)-\frac{7}{2}(x+2)^2+\frac{4}{3}(x+2)^3-\frac{1}{24}(x+2)^4+\frac{1}{60}(x+2)^5+\cdots$

22 $y = \backslash\text{dst}2-2(x+3)-(x+3)^2+(x+3)^3-\frac{11}{12}(x+3)^4+\frac{67}{60}(x+3)^5+\cdots$

23 $y = \backslash\text{dst}-1+2x+\frac{1}{3}x^3-\frac{5}{12}x^4+\frac{2}{5}x^5+\cdots$

24 $y = \backslash\text{dst}2-3(x+1)+\frac{7}{2}(x+1)^2-5(x+1)^3+\frac{197}{24}(x+1)^4-\frac{287}{20}(x+1)^5+\cdots$

25 $y = \backslash\text{dst}-2+3(x+2)-\frac{9}{2}(x+2)^2+\frac{11}{6}(x+2)^3+\frac{5}{24}(x+2)^4+\frac{7}{20}(x+2)^5+\cdots$

26 $y = \backslash\text{dst}2-4(x-2)-\frac{1}{2}(x-2)^2+\frac{2}{9}(x-2)^3+\frac{49}{432}(x-2)^4+\frac{23}{1080}(x-2)^5+\cdots$

27 $y =$ \dst$1 + 2(x+4) - \frac{1}{6}(x+4)^2 - \frac{10}{27}(x+4)^3 + \frac{19}{648}(x+4)^4 + \frac{13}{324}(x+4)^5 + \cdots$

28 $y =$ \dst$-1 + 2(x+1) - \frac{1}{4}(x+1)^2 + \frac{1}{2}(x+1)^3 - \frac{65}{96}(x+1)^4 + \frac{67}{80}(x+1)^5 + \cdots$

31 **(a)** $y =$ \dst$\frac{c_1}{1+x} + \frac{c_2}{1+2x}$ **(b)** $y =$ \dst$\frac{c_1}{1-2x} + \frac{c_2}{1-3x}$ **(c)** $y =$ \dst$\frac{c_1}{1-2x} + \frac{c_2 x}{(1-2x)^2}$

(d) y
$=$ \dst$\frac{c_1}{2+x} + \frac{c_2 x}{(2+x)^2}$
(e) y
$=$ \dst$\frac{c_1}{2+x} + \frac{c_2}{2+3x}$

32 $y =$ \dst$1 - 2x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{17}{24}x^4 - \frac{11}{20}x^5 + \cdots$

33 $y =$ \dst$1 - 2x - \frac{5}{2}x^2 + \frac{2}{3}x^3 - \frac{3}{8}x^4 + \frac{1}{3}x^5 + \cdots$

34 $y =$ \dst$6 - 2x + 9x^2 + \frac{2}{3}x^3 - \frac{23}{4}x^4 - \frac{3}{10}x^5 + \cdots$

35 $y =$ \dst$2 - 5x + 2x^2 - \frac{10}{3}x^3 + \frac{3}{2}x^4 - \frac{25}{12}x^5 + \cdots$

36 $y =$ \dst$3 + 6x - 3x^2 + x^3 - 2x^4 - \frac{17}{20}x^5 + \cdots$

37 $y =$ \dst$3 - 2x - 3x^2 + \frac{3}{2}x^3 + \frac{3}{2}x^4 - \frac{49}{80}x^5 + \cdots$

38 $y =$ \dst$-2 + 3x + \frac{4}{3}x^2 - x^3 - \frac{19}{54}x^4 + \frac{13}{60}x^5 + \cdots$

39 \dst$y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m!} = e^{-x^2}, \quad y_2 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{m!} = xe^{-x^2}$

40 $y =$ \dst$-2 + 3x + x^2 - \frac{1}{6}x^3 - \frac{3}{4}x^4 + \frac{31}{120}x^5 + \cdots$

41 $y =$ \dst$2 + 3x - \frac{7}{2}x^2 - \frac{5}{6}x^3 + \frac{41}{24}x^4 + \frac{41}{120}x^5 + \cdots$

42 $y =$ \dst$-3 + 5x - 5x^2 + \frac{23}{6}x^3 - \frac{23}{12}x^4 + \frac{11}{30}x^5 + \cdots$

43 $y =$ \dst$-2 + 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{17}{12}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{8}(x-1)^5 + \cdots$

44 $y =$ \dst$2 - 3(x+2) + \frac{1}{2}(x+2)^2 - \frac{1}{3}(x+2)^3 + \frac{31}{24}(x+2)^4 - \frac{53}{120}(x+2)^5 + \cdots$

45 $y =$ \dst$1 - 2x + \frac{3}{2}x^2 - \frac{11}{6}x^3 + \frac{15}{8}x^4 - \frac{71}{60}x^5 + \cdots$

46 $y =$ \dst$2 - (x+2) - \frac{7}{2}(x+2)^2 - \frac{43}{6}(x+2)^3 - \frac{203}{24}(x+2)^4 - \frac{167}{30}(x+2)^5 + \cdots$

47 $y = \backslash\mathrm{dst}2 - x - x^2 + \frac{7}{6}x^3 - x^4 + \frac{89}{120}x^5 + \cdots$

48 $y = \backslash\mathrm{dst}1 + \frac{3}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^5 + \cdots$

49 $y = \backslash\mathrm{dst}1 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{6}(x-3)^3 + \frac{1}{4}(x-3)^4 - \frac{1}{6}(x-3)^5 + \cdots$

7.4 Regular Singular Points: Euler Equations

1 $y = c_1x^{-4} + c_2x^{-2}$

2 $y = c_1x + c_2x^7$

3 $y = x(c_1 + c_2\ln x)$

4 $y = x^{-2}(c_1 + c_2\ln x)$

5 $y = c_1\cos(\ln x) + c_2\sin(\ln x)$

6 $y = x^2[c_1\cos(3\ln x) + c_2\sin(3\ln x)]$

7 $y = \backslash\mathrm{dst}c_1x + \frac{c_2}{x^3}$

8 $y = c_1x^{2/3} + c_2x^{3/4}$

9 $y = x^{-1/2}(c_1 + c_2\ln x)$

10 $y = c_1x + c_2x^{1/3}$

11 $y = c_1x^2 + c_2x^{1/2}$

12 $y = \backslash\mathrm{dst}\frac{1}{x}\left[c_1\cos(2\ln x) + c_2\sin(2\ln x\right]$

13 $y = x^{-1/3}(c_1 + c_2\ln x)$

14 $y = x\left[c_1\cos(3\ln x) + c_2\sin(3\ln x)\right]$

15 $y = \backslash\mathrm{dst}c_1x^3 + \frac{c_2}{x^2}$

16 $y = \backslash\mathrm{dst}\frac{c_1}{x} + c_2x^{1/2}$

17 $y = x^2(c_1 + c_2\ln x)$

18 $y = \backslash\mathrm{dst}\frac{1}{x^2}\left[c_1\cos\left(\frac{1}{\sqrt{2}}\ln x\right) + c_2\sin\left(\frac{1}{\sqrt{2}}\ln x\right)\right]$

7.5 The Method of Frobenius I

1 $y_1 =$ \dst$x^{1/2}\left(1-\frac{1}{5}x-\frac{2}{35}x^2+\frac{31}{315}x^3+\cdots\right)$ $y_2 =$ \dst$x^{-1}\left(1+x+\frac{1}{2}x^2-\frac{1}{6}x^3+\cdots\right)$;

2 $y_1 =$ \dst$x^{1/3}\left(1-\frac{2}{3}x+\frac{8}{9}x^2-\frac{40}{81}x^3+\cdots\right)$; $y_2 =$ \dst$1-x+\frac{6}{5}x^2-\frac{4}{5}x^3+\cdots$

3 $y_1 =$ \dst$x^{1/3}\left(1-\frac{4}{7}x-\frac{7}{45}x^2+\frac{970}{2457}x^3+\cdots\right)$; $y_2 =$ \dst$x^{-1}\left(1-x^2+\frac{2}{3}x^3+\cdots\right)$

4 $y_1 =$ \dst$x^{1/4}\left(1-\frac{1}{2}x-\frac{19}{104}x^2+\frac{1571}{10608}x^3+\cdots\right)$; $y_2 =$ \dst$x^{-1}\left(1+2x-\frac{11}{6}x^2-\frac{1}{7}x^3+\cdots\right)$

5 $y_1 =$ \dst$x^{1/3}\left(1-x+\frac{28}{31}x^2-\frac{1111}{1333}x^3+\cdots\right)$; $y_2 =$ \dst$x^{-1/4}\left(1-x+\frac{7}{8}x^2-\frac{19}{24}x^3+\cdots\right)$;

6 $y_1 =$ \dst$x^{1/5}\left(1-\frac{6}{25}x-\frac{1217}{625}x^2+\frac{41972}{46875}x^3+\cdots\right)$; $y_2 =$ \dst$x-\frac{1}{4}x^2-\frac{35}{18}x^3+\frac{11}{12}x^4+\cdots$

7 $y_1 =$ \dst$x^{3/2}\left(1-x+\frac{11}{26}x^2-\frac{109}{1326}x^3+\cdots\right)$; $y_2 =$ \dst$x^{1/4}\left(1+4x-\frac{131}{24}x^2+\frac{39}{14}x^3+\cdots\right)$

8 $y_1 =$ \dst$x^{1/3}\left(1-\frac{1}{3}x+\frac{2}{15}x^2-\frac{5}{63}x^3+\cdots\right)$; $y_2 =$ \dst$x^{-1/6}\left(1-\frac{1}{12}x^2+\frac{1}{18}x^3+\cdots\right)$

9 $y_1 =$ \dst$1-\frac{1}{14}x^2+\frac{1}{105}x^3+\cdots$; $y_2 =$ \dst$x^{-1/3}\left(1-\frac{1}{18}x-\frac{71}{405}x^2+\frac{719}{34992}x^3+\cdots\right)$

10 $y_1 =$ \dst$x^{1/5}\left(1+\frac{3}{17}x-\frac{7}{153}x^2-\frac{547}{5661}x^3+\cdots\right)$; $y_2 =$ \dst$x^{-1/2}\left(1+x+\frac{14}{13}x^2-\frac{556}{897}x^3+\cdots\right)$

14 $y_1 =$ \dst$x^{1/2}\sum_{n=0}^\infty \frac{(-2)^n}{\prod_{j=1}^n(2j+3)}x^n$; $y_2 =$ \dst$x^{-1}\sum_{n=0}^\infty \frac{(-1)^n}{n!}x^n$

15 $y_1 =$ \dst$x^{1/3}\sum_{n=0}^\infty \frac{(-1)^n\prod_{j=1}^n(3j+1)}{9^n n!}x^n$; x^{-1}

16 $y_1 =$ \dst$x^{1/2}\sum_{n=0}^\infty \frac{(-1)^n}{2^n n!}x^n$; $y_2 =$ \dst$\frac{1}{x^2}\sum_{n=0}^\infty \frac{(-1)^n}{\prod_{j=1}^n(2j-5)}x^n$

17 $y_1 =$ \dst$x\sum_{n=0}^\infty \frac{(-1)^n}{\prod_{j=1}^n(3j+4)}x^n$; $y_2 =$ \dst$x^{-1/3}\sum_{n=0}^\infty \frac{(-1)^n}{3^n n!}x^n$

18 $y_1 =$ \dst$x\sum_{n=0}^\infty \frac{2^n}{n!\prod_{j=1}^n(2j+1)}x^n$; $y_2 =$ \dst$x^{1/2}\sum_{n=0}^\infty \frac{2^n}{n!\prod_{j=1}^n(2j-1)}x^n$

19 $y_1 =$ \dst$x^{1/3}\sum_{n=0}^\infty \frac{1}{n!\prod_{j=1}^n(3j+2)}x^n$; $y_2 =$ \dst$x^{-1/3}\sum_{n=0}^\infty \frac{1}{n!\prod_{j=1}^n(3j-2)}x^n$

20 $y_1 =$ \dst$x\left(1+\frac{2}{7}x+\frac{1}{70}x^2\right)$; $y_2 =$ \dst$x^{-1/3}\sum_{n=0}^\infty \frac{(-1)^n}{3^n n!}\left(\prod_{j=1}^n \frac{3j-13}{3j-4}\right)x^n$

21 $y_1 =$ \dst$x^{1/2}\sum_{n=0}^\infty (-1)^n\left(\prod_{j=1}^n \frac{2j+1}{6j+1}\right);x^n$ $y_2 =$ \dst$x^{1/3}\sum_{n=0}^\infty \frac{(-1)^n}{9^n n!}\left(\prod_{j=1}^n(3j+1)\right)x^n$

22 $y_1 =$ \dst$x\sum_{n=0}^\infty \frac{(-1)^n(n+2)!}{2\prod_{j=1}^n(4j+3)};x^n$ $y_2 =$ \dst$x^{1/4}\sum_{n=0}^\infty \frac{(-1)^n}{16^n n!}\prod_{j=1}^n(4j+5)x^n$

23 $y_1 =$ \dst$x^{-1/2}\sum_{n=0}^\infty \frac{(-1)^n}{n!\prod_{j=1}^n(2j+1)}x^n$; $y_2 =$ \dst$x^{-1}\sum_{n=0}^\infty \frac{(-1)^n}{n!\prod_{j=1}^n(2j-1)}x^n$

24 $y_1 = \backslash\mathrm{dst} x^{1/3}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\left(\frac{2}{9}\right)^n\left(\prod_{j=1}^n(6j+5)\right)x^n$; $y_2 = \backslash\mathrm{dst} x^{-1}\sum_{n=0}^\infty (-1)^n 2^n\left(\prod_{j=1}^n \frac{2j-1}{3j-4}\right)x^n$

25 $y_1 = 4\backslash\mathrm{dst} x^{1/3}\sum_{n=0}^\infty \frac{1}{6^n n!(3n+4)}x^n$; x^{-1}

28 $y_1 = \backslash\mathrm{dst} x^{1/2}\left(1 - \frac{9}{40}x + \frac{5}{128}x^2 - \frac{245}{39936}x^3 + \cdots\right)$; $y_2 = \backslash\mathrm{dst} x^{1/4}\left(1 - \frac{25}{96}x + \frac{675}{14336}x^2 - \frac{38025}{5046272}x^3 + \cdots\right)$

29 $y_1 = \backslash\mathrm{dst} x^{1/3}\left(1 + \frac{32}{117}x - \frac{28}{1053}x^2 + \frac{4480}{540189}x^3 + \cdots\right)$; $y_2 = \backslash\mathrm{dst} x^{-3}\left(1 + \frac{32}{7}x + \frac{48}{7}x^2\right)$

30 $y_1 = \backslash\mathrm{dst} x^{1/2}\left(1 - \frac{5}{8}x + \frac{55}{96}x^2 - \frac{935}{1536}x^3 + \cdots\right)$; $y_2 = \backslash\mathrm{dst} x^{-1/2}\left(1 + \frac{1}{4}x - \frac{5}{32}x^2 - \frac{55}{384}x^3 + \cdots\right)$.

31 $y_1 = \backslash\mathrm{dst} x^{1/2}\left(1 - \frac{3}{4}x + \frac{5}{96}x^2 + \frac{5}{4224}x^3 + \cdots\right)$; $y_2 = \backslash\mathrm{dst} x^{-2}\left(1 + 8x + 60x^2 - 160x^3 + \cdots\right)$

32 $y_1 = \backslash\mathrm{dst} x^{-1/3}\left(1 - \frac{10}{63}x + \frac{200}{7371}x^2 - \frac{17600}{3781323}x^3; + \cdots\right)$; $y_2 = \backslash\mathrm{dst} x^{-1/2}\left(1 - \frac{3}{20}x + \frac{9}{352}x^2 - \frac{105}{23936}x^3 + \cdots\right)$

33 $y_1 = \backslash\mathrm{dst} x^{1/2}\sum_{m=0}^\infty \frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^m \frac{4j-3}{8j+1}\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{1/4}\sum_{m=0}^\infty \frac{(-1)^m}{16^m m!}\left(\prod_{j=1}^m \frac{8j-7}{8j-1}\right)x^{2m}$

34 $y_1 = \backslash\mathrm{dst} x^{1/2}\sum_{m=0}^\infty \left(\prod_{j=1}^m \frac{8j-3}{8j+1}\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{1/4}\sum_{m=0}^\infty \frac{1}{2^m m!}\left(\prod_{j=1}^m(2j-1)\right)x^{2m}$

35 $y_1 = \backslash\mathrm{dst} x^4\sum_{m=0}^\infty (-1)^m(m+1)x^{2m}$; $y_2 = -\backslash\mathrm{dst} x\sum_{m=0}^\infty (-1)^m(2m-1)x^{2m}$

36 $y_1 = \backslash\mathrm{dst} x^{1/3}\sum_{m=0}^\infty \frac{(-1)^m}{18^m m!}\left(\prod_{j=1}^m(6j-17)\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} 1 + \frac{4}{5}x^2 + \frac{8}{55}x^4$

37 $y_1 = \backslash\mathrm{dst} x^{1/4}\sum_{m=0}^\infty \left(\prod_{j=1}^m \frac{8j+1}{8j+5}\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{-1}\sum_{m=0}^\infty \frac{\prod_{j=1}^m(2j-1)}{2^m m!}x^{2m}$

38 $y_1 = \backslash\mathrm{dst} x^{1/2}\sum_{m=0}^\infty \frac{1}{8^m m!}\left(\prod_{j=1}^m(4j-1)\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{1/3}\sum_{m=0}^\infty 2^m\left(\prod_{j=1}^m \frac{3j-1}{12j-1}\right)x^{2m}$

39 $y_1 = \backslash\mathrm{dst} x^{7/2}\sum_{m=0}^\infty (-1)^m\frac{\prod_{j=1}^m(4j+5)}{8^m m!}x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{1/2}\sum_{m=0}^\infty \frac{(-1)^m}{4^m}\left(\prod_{j=1}^m \frac{4j-1}{2j-3}\right)x^{2m}$

40 $y_1 = \backslash\mathrm{dst} x^{1/2}\sum_{m=0}^\infty \frac{(-1)^m}{4^m}\left(\prod_{j=1}^m \frac{4j-1}{2j+1}\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{-1/2}\sum_{m=0}^\infty \frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^m(4j-3)\right)x^{2m}$

41 $y_1 = \backslash\mathrm{dst} x^{1/2}\sum_{m=0}^\infty \frac{(-1)^m}{m!}\left(\prod_{j=1}^m(2j+1)\right)x^{2m}$; $y_2 = \backslash\mathrm{dst}\frac{1}{x^2}\sum_{m=0}^\infty (-2)^m\left(\prod_{j=1}^m \frac{4j-3}{4j-5}\right)x^{2m}$

42 $y_1 = \backslash\mathrm{dst} x^{1/3}\sum_{m=0}^\infty (-1)^m\left(\prod_{j=1}^m \frac{3j-4}{3j+2}\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{-1}(1+x^2)$

43 $y_1 = \backslash\mathrm{dst}\sum_{m=0}^\infty (-1)^m\frac{2^m(m+1)!}{\prod_{j=1}^m(2j+3)}x^{2m}$; $y_2 = \backslash\mathrm{dst}\frac{1}{x^3}\sum_{m=0}^\infty (-1)^m\frac{\prod_{j=1}^m(2j-1)}{2^m m!}x^{2m}$

44 $y_1 = \backslash\mathrm{dst} x^{1/2}\sum_{m=0}^\infty \frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^m \frac{(4j-3)^2}{4j+3}\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{-1}\sum_{m=0}^\infty \frac{(-1)^m}{2^m m!}\left(\prod_{j=1}^m \frac{(2j-3)^2}{4j-3}\right)x^{2m}$

45 $y_1 = \backslash\mathrm{dst} x\sum_{m=0}^\infty (-2)^m\left(\prod_{j=1}^m \frac{2j+1}{4j+5}\right)x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{-3/2}\sum_{m=0}^\infty \frac{(-1)^m}{4^m m!}\left(\prod_{j=1}^m(4j-3)\right)x^{2m}$

46 $y_1 = \backslash\mathrm{dst} x^{1/3}\sum_{m=0}^\infty \frac{(-1)^m}{2^m\prod_{j=1}^m(3j+1)}x^{2m}$; $y_2 = \backslash\mathrm{dst} x^{-1/3}\sum_{m=0}^\infty \frac{(-1)^m}{6^m m!}x^{2m}$

47 $y_1 = \backslash\mathrm{dst}x^{1/2}\left(1-\frac{6}{13}x^2+\frac{36}{325}x^4-\frac{216}{12025}x^6+\cdots\right);\quad y_2 = \backslash\mathrm{dst}x^{1/3}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{1}{48}x^6+\cdots\right)$

48 $y_1 = \backslash\mathrm{dst}x^{1/4}\left(1-\frac{13}{64}x^2+\frac{273}{8192}x^4-\frac{2639}{524288}x^6+\cdots\right);\quad y_2 = \backslash\mathrm{dst}x^{-1}\left(1-\frac{1}{3}x^2+\frac{2}{33}x^4-\frac{2}{209}x^6+\cdots\right)$

49 $y_1 = \backslash\mathrm{dst}x^{1/3}\left(1-\frac{3}{4}x^2+\frac{9}{14}x^4-\frac{81}{140}x^6+\cdots\right);\quad y_2 = \backslash\mathrm{dst}x^{-1/3}\left(1-\frac{2}{3}x^2+\frac{5}{9}x^4-\frac{40}{81}x^6+\cdots\right)$

50 $y_1 = \backslash\mathrm{dst}x^{1/2}\left(1-\frac{3}{2}x^2+\frac{15}{8}x^4-\frac{35}{16}x^6+\cdots\right);\quad y_2 = \backslash\mathrm{dst}x^{-1/2}\left(1-2x^2+\frac{8}{3}x^4-\frac{16}{5}x^6+\cdots\right)$

51 $y_1 = \backslash\mathrm{dst}x^{1/4}\left(1-x^2+\frac{3}{2}x^4-\frac{5}{2}x^6+\cdots\right);\quad y_2 = \backslash\mathrm{dst}x^{-1/2}\left(1-\frac{2}{5}x^2+\frac{36}{65}x^4-\frac{408}{455}x^6+\cdots\right)$

53 **(a)** $y_1 = \backslash\mathrm{dst}x^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j+\nu)}x^{2m};\ y_2 = \backslash\mathrm{dst}x^{-\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j-\nu)}x^{2m}$

y_1
$= \backslash\mathrm{dst}\frac{\sin x}{\sqrt{x}};$
y_2
$= \backslash\mathrm{dst}\frac{\cos x}{\sqrt{x}}$

61 $y_1 = \backslash\mathrm{dst}\frac{x^{1/2}}{1+x};\ y_2 = \backslash\mathrm{dst}\frac{x}{1+x}$

62 $y_1 = \backslash\mathrm{dst}\frac{x^{1/3}}{1+2x^2};\ y_2 = \backslash\mathrm{dst}\frac{x^{1/2}}{1+2x^2}$

63 $y_1 = \backslash\mathrm{dst}\frac{x^{1/4}}{1-3x};\ y_2 = \backslash\mathrm{dst}\frac{x^2}{1-3x}$

64 $y_1 = \backslash\mathrm{dst}\frac{x^{1/3}}{5+x};\ y_2 = \backslash\mathrm{dst}\frac{x^{-1/3}}{5+x}$

65 $y_1 = \backslash\mathrm{dst}\frac{x^{1/4}}{2-x^2};\ y_2 = \backslash\mathrm{dst}\frac{x^{-1/2}}{2-x^2}$

66 $y_1 = \backslash\mathrm{dst}\frac{x^{1/2}}{1+3x+x^2};\ y_2 = \backslash\mathrm{dst}\frac{x^{3/2}}{1+3x+x^2}$

67 $y_1 = \backslash\mathrm{dst}\frac{x}{(1+x)^2};\ y_2 = \backslash\mathrm{dst}\frac{x^{1/3}}{(1+x)^2}$

68 $y_1 = \backslash\mathrm{dst}\frac{x}{3+2x+x^2};\ y_2 = \backslash\mathrm{dst}\frac{x^{1/4}}{3+2x+x^2}$

7.6 The Method of Frobenius II

1 \dst$y_1 = x\left(1 - x + \frac{3}{4}x^2 - \frac{13}{36}x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + x^2\left(1 - x + \frac{65}{108}x^2 + \cdots\right)$

2 \dst$y_1 = x^{-1}\left(1 - 2x + \frac{9}{2}x^2 - \frac{20}{3}x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + 1 - \frac{15}{4}x + \frac{133}{18}x^2 + \cdots$

3 \dst$y_1 = 1 + x - x^2 + \frac{1}{3}x^3 + \cdots$; \dst$y_2 = y_1 \ln x - x\left(3 - \frac{1}{2}x - \frac{31}{18}x^2 + \cdots\right)$

4 \dst$y_1 = x^{1/2}\left(1 - 2x + \frac{5}{2}x^2 - 2x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + x^{3/2}\left(1 - \frac{9}{4}x + \frac{17}{6}x^2 + \cdots\right)$

5 \dst$y_1 = x\left(1 - 4x + \frac{19}{2}x^2 - \frac{49}{3}x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + x^2\left(3 - \frac{43}{4}x + \frac{208}{9}x^2 + \cdots\right)$

6 \dst$y_1 = x^{-1/3}\left(1 - x + \frac{5}{6}x^2 - \frac{1}{2}x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + x^{2/3}\left(1 - \frac{11}{12}x + \frac{25}{36}x^2 + \cdots\right)$

7 \dst$y_1 = 1 - 2x + \frac{7}{4}x^2 - \frac{7}{9}x^3 + \cdots$; \dst$y_2 = y_1 \ln x + x\left(3 - \frac{15}{4}x + \frac{239}{108}x^2 + \cdots\right)$

8 \dst$y_1 = x^{-2}\left(1 - 2x + \frac{5}{2}x^2 - 3x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + \frac{3}{4} - \frac{13}{6}x + \cdots$

9 \dst$y_1 = x^{-1/2}\left(1 - x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + x^{1/2}\left(\frac{3}{2} - \frac{13}{16}x + \frac{1}{54}x^2 + \cdots\right)$

10 \dst$y_1 = x^{-1/4}\left(1 - \frac{1}{4}x - \frac{7}{32}x^2 + \frac{23}{384}x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x + x^{3/4}\left(\frac{1}{4} + \frac{5}{64}x - \frac{157}{2304}x^2 + \cdots\right)$

11 \dst$y_1 = x^{-1/3}\left(1 - x + \frac{7}{6}x^2 - \frac{23}{18}x^3 + \cdots\right)$; \dst$y_2 = y_1 \ln x - x^{5/3}\left(\frac{1}{12} - \frac{13}{108}x \cdots\right)$

12 \dst$y_1 = x^{1/2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}x^n$; \dst$y_2 = y_1 \ln x - 2x^{1/2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\sum_{j=1}^{n} \frac{1}{j}\right)x^n$;

13 \dst$y_1 = x^{1/6}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \frac{\prod_{j=1}^{n}(3j+1)}{n!}x^n$;

\dst$y_2 = y_1 \ln x - x^{1/6}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n \frac{\prod_{j=1}^{n}(3j+1)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(3j+1)}\right)x^n$

14 \dst$y_1 = x^2\sum_{n=0}^{\infty}(-1)^n(n+1)^2x^n$; \dst$y_2 = y_1 \ln x - 2x^2\sum_{n=1}^{\infty}(-1)^n n(n+1)x^n$

15 \dst$y_1 = x^3\sum_{n=0}^{\infty}2^n(n+1)x^n$; \dst$y_2 = y_1 \ln x - x^3\sum_{n=1}^{\infty}2^n n x^n$

16 \dst$y_1 = x^{1/5}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(5j+1)}{125^n(n!)^2}x^n$;

\dst$y_2 = y_1 \ln x - x^{1/5}\sum_{n=1}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(5j+1)}{125^n(n!)^2}\left(\sum_{j=1}^{n} \frac{5j+2}{j(5j+1)}\right)x^n$

17 \dst$y_1 = x^{1/2}\sum_{n=0}^\infty \frac{(-1)^n\prod_{j=1}^n(2j-3)}{4^n n!}x^n$;

\dst$y_2 = y_1\ln x + 3x^{1/2}\sum_{n=1}^\infty \frac{(-1)^n\prod_{j=1}^n(2j-3)}{4^n n!}\left(\sum_{j=1}^n \frac{1}{j(2j-3)}\right)x^n$

18 \dst$y_1 = x^{1/3}\sum_{n=0}^\infty \frac{(-1)^n\prod_{j=1}^n(6j-7)^2}{81^n(n!)^2}x^n$;

\dst$y_2 = y_1\ln x + 14x^{1/3}\sum_{n=1}^\infty \frac{(-1)^n\prod_{j=1}^n(6j-7)^2}{81^n(n!)^2}\left(\sum_{j=1}^n \frac{1}{j(6j-7)})\right)x^n$

19 \dst$y_1 = x^2\sum_{n=0}^\infty \frac{(-1)^n\prod_{j=1}^n(2j+5)}{(n!)^2}x^n$;

\dst$y_2 = y_1\ln x - 2x^2\sum_{n=1}^\infty \frac{(-1)^n\prod_{j=1}^n(2j+5)}{(n!)^2}\left(\sum_{j=1}^n \frac{(j+5)}{j(2j+5)}\right)x^n$

20 \dst$y_1 = \frac{1}{x}\sum_{n=0}^\infty \frac{2^n\prod_{j=1}^n(2j-1)}{n!}x^n$;

\dst$y_2 = y_1\ln x + \frac{1}{x}\sum_{n=1}^\infty \frac{2^n\prod_{j=1}^n(2j-1)}{n!}\left(\sum_{j=1}^n \frac{1}{j(2j-1)}\right)x^n$

21 \dst$y_1 = \frac{1}{x}\sum_{n=0}^\infty \frac{(-1)^n\prod_{j=1}^n(2j-5)}{n!}x^n$;

\dst$y_2 = y_1\ln x + \frac{5}{x}\sum_{n=1}^\infty \frac{(-1)^n\prod_{j=1}^n(2j-5)}{n!}\left(\sum_{j=1}^n \frac{1}{j(2j-5)}\right)x^n$

22 \dst$y_1 = x^2\sum_{n=0}^\infty \frac{(-1)^n\prod_{j=1}^n(2j+3)}{2^n n!}x^n$;

\dst$y_2 = y_1\ln x - 3x^2\sum_{n=0}^\infty \frac{(-1)^n\prod_{j=1}^n(2j+3)}{2^n n!}\left(\sum_{j=1}^n \frac{1}{j(2j+3)}\right)x^n$

23 \dst$y_1 = x^{-2}\left(1+3x+\frac{3}{2}x^2-\frac{1}{2}x^3+\cdots\right)$; \dst$y_2 = y_1\ln x - 5x^{-1}\left(1+\frac{5}{4}x-\frac{1}{4}x^2+\cdots\right)$

24 \dst$y_1 = x^3(1+20x+180x^2+1120x^3+\cdots$; \dst$y_2 = y_1\ln x - x^4\left(26+324x+\frac{6968}{3}x^2+\cdots\right)$

25 \dst$y_1 = x\left(1-5x+\frac{85}{4}x^2-\frac{3145}{36}x^3+\cdots\right)$; \dst$y_2 = y_1\ln x + x^2\left(2-\frac{39}{4}x+\frac{4499}{108}x^2+\cdots\right)$

26 \dst$y_1 = 1-x+\frac{3}{4}x^2-\frac{7}{12}x^3+\cdots$; \dst$y_2 = y_1\ln x + x\left(1-\frac{3}{4}x+\frac{5}{9}x^2+\cdots\right)$

27 \dst$y_1 = x^{-3}(1+16x+36x^2+16x^3+\cdots)$; \dst$y_2 = y_1\ln x - x^{-2}\left(40+150x+\frac{280}{3}x^2+\cdots\right)$

28 \dst$y_1 = x\sum_{m=0}^\infty \frac{(-1)^m}{2^m m!}x^{2m}$; \dst$y_2 = y_1\ln x - \frac{x}{2}\sum_{m=1}^\infty \frac{(-1)^m}{2^m m!}\left(\sum_{j=1}^m \frac{1}{j}\right)x^{2m}$

29 \dst$y_1 = x^2 \sum_{m=0}^\infty (-1)^m (m+1) x^{2m}$; \dst$y_2 = y_1 \ln x - \frac{x^2}{2} \sum_{m=1}^\infty (-1)^m m x^{2m}$

30 \dst$y_1 = x^{1/2} \sum_{m=0}^\infty \frac{(-1)^m}{4^m m!} x^{2m}$; \dst$y_2 = y_1 \ln x - \frac{x^{1/2}}{2} \sum_{m=1}^\infty \frac{(-1)^m}{4^m m!} \left(\sum_{j=1}^m \frac{1}{j}\right) x^{2m}$

31 \dst$y_1 = x \sum_{m=0}^\infty \frac{(-1)^m \prod_{j=1}^m (2j-1)}{2^m m!} x^{2m}$;

\dst$y_2 = y_1 \ln x + \frac{x}{2} \sum_{m=1}^\infty \frac{(-1)^m \prod_{j=1}^m (2j-1)}{2^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j-1)}\right) x^{2m}$

32 \dst$y_1 = x^{1/2} \sum_{m=0}^\infty \frac{(-1)^m \prod_{j=1}^m (4j-1)}{8^m m!} x^{2m}$;

\dst$y_2 = y_1 \ln x + \frac{x^{1/2}}{2} \sum_{m=1}^\infty \frac{(-1)^m \prod_{j=1}^m (4j-1)}{8^m m!} \left(\sum_{j=1}^m \frac{1}{j(4j-1)}\right) x^{2m}$

33 \dst$y_1 = x \sum_{m=0}^\infty \frac{(-1)^m \prod_{j=1}^m (2j+1)}{2^m m!} x^{2m}$;

\dst$y_2 = y_1 \ln x - \frac{x}{2} \sum_{m=1}^\infty \frac{(-1)^m \prod_{j=1}^m (2j+1)}{2^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j+1)}\right) x^{2m}$

34 \dst$y_1 = x^{-1/4} \sum_{m=0}^\infty \frac{(-1)^m \prod_{j=1}^m (8j-13)}{(32)^m m!} x^{2m}$;

\dst$y_2 = y_1 \ln x + \frac{13}{2} x^{-1/4} \sum_{m=1}^\infty \frac{(-1)^m \prod_{j=1}^m (8j-13)}{(32)^m m!} \left(\sum_{j=1}^m \frac{1}{j(8j-13)}\right) x^{2m}$

35 \dst$y_1 = x^{1/3} \sum_{m=0}^\infty \frac{(-1)^m \prod_{j=1}^m (3j-1)}{9^m m!} x^{2m}$;

\dst$y_2 = y_1 \ln x + \frac{x^{1/3}}{2} \sum_{m=1}^\infty \frac{(-1)^m \prod_{j=1}^m (3j-1)}{9^m m!} \left(\sum_{j=1}^m \frac{1}{j(3j-1)}\right) x^{2m}$

36 \dst$y_1 = x^{1/2} \sum_{m=0}^\infty \frac{(-1)^m \prod_{j=1}^m (4j-3)(4j-1)}{4^m (m!)^2} x^{2m}$;

\dst$y_2 = y_1 \ln x + x^{1/2} \sum_{m=1}^\infty \frac{(-1)^m \prod_{j=1}^m (4j-3)(4j-1)}{4^m (m!)^2} \left(\sum_{j=1}^m \frac{8j-3}{j(4j-3)(4j-1)}\right) x^{2m}$

37 \dst$y_1 = x^{5/3} \sum_{m=0}^\infty \frac{(-1)^m}{3^m m!} x^{2m}$; \dst$y_2 = y_2 1 \ln x - \frac{x^{5/3}}{2} \sum_{m=1}^\infty \frac{(-1)^m}{3^m m!} \left(\sum_{j=1}^m \frac{1}{j}\right) x^{2m}$

38 \dst$y_1 = \frac{1}{x} \sum_{m=0}^\infty \frac{(-1)^m \prod_{j=1}^m (4j-7)}{2^m m!} x^{2m}$;

\dst$y_2 = y_1 \ln x + \frac{7}{2x} \sum_{m=1}^\infty \frac{(-1)^m \prod_{j=1}^m (4j-7)}{2^m m!} \left(\sum_{j=1}^m \frac{1}{j(4j-7)}\right) x^{2m}$

39 \dst$y_1 = x^{-1}\left(1-\frac{3}{2}x^2+\frac{15}{8}x^4-\frac{35}{16}x^6+\cdots\right)$

;

\dst$y_2 = y_1\ln x + x\left(\frac{1}{4}-\frac{13}{32}x^2+\frac{101}{192}x^4+\cdots\right)$

40 \dst$y_1 = x\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{1}{48}x^6+\cdots\right)$; \dst$y_2 = y_1\ln x + x^3\left(\frac{1}{4}-\frac{3}{32}x^2+\frac{11}{576}x^4+\cdots\right)$

41 \dst$y_1 = x^{-2}\left(1-\frac{3}{4}x^2-\frac{9}{64}x^4-\frac{25}{256}x^6+\cdots\right)$; \dst$y_2 = y_1\ln x + \frac{1}{2}-\frac{21}{128}x^2-\frac{215}{1536}x^4+\cdots$

42 \dst$y_1 = x^{-3}\left(1-\frac{17}{8}x^2+\frac{85}{256}x^4-\frac{85}{18432}x^6+\cdots\right)$; \dst$y_2 = y_1\ln x + x^{-1}\left(\frac{25}{8}-\frac{471}{512}x^2+\frac{1583}{110592}x^4+\cdots\right)$

43 \dst$y_1 = x^{-1}\left(1-\frac{3}{4}x^2+\frac{45}{64}x^4-\frac{175}{256}x^6+\cdots\right)$; \dst$y_2 = y_1\ln x - x\left(\frac{1}{4}-\frac{33}{128}x^2+\frac{395}{1536}x^4+\cdots\right)$

44 \dst$y_1 = \frac{1}{x}$; \dst$y_2 = y_1\ln x - 6 + 6x - \frac{8}{3}x^2$

45 \dst$y_1 = 1-x$; \dst$y_2 = y_1\ln x + 4x$

46 \dst$y_1 = \frac{(x-1)^2}{x}$; \dst$y_2 = y_1\ln x + 3 - 3x + 2\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}x^n$

47 \dst$y_1 = x^{1/2}(x+1)^2$; \dst$y_2 = y_1\ln x - x^{3/2}\left(3+3x+2\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n^2-1)}x^n\right)$

48 \dst$y_1 = x^2(1-x)^3$; \dst$y_2 = y_1\ln x + x^3\left(4-7x+\frac{11}{3}x^2-6\sum_{n=3}^{\infty}\frac{1}{n(n-2)(n^2-1)}x^n\right)$

49 \dst$y_1 = x-4x^3+x^5$; \dst$y_2 = y_1\ln x + 6x^3 - 3x^5$

50 \dst$y_1 = x^{1/3}\left(1-\frac{1}{6}x^2\right)$; \dst$y_2 = y_1\ln x + x^{7/3}\left(\frac{1}{4}-\frac{1}{12}\sum_{m=1}^{\infty}\frac{1}{6^m m(m+1)(m+1)!}x^{2m}\right)$

51 \dst$y_1 = (1+x^2)^2$; \dst$y_2 = y_1\ln x - \frac{3}{2}x^2 - \frac{3}{2}x^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m(m-1)(m-2)}x^{2m}$

52 \dst$y_1 = x^{-1/2}\left(1-\frac{1}{2}x^2+\frac{1}{32}x^4\right)$; \dst$y_2 = y_1\ln x + x^{3/2}\left(\frac{5}{8}-\frac{9}{128}x^2+\sum_{m=2}^{\infty}\frac{1}{4^{m+1}(m-1)m(m+1)(m+1)!}x^{2m}\right)$.

56 \dst$y_1 = \sum_{m=0}^{\infty}\frac{(-1)^m}{4^m(m!)^2}x^{2m}$; \dst$y_2 = y_1\ln x - \sum_{m=1}^{\infty}\frac{(-1)^m}{4^m(m!)^2}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}$

58 \dst$\frac{x^{1/2}}{1+x}$; \dst$\frac{x^{1/2}\ln x}{1+x}$

59 \dst$\frac{x^{1/3}}{3+x}$; \dst$\frac{x^{1/3}\ln x}{3+x}$

60 \dst$\frac{x}{2-x^2}$; \dst$\frac{x\ln x}{2-x^2}$

61 \dst$\frac{x^{1/4}}{1+x^2}$; \dst$\frac{x^{1/4}\ln x}{1+x^2}$

62 \dst $\frac{x}{4+3x}$; \dst $\frac{x\ln x}{4+3x}$

63 \dst $\frac{x^{1/2}}{1+3x+x^2}$; \dst $\frac{x^{1/2}\ln x}{1+3x+x^2}$

64 \dst $\frac{x}{(1-x)^2}$; \dst $\frac{x\ln x}{(1-x)^2}$

65 \dst $\frac{x^{1/3}}{1+x+x^2}$; \dst $\frac{x^{1/3}\ln x}{1+x+x^2}$

7.7 The Method of Frobenius III

1 \dst$y_1 = 2x^3 \sum_{n=0}^{\infty} \frac{(-4)^n}{n!(n+2)!} x^n;$ \dst$y_2 = x + 4x^2 - 8\left(y_1 \ln x - 4\sum_{n=1}^{\infty} \frac{(-4)^n}{n!(n+2)!}\left(\sum_{j=1}^{n} \frac{j+1}{j(j+2)}\right) x^n\right)$

2 \dst$y_1 = x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n;$ \dst$y_2 = 1 - y_1 \ln x + x\sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\sum_{j=1}^{n} \frac{2j+1}{j(j+1)}\right) x^n$

3 $y_1 = x^{1/2};$ \dst$y_2 = x^{-1/2} + y_1 \ln x + x^{1/2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$

4 \dst$y_1 = x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = xe^{-x};$ \dst$y_2 = 1 - y_1 \ln x + x\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\sum_{j=1}^{n} \frac{1}{j}\right) x^n$

5 \dst$y_1 = x^{1/2}\sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n \frac{\prod_{j=1}^{n}(2j+1)}{n!} x^n;$

\dst$y_2 = x^{-1/2} - \frac{3}{4}\left(y_1 \ln x - x^{1/2}\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^n \frac{\prod_{j=1}^{n}(2j+1)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j+1)}\right) x^n\right)$

6 \dst$y_1 = x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = xe^{-x};$ \dst$y_2 = x^{-2}\left(1 + \frac{1}{2}x + \frac{1}{2}x^2\right) - \frac{1}{2}\left(y_1 \ln x - x\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\sum_{j=1}^{n} \frac{1}{j}\right) x^n\right)$

7 \dst$y_1 = 6x^{3/2}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+3)!} x^n;$

\dst$y_2 = x^{-3/2}\left(1 + \frac{1}{8}x + \frac{1}{64}x^2\right) - \frac{1}{768}\left(y_1 \ln x - 6x^{3/2}\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!(n+3)!}\left(\sum_{j=1}^{n} \frac{2j+3}{j(j+3)}\right) x^n\right)$

8 \dst$y_1 = \frac{120}{x^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+5)!} x^n;$

\dst$y_2 = x^{-7}\left(1 + \frac{1}{4}x + \frac{1}{24}x^2 + \frac{1}{144}x^3 + \frac{1}{576}x^4\right) - \frac{1}{2880}\left(y_1 \ln x - \frac{120}{x^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+5)!}\left(\sum_{j=1}^{n} \frac{2j+5}{j(j+5)}\right) x^n\right)$

9 \dst$y_1 = \frac{x^{1/2}}{6}\sum_{n=0}^{\infty} (-1)^n (n+1)(n+2)(n+3) x^n;$

\dst$y_2 = x^{-5/2}\left(1 + \frac{1}{2}x + x^2\right) - 3y_1 \ln x + \frac{3}{2}x^{1/2}\sum_{n=1}^{\infty} (-1)^n (n+1)(n+2)(n+3)\left(\sum_{j=1}^{n} \frac{1}{j(j+3)}\right) x^n$

10 \dst$y_1 = x^4\left(1 - \frac{2}{5}x\right)$ \dst$y_2 = 1 + 10x + 50x^2 + 200x^3 - 300\left(y_1 \ln x + \frac{27}{25}x^5 - \frac{1}{30}x^6\right)$

11 $y_1 = x^3;\ y_2 =$ \dst$x^{-3}\left(1 - \frac{6}{5}x + \frac{3}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{1}{20}x^5\right) - \frac{1}{120}\left(y_1 \ln x + x^3\sum_{n=1}^{\infty} \frac{(-1)^n 6!}{n(n+6)!} x^n\right)$

12 \dst$y_1 = x^2\sum_{n=0}^{\infty} \frac{1}{n!}\left(\prod_{j=1}^{n} \frac{2j+3}{j+4}\right) x^n;$

\dst$y_2 = x^{-2}\left(1 + x + \frac{1}{4}x^2 - \frac{1}{12}x^3\right) - \frac{1}{16}y_1 \ln x + \frac{x^2}{8}\sum_{n=1}^{\infty} \frac{1}{n!}\left(\prod_{j=1}^{n} \frac{2j+3}{j+4}\right)\left(\sum_{j=1}^{n} \frac{(j^2+3j+6)}{j(j+4)(2j+3)}\right) x^n$

13 $y_1 =$ \dst$x^5 \sum_{n=0}^{\infty} (-1)^n (n+1)(n+2)x^n;\ y_2 =$ \dst$1 - \frac{x}{2} + \frac{x^2}{6}$

14 $y_1 =$ \dst$\frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\prod_{j=1}^{n} \frac{(j+3)(2j-3)}{j+6}\right) x^n;\ y_2 =$ \dst$x^{-7}\left(1 + \frac{26}{5}x + \frac{143}{20}x^2\right)$

15 $y_1 =$ \dst$x^{7/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+4)!} x^n;\ y_2 =$ \dst$x^{-1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3\right)$

16 $y_1 =$ \dst$x^{10/3} \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{9^n} \left(\prod_{j=1}^{n} \frac{3j+7}{j+4}\right) x^n;\ y_2 =$ \dst$x^{-2/3}\left(1 + \frac{4}{27}x - \frac{1}{243}x^2\right)$

17 $y_1 =$ \dst$x^3 \sum_{n=0}^{7} (-1)^n (n+1) \left(\prod_{j=1}^{n} \frac{j-8}{j+6}\right) x^n;\ y_2 =$ \dst$x^{-3}\left(1 + \frac{52}{5}x + \frac{234}{5}x^2 + \frac{572}{5}x^3 + 143x^4\right)$

18 $y_1 =$ \dst$x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\prod_{j=1}^{n} \frac{(j+3)^2}{j+5}\right) x^n; \quad y_2 =$ \dst$x^{-2}\left(1 + \frac{1}{4}x\right)$

19 $y_1 =$ \dst$x^6 \sum_{n=0}^{4} (-1)^n 2^n \left(\prod_{j=1}^{n} \frac{j-5}{j+5}\right) x^n;\ y_2 = x(1 + 18x + 144x^2 + 672x^3 + 2016x^4)$

20 $y_1 =$ \dst$x^6\left(1 + \frac{2}{3}x + \frac{1}{7}x^2\right);\ y_2 =$ \dst$x\left(1 + \frac{21}{4}x + \frac{21}{2}x^2 + \frac{35}{4}x^3\right)$

21 $y_1 =$ \dst$x^{7/2} \sum_{n=0}^{\infty} (-1)^n (n+1)x^n;\ y_2 = x^{-7/2}$\dst$\left(1 - \frac{5}{6}x + \frac{2}{3}x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5\right)$

22 $y_1 =$ \dst$\frac{x^{10}}{6} \sum_{n=0}^{\infty} (-1)^n 2^n (n+1)(n+2)(n+3)x^n;$

y_2
$=$ \dst$\left(1 - \frac{4}{3}x + \frac{5}{3}x^2 - \frac{40}{21}x^3 + \frac{40}{21}x^4 - \frac{32}{21}x^5 + \frac{16}{21}x^6\right)$

23 \dst$y_1 = x^6 \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+5)}{2^m m!} x^{2m};$

\dst$y_2 = x^2\left(1 + \frac{3}{2}x^2\right) - \frac{15}{2}y_1 \ln x + \frac{75}{2}x^6 \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+5)}{2^{m+1}m!} \left(\sum_{j=1}^{m} \frac{1}{j(2j+5)}\right) x^{2m}$

24 \dst$y_1 = x^6 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m} = x^6 e^{-x^2/2};$

\dst$y_2 = x^2\left(1 + \frac{1}{2}x^2\right) - \frac{1}{2}y_1 \ln x + \frac{x^6}{4} \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

25 \dst$y_1 = 6x^6 \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(m+3)!} x^{2m};$

\dst$y_2 = 1 + \frac{1}{8}x^2 + \frac{1}{64}x^4 - \frac{1}{384}\left(y_1 \ln x - 3x^6 \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m!(m+3)!} \left(\sum_{j=1}^{m} \frac{2j+3}{j(j+3)}\right) x^{2m}\right)$

26 \dst$y_1 = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m(m+2)}{m!} x^{2m};$

\dst$y_2 = x^{-1} - 4y_1 \ln x + x \sum_{m=1}^{\infty} \frac{(-1)^m(m+2)}{m!} \left(\sum_{j=1}^{m} \frac{j^2+4j+2}{j(j+1)(j+2)}\right) x^{2m}$

27 \dst $y_1 = 2x^3 \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(m+2)!} x^{2m};$

\dst $y_2 = x^{-1}\left(1 + \frac{1}{4}x^2\right) - \frac{1}{16}\left(y_1 \ln x - 2x^3 \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m!(m+2)!}\left(\sum_{j=1}^{m} \frac{j+1}{j(j+2)}\right) x^{2m}\right)$

28 \dst $y_1 = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{8^m m!(m+1)!} x^{2m};$

\dst $y_2 = x^{-5/2} + \frac{1}{4} y_1 \ln x - x^{-1/2} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{8^{m+1} m!(m+1)!}\left(\sum_{j=1}^{m} \frac{2j^2-2j-1}{j(j+1)(2j-1)}\right) x^{2m}$

29 \dst $y_1 = x \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m} = xe^{-x^2/2};$ \dst $y_2 = x^{-1} - y_1 \ln x + \frac{x}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!}\left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

30 \dst $y_1 = x^2 \sum_{m=0}^{\infty} \frac{1}{m!} x^{2m} = x^2 e^{x^2};$ \dst $y_2 = x^{-2}(1 - x^2) - 2y_1 \ln x + x^2 \sum_{m=1}^{\infty} \frac{1}{m!}\left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

31 \dst $y_1 = 6x^{5/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{16^m m!(m+3)!} x^{2m};$

\dst $y_2 = x^{-7/2}\left(1 + \frac{1}{32}x^2 + \frac{1}{1024}x^4\right) - \frac{1}{24576}\left(y_1 \ln x - 3x^{5/2} \sum_{m=1}^{\infty} \frac{(-1)^m}{16^m m!(m+3)!}\left(\sum_{j=1}^{m} \frac{2j+3}{j(j+3)}\right) x^{2m}\right)$

32 \dst $y_1 = 2x^{13/3} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!} x^{2m};$

\dst $y_2 = x^{1/3}\left(1 + \frac{2}{9}x^2\right) + \frac{2}{81}\left(y_1 \ln x - x^{13/3} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!}\left(\sum_{j=1}^{m} \frac{3j^2+2j+2}{j(j+2)(3j+1)}\right) x^{2m}\right)$

33 \dst $y_1 = x^2;$ \dst $y_2 = x^{-2}(1 + 2x^2) - 2\left(y_1 \ln x + x^2 \sum_{m=1}^{\infty} \frac{1}{m(m+2)!} x^{2m}\right)$

34 \dst $y_1 = x^2\left(1 - \frac{1}{2}x^2\right);$ \dst $y_2 = x^{-2}\left(1 + \frac{9}{2}x^2\right) - \frac{27}{2}\left(y_1 \ln x + \frac{7}{12}x^4 - x^2 \sum_{m=2}^{\infty} \frac{\left(\frac{3}{2}\right)^m}{m(m-1)(m+2)!} x^{2m}\right)$

35 $y_1 =$ \dst $\sum_{m=0}^{\infty} (-1)^m (m+1) x^{2m};$ $y_2 =$ \dst x^{-4}

36 $y_1 =$ \dst $x^{5/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)(m+3)} x^{2m};$ $y_2 =$ \dst $x^{-7/2}(1 + x^2)^2$

37 $y_1 =$ \dst $\frac{x^7}{5} \sum_{m=0}^{\infty} (-1)^m (m+5) x^{2m};$ $y_2 =$ \dst $x^{-1}\left(1 - 2x^2 + 3x^4 - 4x^6\right)$

38 $y_1 =$ \dst $x^3 \sum_{m=0}^{\infty} (-1)^m \frac{m+1}{2^m}\left(\prod_{j=1}^{m} \frac{2j+1}{j+5}\right) x^{2m};$ $y_2 =$ \dst $x^{-7}\left(1 + \frac{21}{8}x^2 + \frac{35}{16}x^4 + \frac{35}{64}x^6\right)$

39 $y_1 =$ \dst $2x^4 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(4j+5)}{2^m (m+2)!} x^{2m};$ $y_2 =$ \dst $1 - \frac{1}{2}x^2$

40 $y_1 =$ \dst $x^{3/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{2^{m-1}(m+2)!} x^{2m};$ $y_2 =$ \dst $x^{-5/2}\left(1 + \frac{3}{2}x^2\right)$

42 \dst$y_1 = x^\nu \sum_{m=0}^\infty \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j+\nu)} x^{2m}$;

\dst$y_2 = x^{-\nu} \sum_{m=0}^{\nu-1} \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j-\nu)} x^{2m} - \frac{2}{4^\nu \nu!(\nu-1)!} \left(y_1 \ln x - \frac{x^\nu}{2} \sum_{m=1}^\infty \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j+\nu)} \left(\sum_{j=1}^m \frac{2j+\nu}{j(j+\nu)} \right) x^{2m} \right)$