6 Applications of Linear Second Order Equations

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IN THIS CHAPTER we study applications of linear second order equations.

SECTIONS 6.1 AND 6.2 is about spring–mass systems.

SECTION 6.2 is about RLC circuits, the electrical analogs of spring–mass systems.

SECTION 6.3 is about motion of an object under a central force, which is particularly relevant in the space age, since, for example, a satellite moving in orbit subject only to Earth's gravity is experiencing motion under a central force.

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Answers to selected exercises in this chapter

6.1 Spring Problems I

We consider the motion of an object of mass m, suspended from a spring of negligible mass. We say that the spring–mass system is in equilibrium when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the equilibrium position. We define y to be the displacement of the object from its equilibrium position (Figure 6.1.1), measured positive upward.

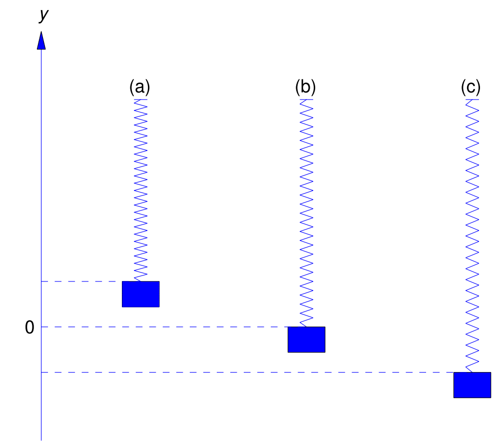

Figure 6.1.1. ***(a)*** $y > 0$ ***(b)*** $y = 0,$ ***(c)*** $y < 0$

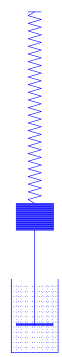

Figure 6.1.2. *A spring – mass system with damping*

Our model accounts for the following kinds of forces acting on the object:

- The force $-mg$, due to gravity.
- A force F_s exerted by the spring resisting change in its length. The natural length of the spring is its length with no mass attached. We assume that the spring obeys Hooke's law (http://www-history.mcs.st-and.ac.uk/Mathematicians/Hooke.html): If the length of the spring is changed by an amount ΔL from its natural length, then the spring exerts a force $F_s = k\Delta L$, where k is a positive number called the spring constant. If the spring is stretched then $\Delta L > 0$ and $F_s > 0$, so the spring force is upward, while if the spring is compressed then $\Delta L < 0$ and $F_s < 0$, so the spring force is downward.
- A damping force $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to air resistance or friction in the spring. However, a convenient way to visualize a damping force is to assume that the object is rigidly attached to a piston with negligible mass immersed in a cylinder (called a dashpot) filled with a viscous liquid (Figure 6.1.2). As the piston moves, the liquid exerts a damping force. We say that the motion is undamped if $c = 0$, or damped if $c > 0$.
- An external force F, other than the force due to gravity, that may vary with t, but is independent of displacement and velocity. We say that the motion is free if $F \equiv 0$, or forced if $F \not\equiv 0$.

From Newton's second law of motion,

$$my'' = -mg + F_d + F_s + F = -mg - cy' + F_s + F. \tag{6.1.1}$$

We must now relate F_s to y. In the absence of external forces the object stretches the spring by an amount Δl to assume its equilibrium position (Figure 6.1.3). Since the sum of the forces acting on the object is then zero, Hooke's Law implies that $mg = k\Delta l$. If the object is displaced y units from its equilibrium position, the total change in the length of the spring is $\Delta L = \Delta l - y$, so Hooke's law implies that

$$F_s = k\Delta L = k\Delta l - ky.$$

Substituting this into (6.1.1) yields

$$my'' = -mg - cy' + k\Delta L - ky + F.$$

Since $mg = k\Delta l$ this can be written as

$$my'' + cy' + ky = F. \tag{6.1.2}$$

We call this the equation of motion.

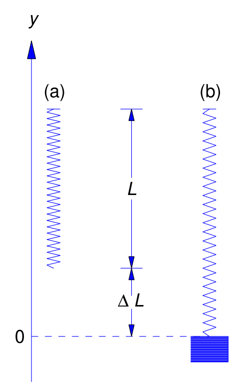

***Figure 6.1.3. (a)** Natural length of spring* ***(b)** Spring stretched by mass*

Simple Harmonic Motion

Throughout the rest of this section we'll consider spring–mass systems without damping; that is, $c = 0$. We'll consider systems with damping in the next section.

We first consider the case where the motion is also free; that is, F=0. We begin with an example.

EXAMPLE 6.1.1

An object stretches a spring 6 inches in equilibrium.

a. Set up the equation of motion and find its general solution.
b. Find the displacement of the object for $t > 0$ if it's initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

SOLUTION (A) Setting $c = 0$ and $F = 0$ in (6.1.2) yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m} y = 0. \tag{6.1.3}$$

Although we would need the weight of the object to obtain k from the equation $mg = k\Delta l$ we can obtain k/m from Δl alone; thus, $k/m = g/\Delta l$. Consistent with the units used in the problem statement, we take $g = 32$ ft/s^2. Although Δl is stated in inches, we must convert it to feet to be consistent with this choice of g; that is, $\Delta l = 1/2$ ft. Therefore

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

and (6.1.3) becomes

$$y'' + 64y = 0. \tag{6.1.4}$$

The characteristic equation of (6.1.4) is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of (6.1.4) is

$$y = c_1 \cos 8t + c_2 \sin 8t. \tag{6.1.5}$$

Solution (b) The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating (6.1.5) yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \tag{6.1.6}$$

Setting $t = 0$ in (6.1.5) and (6.1.6) and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t,$$

where y is in feet (Figure 6.1.4).

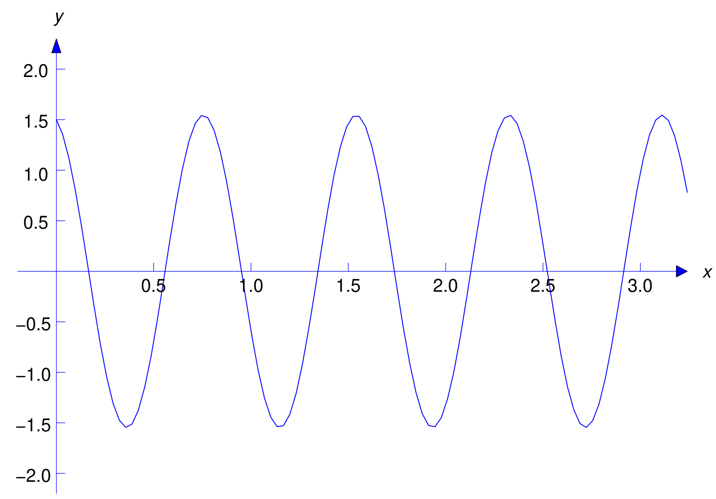

Figure 6.1.4. $y = \dst\frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t$

We'll now consider the equation

$$my'' + ky = 0$$

where m and k are arbitrary positive numbers. Dividing through by m and defining $\omega_0 = \sqrt{k/m}$ yields

$$y'' + \omega_0^2 y = 0.$$

The general solution of this equation is

$$y = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t. \tag{6.1.7}$$

We can rewrite this in a more useful form by defining

$$R = \sqrt{c_1^2 + c_2^2}, \tag{6.1.8}$$

and

$$c_1 = R\cos\phi \quad \text{and} \quad c_2 = R\sin\phi. \tag{6.1.9}$$

Substituting from (6.1.9) into (6.1.7) and applying the identity

$$\cos\omega_0 t\cos\phi + \sin\omega_0 t\sin\phi = \cos(\omega_0 t - \phi)$$

yields

$$y = R\cos(\omega_0 t - \phi). \tag{6.1.10}$$

From (6.1.8) and (6.1.9) we see that the R and ϕ can be interpreted as polar coordinates of the point with rectangular coordinates (c_1, c_2) (Figure 6.1.5). Given c_1 and c_2, we can compute R from (6.1.8). From (6.1.8) and (6.1.9), we see that ϕ is related to c_1 and c_2 by

$$\cos\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

There are infinitely many angles ϕ, differing by integer multiples of 2π, that satisfy these equations. We will always choose ϕ so that $-\pi \le \phi < \pi$.

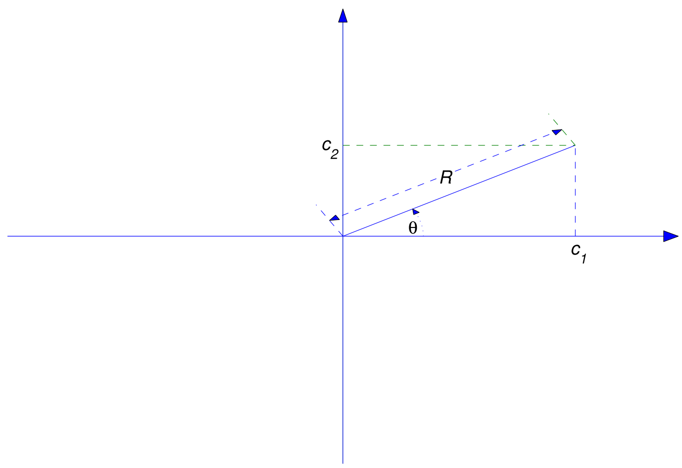

Figure 6.1.5. $R = \protect\sqrt{c_1^2 + c_2^2}$; $\quad c_1 = R\cos\phi$; $\quad c_2 = R\sin\phi$

The motion described by (6.1.7) or (6.1.10) is simple harmonic motion. We see from either of these equations that the motion is periodic, with period

$$T = 2\pi/\omega_0.$$

This is the time required for the object to complete one full cycle of oscillation (for example, to move from its highest position to its lowest position and back to its highest position). Since the highest and lowest positions of the object are $y = R$ and $y = -R$, we say that R is the amplitude of the oscillation. The angle ϕ in (6.1.10) is the phase angle. It's measured in radians. Equation (6.1.10) is the amplitude–

phase form of the displacement. If t is in seconds then ω_0 is in radians per second (rad/s); it's the frequency of the motion. It is also called the natural frequency of the spring–mass system without damping.

Example 6.1.2

We found the displacement of the object in Example 6.1.1 to be

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

Solution The frequency is $\omega_0 = 8$ rad/s, and the period is $T = 2\pi/\omega_0 = \pi/4$ s. Since $c_1 = 3/2$ and $c_2 = -3/8$, the amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{8}\right)^2} = \frac{3}{8}\sqrt{17}.$$

The phase angle is determined by

$$\cos\phi = \frac{\frac{3}{2}}{\frac{3}{8}\sqrt{17}} = \frac{4}{\sqrt{17}} \tag{6.1.11}$$

and

$$\sin\phi = \frac{-\frac{3}{8}}{\frac{3}{8}\sqrt{17}} = -\frac{1}{\sqrt{17}}. \tag{6.1.12}$$

Using a calculator, we see from (6.1.11) that

$$\phi \approx \pm .245 \text{ rad}.$$

Since $\sin\phi < 0$ (see (6.1.12)), the minus sign applies here; that is,

$$\phi \approx -.245 \text{ rad}.$$

Example 6.1.3

The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

Solution In cgs units $g = 980$ cm/s^2. Since $\Delta l = 2$ cm, $\omega_0^2 = g/\Delta l = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10}\left(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t\right).$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}} \sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10}$ rad/s, and the period is $T = 2\pi/(7\sqrt{10})$ s. The amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}} \text{ cm}.$$

The phase angle is determined by

$$\cos\phi = \frac{c_1}{R} = -\sqrt{\frac{5}{7}} \quad \text{and} \quad \sin\phi = \frac{c_2}{R} = \sqrt{\frac{2}{7}}.$$

Therefore ϕ is in the second quadrant and

$$\phi = \cos^{-1}\left(-\sqrt{\frac{5}{7}}\right) \approx 2.58 \text{ rad.}$$

Undamped Forced Oscillation

In many mechanical problems a device is subjected to periodic external forces. For example, soldiers marching in cadence on a bridge cause periodic disturbances in the bridge, and the engines of a propeller driven aircraft cause periodic disturbances in its wings. In the absence of sufficient damping forces, such disturbances – even if small in magnitude – can cause structural breakdown if they are at certain critical frequencies. To illustrate, this we'll consider the motion of an object in a spring–mass system without damping, subject to an external force

$$F(t) = F_0 \cos \omega t$$

where F_0 is a constant. In this case the equation of motion (6.1.2) is

$$my'' + ky = F_0 \cos \omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t \tag{6.1.13}$$

with $\omega_0 = \sqrt{k/m}$. We'll see from the next two examples that the solutions of (6.1.13) with $\omega \neq \omega_0$ behave very differently from the solutions with $\omega = \omega_0$.

Example 6.1.4

Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t, \quad y(0) = 0, \quad y'(0) = 0, \tag{6.1.14}$$

given that $\omega \neq \omega_0$.

Solution We first obtain a particular solution of (6.1.13) by the method of undetermined coefficients. Since $\omega \neq \omega_0$, $\cos \omega t$ isn't a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

Therefore (6.1.13) has a particular solution of the form

$$y_p = A\cos\omega t + B\sin\omega t.$$

Since

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m}\cos\omega t$$

if and only if

$$(\omega_0^2 - \omega^2)\left(A\cos\omega t + B\sin\omega t\right) = \frac{F_0}{m}\cos\omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t.$$

The general solution of (6.1.13) is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t, \tag{6.1.15}$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)}\sin\omega t + \omega_0(-c_1\sin\omega_0 t + c_2\cos\omega_0 t).$$

The initial conditions $y(0)=0$ and $y'(0)=0$ in (6.1.14) imply that

$$c_1=-\frac{F_0}{m(\omega_0^2-\omega^2)} \quad \text{and} \quad c_2=0.$$

Substituting these into (6.1.15) yields

$$y=\frac{F_0}{m(\omega_0^2-\omega^2)}(\cos\omega t-\cos\omega_0 t). \tag{6.1.16}$$

It is revealing to write this in a different form. We start with the trigonometric identities

$$\begin{aligned}\cos(\alpha-\beta)&=\cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos(\alpha+\beta)&=\cos\alpha\cos\beta-\sin\alpha\sin\beta.\end{aligned}$$

Subtracting the second identity from the first yields

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta \tag{6.1.17}$$

Now let

$$\alpha-\beta=\omega t \quad \text{and} \quad \alpha+\beta=\omega_0 t, \tag{6.1.18}$$

so that

$$\alpha=\frac{(\omega_0+\omega)t}{2} \quad \text{and} \quad \beta=\frac{(\omega_0-\omega)t}{2}. \tag{6.1.19}$$

Substituting (6.1.18) and (6.1.19) into (6.1.17) yields

$$\cos\omega t-\cos\omega_0 t=2\sin\frac{(\omega_0-\omega)t}{2}\sin\frac{(\omega_0+\omega)t}{2},$$

and substituting this into (6.1.16) yields

$$y=R(t)\sin\frac{(\omega_0+\omega)t}{2}, \tag{6.1.20}$$

where

$$R(t)=\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}. \tag{6.1.21}$$

From (6.1.20) we can regard y as a sinusoidal variation with frequency $(\omega_0+\omega)/2$ and variable amplitude $|R(t)|$. In Figure 6.1.6 the dashed curve above the t axis is $y=|R(t)|$, the dashed curve below the t axis is $y=-|R(t)|$, and the displacement y appears as an oscillation bounded by them. The oscillation of y for t on an interval between successive zeros of $R(t)$ is called a beat.

Figure 6.1.6. *Undamped oscillation with beats*

You can see from (6.1.20) and (6.1.21) that

$$|y(t)|\le\frac{2|F_0|}{m|\omega_0^2-\omega^2|};$$

moreover, if $\omega+\omega_0$ is sufficiently large compared with $\omega-\omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all t. (This assumes that the spring can withstand deflections of this size and continue to obey Hooke's law.) The next example shows that this isn't so if $\omega=\omega_0$.

Example 6.1.5

Find the general solution of

$$y''+\omega_0^2y=\frac{F_0}{m}\cos\omega_0t. \tag{6.1.22}$$

Solution We first obtain a particular solution y_p of (6.1.22). Since $\cos\omega_0t$ is a solution of the complementary equation, the form for y_p is

$$y_p=t(A\cos\omega_0t+B\sin\omega_0t). \tag{6.1.23}$$

Then

$$y_p'=A\cos\omega_0t+B\sin\omega_0t+\omega_0t(-A\sin\omega_0t+B\cos\omega_0t)$$

and

$$y_p'' = 2\omega_0(-A\sin\omega_0 t + B\cos\omega_0 t) - \omega_0^2 t(A\cos\omega_0 t + B\sin\omega_0 t). \tag{6.1.24}$$

From (6.1.23) and (6.1.24), we see that y_p satisfies (6.1.22) if

$$-2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$$

is a particular solution of (6.1.22). The general solution of (6.1.22) is

$$y = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t + c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

The graph of y_p is shown in Figure 6.1.7, where it can be seen that y_p oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches ∞ as $t \to \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break. ∎

This phenomenon of unbounded displacements of a spring–mass system in response to a periodic forcing function at its natural frequency is called resonance. More complicated mechanical structures can also exhibit resonance–like phenomena. For example, rhythmic oscillations of a suspension bridge by wind forces or of an airplane wing by periodic vibrations of reciprocating engines can cause damage or even failure if the frequencies of the disturbances are close to critical frequencies determined by the parameters of the mechanical system in question.

Figure 6.1.7. *Unbounded displacement due to resonance*

6.1 Exercises

In the following exercises assume that there's no damping.

1. C/G An object stretches a spring 4 inches in equilibrium. Find and graph its displacement for $t > 0$ if it's initially displaced 36 inches above equilibrium and given a downward velocity of 2 ft/s.

   Show answer

2. An object stretches a string 1.2 inches in equilibrium. Find its displacement for $t > 0$ if it's initially displaced 3 inches below equilibrium and given a downward velocity of 2 ft/s.

   Show answer

3. A spring with natural length .5 m has length 50.5 cm with a mass of 2 gm suspended from it. The mass is initially displaced 1.5 cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

   Show answer

4. An object stretches a spring 6 inches in equilibrium. Find its displacement for $t > 0$ if it's initially displaced 3 inches above equilibrium and given a downward velocity of 6 inches/s. Find the frequency, period, amplitude and phase angle of the motion.

   Show answer

5. C/G An object stretches a spring 5 cm in equilibrium. It is initially displaced 10 cm above equilibrium and given an upward velocity of .25 m/s. Find and graph its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

   Show answer

6. A 10 kg mass stretches a spring 70 cm in equilibrium. Suppose a 2 kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

   Show answer

7. A weight stretches a spring 1.5 inches in equilibrium. The weight is initially displaced 8 inches above equilibrium and given a downward velocity of 4 ft/s. Find its displacement for $t > 0$.

   Show answer

8. A weight stretches a spring 6 inches in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/s. Find its displacement for $t > 0$.

   Show answer

9. A spring–mass system has natural frequency $7\sqrt{10}$ rad/s. The natural length of the spring is .7 m. What is the length of the spring when the mass is in equilibrium?

   Show answer

10. A 64 lb weight is attached to a spring with constant $k = 8$ lb/ft and subjected to an external force $F(t) = 2\sin t$. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 2 ft/s. Find its displacement for $t > 0$.

Show answer

11. A unit mass hangs in equilibrium from a spring with constant $k = 1/16$. Starting at $t = 0$, a force $F(t) = 3\sin t$ is applied to the mass. Find its displacement for $t > 0$.

Show answer

12. C/G A 4 lb weight stretches a spring 1 ft in equilibrium. An external force $F(t) = .25\sin 8t$ lb is applied to the weight, which is initially displaced 4 inches above equilibrium and given a downward velocity of 1 ft/s. Find and graph its displacement for $t > 0$.

Show answer

13. A 2 lb weight stretches a spring 6 inches in equilibrium. An external force $F(t) = \sin 8t$ lb is applied to the weight, which is released from rest 2 inches below equilibrium. Find its displacement for $t > 0$.

Show answer

14. A 10 gm mass suspended on a spring moves in simple harmonic motion with period 4 s. Find the period of the simple harmonic motion of a 20 gm mass suspended from the same spring.

Show answer

15. A 6 lb weight stretches a spring 6 inches in equilibrium. Suppose an external force $F(t) = \backslash\mathrm{dst}\frac{3}{16}\sin\omega t + \backslash\mathrm{dst}\frac{3}{8}\cos\omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

Show answer

16. C/G A 6 lb weight stretches a spring 4 inches in equilibrium. Suppose an external force $F(t) = 4\sin\omega t - 6\cos\omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find and graph the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

Show answer

17. A mass of one kg is attached to a spring with constant $k = 4$ N/m. An external force $F(t) = -\cos\omega t - 2\sin\omega t$ n is applied to the mass. Find the displacement y for $t > 0$ if ω equals the natural frequency of the spring–mass system. Assume that the mass is initially displaced 3 m above equilibrium and given an upward velocity of 450 cm/s.

Show answer

18. An object is in simple harmonic motion with frequency ω_0, with $y(0) = y_0$ and $y'(0) = v_0$. Find its displacement for $t > 0$. Also, find the amplitude of the oscillation and give formulas for the sine and cosine of the initial phase angle.

Show answer

19. Two objects suspended from identical springs are set into motion. The period of one object is twice the period of the other. How are the weights of the two objects related?

Show answer

20. Two objects suspended from identical springs are set into motion. The weight of one object is twice the weight of the other. How are the periods of the resulting motions related?

Show answer

21. Two identical objects suspended from different springs are set into motion. The period of one motion is 3 times the period of the other. How are the two spring constants related?

Show answer

6.2 Spring Problems II

Free Vibrations With Damping

In this section we consider the motion of an object in a spring–mass system with damping. We start with unforced motion, so the equation of motion is

$$my'' + cy' + ky = 0. \tag{6.2.1}$$

Now suppose the object is displaced from equilibrium and given an initial velocity. Intuition suggests that if the damping force is sufficiently weak the resulting motion will be oscillatory, as in the undamped case considered in the previous section, while if it's sufficiently strong the object may just move slowly toward the equilibrium position without ever reaching it. We'll now confirm these intuitive ideas mathematically. The characteristic equation of (6.2.1) is

$$mr^2 + cr + k = 0.$$

The roots of this equation are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}. \tag{6.2.2}$$

In Section 5.2 we saw that the form of the solution of (6.2.1) depends upon whether $c^2 - 4mk$ is positive, negative, or zero. We'll now consider these three cases.

Underdamped Motion

We say the motion is underdamped if $c < \sqrt{4mk}$. In this case r_1 and r_2 in (6.2.2) are complex conjugates, which we write as

$$r_1 = -\frac{c}{2m} - i\omega_1 \quad \text{and} \quad r_2 = -\frac{c}{2m} + i\omega_1,$$

where

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}.$$

The general solution of (6.2.1) in this case is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

By the method used in Section 6.1 to derive the amplitude–phase form of the displacement of an object in simple harmonic motion, we can rewrite this equation as

$$y = Re^{-ct/2m} \cos(\omega_1 t - \phi), \tag{6.2.3}$$

where

$$R = \sqrt{c_1^2 + c_2^2}, \quad R\cos\phi = c_1, \quad \text{and} \quad R\sin\phi = c_2.$$

The factor $Re^{-ct/2m}$ in (6.2.3) is called the time–varying amplitude of the motion, the quantity ω_1 is called the frequency, and $T = 2\pi/\omega_1$ (which is the period of the cosine function in (6.2.3) is called the quasi–period. A typical graph of (6.2.3) is shown in Figure 6.2.1. As illustrated in that figure, the graph of y oscillates between the dashed exponential curves $y = \pm Re^{-ct/2m}$.

Figure 6.2.1. *Underdamped motion*

Overdamped Motion

We say the motion is overdamped if $c > \sqrt{4mk}$. In this case the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$ (see (6.2.2)), and the general solution of (6.2.1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Again $\lim_{t\to\infty} y(t) = 0$ as in the underdamped case, but the motion isn't oscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (Exercise 23.)

Critically Damped Motion

We say the motion is critically damped if $c = \sqrt{4mk}$. In this case $r_1 = r_2 = -c/2m$ and the general solution of (6.2.1) is

$$y = e^{-ct/2m}(c_1 + c_2 t).$$

Again $\lim_{t\to\infty} y(t) = 0$ and the motion is nonoscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (Exercise 22).

Example 6.2.1

Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and a dashpot provides a damping force of c lb for each ft/sec of velocity.

a. Write the equation of motion of the object and determine the value of c for which the motion is critically damped.

b. Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = 20$.

c. Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = -20$.

Solution (a) Here $m = 2$ slugs and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion (6.2.1) is

$$2y'' + cy' + 128y = 0. \tag{6.2.4}$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb--sec/ft}.$$

Solution (b) Setting $c = 32$ in (6.2.4) and cancelling the common factor 2 yields

$$y'' + 16y + 64y = 0.$$

The characteristic equation is

$$r^2 + 16r + 64y = (r+8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \tag{6.2.5}$$

Differentiating this yields

$$y' = -8y + c_2 e^{-8t}. \tag{6.2.6}$$

Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$. Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \to \infty$. There's no oscillation. **Solution (c)** Imposing the initial conditions $y(0) = 1$ and $y'(0) = -20$ in (6.2.5) and (6.2.6) yields $1 = c_1$ and $-20 = -8 + c_2$. Hence, the solution of this initial value problem is

$$y = e^{-8t}(1 - 12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t \to \infty$. Again, there's no oscillation. The solutions of these two initial value problems are graphed in Figure 6.2.2.

Figure 6.2.2. (a) $y = e^{-8t}(1 + 28t)$ ***(b)*** $y = e^{-8t}(1 - 12t)$

Example 6.2.2

Find the displacement of the object in Example 6.2.1 if the damping constant is $c = 4$ lb–sec/ft and the initial conditions are $y(0) = 1.5$ ft and $y'(0) = -3$ ft/sec.

Solution With $c = 4$, the equation of motion (6.2.4) becomes

$$y'' + 2y' + 64y = 0 \tag{6.2.7}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 2r + 64 = 0$$

has complex conjugate roots

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 64}}{2} = -1 \pm 3\sqrt{7}i.$$

Therefore the motion is underdamped and the general solution of (6.2.7) is

$$y = e^{-t}(c_1 \cos 3\sqrt{7}t + c_2 \sin 3\sqrt{7}t).$$

Differentiating this yields

$$y' = -y + 3\sqrt{7}e^{-t}(-c_1 \sin 3\sqrt{7}t + c_2 \cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0) = 1.5$ and $y'(0) = -3$ in the last two equations yields $1.5 = c_1$ and $-3 = -1.5 + 3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y=e^{-t}\left(\frac{3}{2}\cos 3\sqrt{7}t-\frac{1}{2\sqrt{7}}\sin 3\sqrt{7}t\right). \tag{6.2.8}$$

The amplitude of the function in parentheses is

$$R=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{1}{2\sqrt{7}}\right)^2}=\sqrt{\frac{9}{4}+\frac{1}{4\cdot 7}}=\sqrt{\frac{64}{4\cdot 7}}=\frac{4}{\sqrt{7}}.$$

Therefore we can rewrite (6.2.8) as

$$y=\frac{4}{\sqrt{7}}e^{-t}\cos(3\sqrt{7}t-\phi),$$

where

$$\cos\phi=\frac{3}{2R}=\frac{3\sqrt{7}}{8}\quad\text{and}\quad\sin\phi=-\frac{1}{2\sqrt{7}R}=-\frac{1}{8}.$$

Therefore $\phi\cong -.125$ radians.

Example 6.2.3

Let the damping constant in Example 1 be $c=40$ lb–sec/ft. Find the displacement y for $t>0$ if $y(0)=1$ and $y'(0)=1$.

Solution With $c=40$, the equation of motion (6.2.4) reduces to

$$y''+20y'+64y=0 \tag{6.2.9}$$

after cancelling the common factor 2. The characteristic equation

$$r^2+20r+64=(r+16)(r+4)=0$$

has the roots $r_1=-4$ and $r_2=-16$. Therefore the general solution of (6.2.9) is

$$y=c_1e^{-4t}+c_2e^{-16t}. \tag{6.2.10}$$

Differentiating this yields

$$y' = -4e^{-4t} - 16c_2e^{-16t}.$$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{array}{rcrl} c_1 & + & c_2 & = 1 \\ -4c_1 & - & 16c_2 & = 1. \end{array}$$

The solution of this system is $c_1 = 17/12$, $c_2 = -5/12$. Substituting these into (6.2.10) yields

$$y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$$

as the solution of the given initial value problem (Figure 6.2.3).

Figure 6.2.3. $y = \dst\frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$

Forced Vibrations With Damping

Now we consider the motion of an object in a spring-mass system with damping, under the influence of a periodic forcing function $F(t) = F_0 \cos \omega t$, so that the equation of motion is

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{6.2.11}$$

In Section 6.1 we considered this equation with $c = 0$ and found that the resulting displacement y assumed arbitrarily large values in the case of resonance (that is, when $\omega = \omega_0 = \sqrt{k/m}$). Here we'll see that in the presence of damping the displacement remains bounded for all t, and the initial conditions have little effect on the motion as $t \to \infty$. In fact, we'll see that for large t the displacement is closely approximated by a function of the form

$$y = R\cos(\omega t - \phi), \tag{6.2.12}$$

where the amplitude R depends upon m, c, k, F_0, and ω. We're interested in the following question:

QUESTION: Assuming that m, c, k, and F_0 are held constant, what value of ω produces the largest amplitude R in (6.2.12), and what is this largest amplitude?

To answer this question, we must solve (6.2.11) and determine R in terms of F_0, ω_0, ω, and c. We can obtain a particular solution of (6.2.11) by the method of undetermined coefficients. Since $\cos\omega t$ does not satisfy the complementary equation

$$my'' + cy' + ky = 0,$$

we can obtain a particular solution of (6.2.11) in the form

$$y_p = A\cos\omega t + B\sin\omega t. \tag{6.2.13}$$

Differentiating this yields

$$y_p' = \omega(-A\sin\omega t + B\cos\omega t)$$

and

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t).$$

From the last three equations,

$$my_p'' + cy_p' + ky_p = (-m\omega^2 A + c\omega B + kA)\cos\omega t + (-m\omega^2 B - c\omega A + kB)\sin\omega t,$$

so y_p satisfies (6.2.11) if

$$\begin{array}{rcl} (k - m\omega^2)A + \quad c\omega B & = & F_0 \\ -c\omega A \quad + (k - m\omega^2)B & = & 0. \end{array}$$

Solving for A and B and substituting the results into (6.2.13) yields

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2}\left[(k - m\omega^2)\cos\omega t + c\omega\sin\omega t\right],$$

which can be written in amplitude–phase form as

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\cos(\omega t - \phi), \tag{6.2.14}$$

where

$$\cos\phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin\phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}. \tag{6.2.15}$$

To compare this with the undamped forced vibration that we considered in Section 6.1 it's useful to write

$$k - m\omega^2 = m\left(\frac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2), \tag{6.2.16}$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the undamped simple harmonic motion of an object with mass m on a spring with constant k. Substituting (6.2.16) into (6.2.14) yields

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi). \tag{6.2.17}$$

The solution of an initial value problem

$$my'' + cy' + ky = F_0 \cos\omega t, \quad y(0) = y_0, \quad y'(0) = v_0,$$

is of the form $y = y_c + y_p$, where y_c has one of the three forms

$$\begin{aligned} y_c &= e^{-ct/2m}(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t), \\ y_c &= e^{-ct/2m}(c_1 + c_2 t), \\ y_c &= c_1 e^{r_1 t} + c_2 e^{r_2 t}\ (r_1, r_2 < 0). \end{aligned}$$

In all three cases $\lim_{t\to\infty} y_c(t) = 0$ for any choice of c_1 and c_2. For this reason we say that y_c is the transient component of the solution y. The behavior of y for large t is determined by y_p, which we call the steady state component of y. Thus, for large t the motion is like simple harmonic motion at the frequency of the external force.

The amplitude of y_p in (6.2.17) is

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \tag{6.2.18}$$

which is finite for all ω; that is, the presence of damping precludes the phenomenon of resonance that we encountered in studying undamped vibrations under a periodic forcing function. We'll now find the value $\omega_{\max}$ of ω for which R is maximized. This is the value of ω for which the function

$$\rho(\omega) = m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2$$

in the denominator of (6.2.18) attains its minimum value. By rewriting this as

$$\rho(\omega)=m^2(\omega_0^4+\omega^4)+(c^2-2m^2\omega_0^2)\omega^2, \tag{6.2.19}$$

you can see that ρ is a strictly increasing function of ω^2 if

$$c \geq \sqrt{2m^2\omega_0^2}=\sqrt{2mk}.$$

(Recall that $\omega_0^2=k/m$). Therefore $\omega_{\max}=0$ if this inequality holds. From (6.2.15), you can see that $\phi=0$ if $\omega=0$. In this case, (6.2.14) reduces to

$$y_p=\frac{F_0}{\sqrt{m^2\omega_0^4}}=\frac{F_0}{k},$$

which is consistent with Hooke's law: if the mass is subjected to a constant force F_0, its displacement should approach a constant y_p such that $ky_p=F_0$. Now suppose $c<\sqrt{2mk}$. Then, from (6.2.19),

$$\rho'(\omega)=2\omega(2m^2\omega^2+c^2-2m^2\omega_0^2),$$

and $\omega_{\max}$ is the value of ω for which the expression in parentheses equals zero; that is,

$$\omega_{\max}=\sqrt{\omega_0^2-\frac{c^2}{2m^2}}=\sqrt{\frac{k}{m}\left(1-\frac{c^2}{2km}\right)}.$$

(To see that $\rho(\omega_{\max})$ is the minimum value of $\rho(\omega)$, note that $\rho'(\omega)<0$ if $\omega<\omega_{\max}$ and $\rho'(\omega)>0$ if $\omega>\omega_{\max}$.) Substituting $\omega=\omega_{\max}$ in (6.2.18) and simplifying shows that the maximum amplitude $R_{\max}$ is

$$R_{\max}=\frac{2mF_0}{c\sqrt{4mk-c^2}} \quad \text{if} \quad c<\sqrt{2mk}.$$

We summarize our results as follows.

THEOREM 6.2.1

Suppose we consider the amplitude R of the steady state component of the solution of

$$my''+cy'+ky=F_0\cos\omega t$$

as a function of ω. (a) If $c \geq \sqrt{2mk}$, the maximum amplitude is $R_{\max} = F_0/k$ and it's attained when $\omega = \omega_{\max} = 0$. (b) If $c < \sqrt{2mk}$, the maximum amplitude is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}}, \tag{6.2.20}$$

and it's attained when

$$\omega = \omega_{\max} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}. \tag{6.2.21}$$

Note that $R_{\max}$ and $\omega_{\max}$ are continuous functions of c, for $c \geq 0$, since (6.2.20) and (6.2.21) reduce to $R_{\max} = F_0/k$ and $\omega_{\max} = 0$ if $c = \sqrt{2km}$.

6.2 Exercises

1. A 64 lb object stretches a spring 4 ft in equilibrium. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement and time–varying amplitude for $t > 0$.

Show answer

2. C/G A 16 lb weight is attached to a spring with natural length 5 ft. With the weight attached, the spring measures 8.2 ft. The weight is initially displaced 3 ft below equilibrium and given an upward velocity of 2 ft/sec. Find and graph its displacement for $t > 0$ if the medium resists the motion with a force of one lb for each ft/sec of velocity. Also, find its time–varying amplitude.

Show answer

3. C/G An 8 lb weight stretches a spring 1.5 inches. It is attached to a dashpot with damping constant c=8 lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given an upward velocity of 6 ft/sec. Find and graph its displacement for $t > 0$.

Show answer

4. A 96 lb weight stretches a spring 3.2 ft in equilibrium. It is attached to a dashpot with damping constant c=18 lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.

Show answer

5. A 16 lb weight stretches a spring 6 inches in equilibrium. It is attached to a damping mechanism with constant c. Find all values of c such that the free vibration of the weight has infinitely many oscillations.

Show answer

6. An 8 lb weight stretches a spring .32 ft. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 4 ft/sec. Find its displacement for $t > 0$ if the medium exerts a damping force of 1.5 lb for each ft/sec of velocity.

Show answer

7. A 32 lb weight stretches a spring 2 ft in equilibrium. It is attached to a dashpot with constant $c = 8$ lb-sec/ft. The weight is initially displaced 8 inches below equilibrium and released from rest. Find its displacement for $t > 0$.

Show answer

8. A mass of 20 gm stretches a spring 5 cm. The spring is attached to a dashpot with damping constant 400 dyne sec/cm. Determine the displacement for $t > 0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.

Show answer

9. A 64 lb weight is suspended from a spring with constant $k = 25$ lb/ft. It is initially displaced 18 inches above equilibrium and released from rest. Find its displacement for $t > 0$ if the medium resists the motion with 6 lb of force for each ft/sec of velocity.

Show answer

10. A 32 lb weight stretches a spring 1 ft in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/sec. Find its displacement for $t > 0$ if the medium resists the motion with a force equal to 3 times the speed in ft/sec.

Show answer

11. An 8 lb weight stretches a spring 2 inches. It is attached to a dashpot with damping constant c=4 lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for $t > 0$.

Show answer

12. C/G A 2 lb weight stretches a spring .32 ft. The weight is initially displaced 4 inches below equilibrium and given an upward velocity of 5 ft/sec. The medium provides damping with constant $c = 1/8$ lb-sec/ft. Find and graph the displacement for $t > 0$.

Show answer

13. An 8 lb weight stretches a spring 8 inches in equilibrium. It is attached to a dashpot with damping constant $c = .5$ lb-sec/ft and subjected to an external force $F(t) = 4\cos 2t$ lb. Determine the steady state component of the displacement for $t > 0$.

Show answer

14. A 32 lb weight stretches a spring 1 ft in equilibrium. It is attached to a dashpot with constant $c = 12$ lb-sec/ft. The weight is initially displaced 8 inches above equilibrium and released from rest. Find its displacement for $t > 0$.

Show answer

15. A mass of one kg stretches a spring 49 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 4 N for each m/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

Show answer

16. A mass of 100 grams stretches a spring 98 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 600 dynes for each cm/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

Show answer

17. A 192 lb weight is suspended from a spring with constant $k = 6$ lb/ft and subjected to an external force $F(t) = 8\cos 3t$ lb. Find the steady state component of the displacement for $t > 0$ if the medium resists the motion with a force equal to 8 times the speed in ft/sec.

Show answer

18. A 2 gm mass is attached to a spring with constant 20 dyne/cm. Find the steady state component of the displacement if the mass is subjected to an external force $F(t) = 3\cos 4t - 5\sin 4t$ dynes and a dashpot supplies 4 dynes of damping for each cm/sec of velocity.

Show answer

19. C/G A 96 lb weight is attached to a spring with constant 12 lb/ft. Find and graph the steady state component of the displacement if the mass is subjected to an external force $F(t) = 18\cos t - 9\sin t$ lb and a dashpot supplies 24 lb of damping for each ft/sec of velocity.

Show answer

20. A mass of one kg stretches a spring 49 cm in equilibrium. It is attached to a dashpot that supplies a damping force of 4 N for each m/sec of speed. Find the steady state component of its displacement if it's subjected to an external force $F(t) = 8\sin 2t - 6\cos 2t$ N.

Show answer

21. A mass m is suspended from a spring with constant k and subjected to an external force $F(t) = \alpha\cos\omega_0 t + \beta\sin\omega_0 t$, where ω_0 is the natural frequency of the spring–mass system without damping. Find the steady state component of the displacement if a dashpot with constant c supplies damping.

Show answer

22. Show that if c_1 and c_2 are not both zero then

$$y = e^{r_1 t}(c_1 + c_2 t)$$

can't equal zero for more than one value of t.

23. Show that if c_1 and c_2 are not both zero then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

can't equal zero for more than one value of t.

24. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is underdamped, so the general solution of the equation is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

Show answer

25. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is overdamped, so the general solution of the equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}\ (r_1, r_2 < 0).$$

Show answer

26. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is critically damped, so that the general solution of the equation is of the form

$$y = e^{r_1 t}(c_1 + c_2 t)\,(r_1 < 0).$$

Show answer

6.3 The RLC Circuit

In this section we consider the RLC circuit, shown schematically in Figure 6.3.1. As we'll see, the RLC circuit is an electrical analog of a spring-mass system with damping.

Nothing happens while the switch is open (dashed line). When the switch is closed (solid line) we say that the circuit is closed. Differences in electrical potential in a closed circuit cause current to flow in the circuit. The battery or generator in Figure 6.3.1 creates a difference in electrical potential $E = E(t)$ between its two terminals, which we've marked arbitrarily as positive and negative. (We could just as well interchange the markings.) We'll say that $E(t) > 0$ if the potential at the positive terminal is greater than the potential at the negative terminal, $E(t) < 0$ if the potential at the positive terminal is less than the potential at the negative terminal, and $E(t) = 0$ if the potential is the same at the two terminals. We call E the impressed voltage.

Figure 6.3.1. *An RLC circuit*

At any time t, the same current flows in all points of the circuit. We denote current by $I = I(t)$. We say that $I(t) > 0$ if the direction of flow is around the circuit from the positive terminal of the battery or generator back to the negative terminal, as indicated by the arrows in Figure 6.3.1 $I(t) < 0$ if the flow is in the opposite direction, and $I(t) = 0$ if no current flows at time t.

Differences in potential occur at the resistor, induction coil, and capacitor in Figure 6.3.1. Note that the two sides of each of these components are also identified as positive and negative. The voltage drop across each component is defined to be the potential on the positive side of the component minus the potential on the negative side. This terminology is somewhat misleading, since "drop" suggests a decrease even though changes in potential are signed quantities and therefore may be increases. Nevertheless, we'll go along with tradition and call them voltage drops. The voltage drop across the resistor in Figure 6.3.1 is given by

$$V_R = IR, \tag{6.3.1}$$

where I is current and R is a positive constant, the resistance of the resistor. The voltage drop across the induction coil is given by

$$V_I = L\frac{dI}{dt} = LI', \tag{6.3.2}$$

where L is a positive constant, the inductance of the coil.

A capacitor stores electrical charge $Q = Q(t)$, which is related to the current in the circuit by the equation

$$Q(t) = Q_0 + \int_0^t I(\tau)\, d\tau, \tag{6.3.3}$$

where Q_0 is the charge on the capacitor at $t = 0$. The voltage drop across a capacitor is given by

$$V_C = \frac{Q}{C}, \tag{6.3.4}$$

where C is a positive constant, the capacitance of the capacitor.

Table 6.3.1 names the units for the quantities that we've discussed. The units are defined so that

$$\begin{aligned} 1\,\text{volt} &= 1\,\text{ampere}\cdot 1\,\text{ohm} \\ &= 1\,\text{henry}\cdot 1\,\text{ampere/second} \\ &= 1\,\text{coulomb/ farad} \end{aligned}$$

$$\backslash\text{arraytext}\,and$$

$$1\,\text{ampere} = 1\,\text{coulomb/second}.$$

Table 6.3.1. *Electrical Units*

Symbol	**Name**	**Unit**
E	Impressed Voltage	volt
I	Current	ampere
Q	Charge	coulomb
R	Resistance	ohm
L	Inductance	henry
C	Capacitance	farad

According to Kirchoff's law (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kirchhoff.html), the sum of the voltage drops in a closed RLC circuit equals the impressed voltage. Therefore, from (6.3.1), (6.3.2), and (6.3.4),

$$LI' + RI + \frac{1}{C}Q = E(t). \tag{6.3.5}$$

This equation contains two unknowns, the current I in the circuit and the charge Q on the capacitor. However, (6.3.3) implies that $Q' = I$, so (6.3.5) can be converted into the second order equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{6.3.6}$$

in Q. To find the current flowing in an RLC circuit, we solve (6.3.6) for Q and then differentiate the solution to obtain I.

In Sections 6.1 and 6.2 we encountered the equation

$$my'' + cy' + ky = F(t) \tag{6.3.7}$$

in connection with spring-mass systems. Except for notation this equation is the same as (6.3.6). The correspondence between electrical and mechanical quantities connected with (6.3.6) and (6.3.7) is shown in Table 6.3.2.

Table 6.3.2. *Electrical and Mechanical Units*

Electrical	Mechanical		
charge	Q	displacement	y
curent	I	velocity	y'
impressed voltage	$E(t)$	external force	$F(t)$
inductance	L	Mass	m
resistance	R	damping	c
1/capacitance	$1/C$	cpring constant	k

The equivalence between (6.3.6) and (6.3.7) is an example of how mathematics unifies fundamental similarities in diverse physical phenomena. Since we've already studied the properties of solutions of (6.3.7) in Sections 6.1 and 6.2, we can obtain results concerning solutions of (6.3.6) by simpling changing notation, according to Table 6.3.1.

Free Oscillations

We say that an RLC circuit is in free oscillation if $E(t) = 0$ for $t > 0$, so that (6.3.6) becomes

$$LQ'' + RQ' + \frac{1}{C}Q = 0. \tag{6.3.8}$$

The characteristic equation of (6.3.8) is

$$Lr^2 + Rr + \frac{1}{C} = 0,$$

with roots

$$r_1 = \frac{-R - \sqrt{R^2 - 4L/C}}{2L} \quad \text{and} \quad r_2 = \frac{-R + \sqrt{R^2 - 4L/C}}{2L}. \tag{6.3.9}$$

There are three cases to consider, all analogous to the cases considered in Section 6.2 for free vibrations of a damped spring-mass system.

Case 1. The oscillation is underdamped if $R < \sqrt{4L/C}$. In this case, r_1 and r_2 in (6.3.9) are complex conjugates, which we write as

$$r_1 = -\frac{R}{2L} + i\omega_1 \quad \text{and} \quad r_2 = -\frac{R}{2L} - i\omega_1,$$

where

$$\omega_1 = \frac{\sqrt{4L/C - R^2}}{2L}.$$

The general solution of (6.3.8) is

$$Q = e^{-Rt/2L}(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t),$$

which we can write as

$$Q = Ae^{-Rt/2L}\cos(\omega_1 t - \phi), \tag{6.3.10}$$

where

$$A = \sqrt{c_1^2 + c_2^2}, \quad A\cos\phi = c_1, \quad \text{and} \quad A\sin\phi = c_2.$$

In the idealized case where $R = 0$, the solution (6.3.10) reduces to

$$Q = A\cos\left(\frac{t}{\sqrt{LC}} - \phi\right),$$

which is analogous to the simple harmonic motion of an undamped spring-mass system in free vibration.

Actual RLC circuits are usually underdamped, so the case we've just considered is the most important. However, for completeness we'll consider the other two possibilities.

Case 2. The oscillation is overdamped if $R > \sqrt{4L/C}$. In this case, the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$ (see (6.3.9)), and the general solution of (6.3.8) is

$$Q = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{6.3.11}$$

Case 3. The oscillation is critically damped if $R = \sqrt{4L/C}$. In this case, $r_1 = r_2 = -R/2L$ and the general solution of (6.3.8) is

$$Q = e^{-Rt/2L}(c_1 + c_2 t). \tag{6.3.12}$$

If $R \neq 0$, the exponentials in (6.3.10), (6.3.11), and (6.3.12) are negative, so the solution of any homogeneous initial value problem

$$LQ'' + RQ' + \frac{1}{C}Q = 0, \quad Q(0) = Q_0, \quad Q'(0) = I_0,$$

approaches zero exponentially as $t \to \infty$. Thus, all such solutions are transient, in the sense defined Section 6.2 in the discussion of forced vibrations of a spring-mass system with damping.

Example 6.3.1

At $t = 0$ a current of 2 amperes flows in an RLC circuit with resistance $R = 40$ ohms, inductance $L = .2$ henrys, and capacitance $C = 10^{-5}$ farads. Find the current flowing in the circuit at $t > 0$ if the initial charge on the capacitor is 1 coulomb. Assume that $E(t) = 0$ for $t > 0$.

Solution The equation for the charge Q is

$$\frac{1}{5}Q'' + 40Q' + 10000Q = 0,$$

or

$$Q'' + 200Q' + 50000Q = 0. \tag{6.3.13}$$

Therefore we must solve the initial value problem

$$Q'' + 200Q' + 50000Q = 0, \quad Q(0) = 1, \quad Q'(0) = 2. \tag{6.3.14}$$

The desired current is the derivative of the solution of this initial value problem.

The characteristic equation of (6.3.13) is

$$r^2 + 200r + 50000 = 0,$$

which has complex zeros $r = -100 \pm 200i$. Therefore the general solution of (6.3.13) is

$$Q = e^{-100t}(c_1 \cos 200t + c_2 \sin 200t). \tag{6.3.15}$$

Differentiating this and collecting like terms yields

$$Q' = -e^{-100t}\left[(100c_1 - 200c_2)\cos 200t + (100c_2 + 200c_1)\sin 200t\right]. \tag{6.3.16}$$

To find the solution of the initial value problem (6.3.14), we set $t = 0$ in (6.3.15) and (6.3.16) to obtain

$$c_1 = Q(0) = 1 \quad \text{and} \quad -100c_1 + 200c_2 = Q'(0) = 2;$$

therefore, $c_1 = 1$ and $c_2 = 51/100$, so

$$Q = e^{-100t}\left(\cos 200t + \frac{51}{100}\sin 200t\right)$$

is the solution of (6.3.14). Differentiating this yields

$$I = e^{-100t}(2\cos 200t - 251\sin 200t).$$

Forced Oscillations With Damping

An initial value problem for (6.3.6) has the form

$$LQ'' + RQ' + \frac{1}{C}Q = E(t), \quad Q(0) = Q_0, \quad Q'(0) = I_0, \tag{6.3.17}$$

where Q_0 is the initial charge on the capacitor and I_0 is the initial current in the circuit. We've already seen that if $E \equiv 0$ then all solutions of (6.3.17) are transient. If $E \not\equiv 0$, we know that the solution of (6.3.17) has the form $Q = Q_c + Q_p$, where Q_c satisfies the complementary equation, and approaches zero exponentially as $t \to \infty$ for any initial conditions , while Q_p depends only on E and is independent of the initial conditions. As in the case of forced oscillations of a spring-mass system with damping, we call Q_p the steady state charge on the capacitor of the RLC circuit. Since $I = Q' = Q_c' + Q_p'$ and Q_c' also tends to zero exponentially as $t \to \infty$, we say that $I_c = Q_c'$ is the transient current and $I_p = Q_p'$ is the steady state current. In most applications we're interested only in the steady state charge and current.

Example 6.3.2

Find the amplitude-phase form of the steady state current in the RLC circuit in Figure 6.3.1 if the impressed voltage, provided by an alternating current generator, is $E(t) = E_0 \cos \omega t$.

Solution We'll first find the steady state charge on the capacitor as a particular solution of

$$LQ'' + RQ' + \frac{1}{C}Q = E_0 \cos \omega t.$$

To do, this we'll simply reinterpret a result obtained in Section 6.2, where we found that the steady state solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

is

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos \phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}.$$

(See Equations (6.2.14) and (6.2.15).) By making the appropriate changes in the symbols (according to Table 2) yields the steady state charge

$$Q_p = \frac{E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos\phi = \frac{1/C - L\omega^2}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \quad \text{and} \quad \sin\phi = \frac{R\omega}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}}.$$

Therefore the steady state current in the circuit is

$$I_p = Q_p' = -\frac{\omega E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}}\sin(\omega t - \phi).$$

6.3 Exercises

In Exercises 1-5 find the current in the RLC circuit, assuming that $E(t) = 0$ for $t > 0$.

1. $R = 3$ ohms; $L = .1$ henrys; $C = .01$ farads; $Q_0 = 0$ coulombs; $I_0 = 2$ amperes.

Show answer

2. $R = 2$ ohms; $L = .05$ henrys; $C = .01$ farads'; $Q_0 = 2$ coulombs; $I_0 = -2$ amperes.

Show answer

3. $R = 2$ ohms; $L = .1$ henrys; $C = .01$ farads; $Q_0 = 2$ coulombs; $I_0 = 0$ amperes.

Show answer

4. $R = 6$ ohms; $L = .1$ henrys; $C = .004$ farads'; $Q_0 = 3$ coulombs; $I_0 = -10$ amperes.

Show answer

5. $R = 4$ ohms; $L = .05$ henrys; $C = .008$ farads; $Q_0 = -1$ coulombs; $I_0 = 2$ amperes.

Show answer

In Exercises 6-10 find the steady state current in the circuit described by the equation.

6. $\dst\frac{1}{10}Q''+3Q'+100Q=5\cos 10t-5\sin 10t$

Show answer

7. $\dst\frac{1}{20}Q''+2Q'+100Q=10\cos 25t-5\sin 25t$

Show answer

8. $\dst\frac{1}{10}Q''+2Q'+100Q=3\cos 50t-6\sin 50t$

Show answer

9. $\dst\frac{1}{10}Q''+6Q'+250Q=10\cos 100t+30\sin 100t$

Show answer

10. $\dst\frac{1}{20}Q''+4Q'+125Q=15\cos 30t-30\sin 30t$

Show answer

11. Show that if $E(t)=U\cos\omega t+V\sin\omega t$ where U and V are constants then the steady state current in the RLC circuit shown in Figure 6.3.1 is

$$I_p=\frac{\omega^2RE(t)+(1/C-L\omega^2)E'(t)}{\Delta},$$

where

$$\Delta=(1/C-L\omega^2)^2+R^2\omega^2.$$

12. Find the amplitude of the steady state current I_p in the RLC circuit shown in Figure 6.3.1 if $E(t)=U\cos\omega t+V\sin\omega t$, where U and V are constants. Then find the value ω_0 of ω maximizes the amplitude, and find the maximum amplitude.

Show answer

In Exercises 13-17 plot the amplitude of the steady state current against ω. Estimate the value of ω that maximizes the amplitude of the steady state current, and estimate this maximum amplitude. Hint: You can confirm your results by doing Exercise ???.

13. L \dst $\frac{1}{10}Q'' + 3Q' + 100Q = U\cos\omega t + V\sin\omega t$

14. L \dst $\frac{1}{20}Q'' + 2Q' + 100Q = U\cos\omega t + V\sin\omega t$

15. L \dst $\frac{1}{10}Q'' + 2Q' + 100Q = U\cos\omega t + V\sin\omega t$

16. L \dst $\frac{1}{10}Q'' + 6Q' + 250Q = U\cos\omega t + V\sin\omega t$

17. L \dst $\frac{1}{20}Q'' + 4Q' + 125Q = U\cos\omega t + V\sin\omega t$

6.4 Motion Under a Central Force

We'll now study the motion of a object moving under the influence of a central force; that is, a force whose magnitude at any point P other than the origin depends only on the distance from P to the origin, and whose direction at P is parallel to the line connecting P and the origin, as indicated in Figure 6.4.1 for the case where the direction of the force at every point is toward the origin. Gravitational forces are central forces; for example, as mentioned in Section 4.3, if we assume that Earth is a perfect sphere with constant mass density then Newton's law of gravitation asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth, which we take to be the origin.

If the initial position and velocity vectors of an object moving under a central force are parallel, then the subsequent motion is along the line from the origin to the initial position. Here we'll assume that the initial position and velocity vectors are not parallel; in this case the subsequent motion is in the plane determined by them. For convenience we take this to be the xy-plane. We'll consider the problem of determining the curve traversed by the object. We call this curve the orbit.

Figure 6.4.1.

We can represent a central force in terms of polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

as

$$\mathbf{F}(r,\theta) = f(r)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).$$

We assume that f is continuous for all $r > 0$. The magnitude of $\mathbf{F}$ at $(x,y) = (r\cos\theta, r\sin\theta)$ is $|f(r)|$, so it depends only on the distance r from the point to the origin the direction of $\mathbf{F}$ is from the point to the origin if $f(r) < 0$, or from the origin to the point if $f(r) > 0$. We'll show that the orbit of an object with mass m moving under this force is given by

$$r(\theta) = \frac{1}{u(\theta)},$$

where u is solution of the differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2} f(1/u), \tag{6.4.1}$$

and h is a constant defined below.

Newton's second law of motion ($\mathbf{F} = m\mathbf{a}$) says that the polar coordinates $r = r(t)$ and $\theta = \theta(t)$ of the particle satisfy the vector differential equation

$$m(r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j})'' = f(r)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}). \tag{6.4.2}$$

To deal with this equation we introduce the unit vectors

$$\mathbf{e}_1 = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \quad \text{and} \quad \mathbf{e}_2 = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

Note that $\mathbf{e}_1$ points in the direction of increasing r and $\mathbf{e}_2$ points in the direction of increasing θ (Figure 6.4.2); moreover,

$$\frac{d\mathbf{e}_1}{d\theta} = \mathbf{e}_2, \quad \frac{d\mathbf{e}_2}{d\theta} = -\mathbf{e}_1, \tag{6.4.3}$$

and

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos\theta(-\sin\theta) + \sin\theta\cos\theta = 0,$$

so $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular. Recalling that the single prime $(')$ stands for differentiation with respect to t, we see from (6.4.3) and the chain rule that

$$\mathbf{e}_1' = \theta'\mathbf{e}_2 \quad \text{and} \quad \mathbf{e}_2' = -\theta'\mathbf{e}_1. \tag{6.4.4}$$

Figure 6.4.2.

Now we can write (6.4.2) as

$$m(r\mathbf{e}_1)'' = f(r)\mathbf{e}_1. \tag{6.4.5}$$

But

$$(r\mathbf{e}_1)' = r'\mathbf{e}_1 + r\mathbf{e}_1' = r'\mathbf{e}_1 + r\theta'\mathbf{e}_2$$

(from (6.4.4)), and

$$\begin{aligned}(r\mathbf{e}_1)'' &= (r'\mathbf{e}_1 + r\theta'\mathbf{e}_2)' \\ &= r''\mathbf{e}_1 + r'\mathbf{e}_1' + (r\theta'' + r'\theta')\mathbf{e}_2 + r\theta'\mathbf{e}_2' \\ &= r''\mathbf{e}_1 + r'\theta'\mathbf{e}_2 + (r\theta'' + r'\theta')\mathbf{e}_2 - r(\theta')^2\mathbf{e}_1 \quad \text{(from (???))} \\ &= \left(r'' - r(\theta')^2\right)\mathbf{e}_1 + (r\theta'' + 2r'\theta')\mathbf{e}_2.\end{aligned}$$

Substituting this into (6.4.5) yields

$$m\left(r'' - r(\theta')^2\right)\mathbf{e}_1 + m(r\theta'' + 2r'\theta')\mathbf{e}_2 = f(r)\mathbf{e}_1.$$

By equating the coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$ on the two sides of this equation we see that

$$m\left(r'' - r(\theta')^2\right) = f(r) \tag{6.4.6}$$

and

$$r\theta'' + 2r'\theta' = 0.$$

Multiplying the last equation by r yields

$$r^2\theta'' + 2rr'\theta' = (r^2\theta')' = 0,$$

so

$$r^2\theta' = h, \tag{6.4.7}$$

where h is a constant that we can write in terms of the initial conditions as

$$h = r^2(0)\theta'(0).$$

Since the initial position and velocity vectors are

$$r(0)\mathbf{e}_1(0) \quad \text{and} \quad r'(0)\mathbf{e}_1(0)+r(0)\theta'(0)\mathbf{e}_2(0),$$

our assumption that these two vectors are not parallel implies that $\theta'(0)\neq 0$, so $h\neq 0$.

Now let $u=1/r$. Then $u^2=\theta'/h$ (from (6.4.7)) and

$$r' = -\text{\dst}\frac{u'}{u^2} = -h\left(\frac{u'}{\theta'}\right),$$

which implies that

$$r' = -h\frac{du}{d\theta}, \tag{6.4.8}$$

since

$$\frac{u'}{\theta'} = \frac{du}{dt}\bigg/\frac{d\theta}{dt} = \frac{du}{d\theta}.$$

Differentiating (6.4.8) with respect to t yields

$$r'' = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\theta',$$

which implies that

$$r'' = -h^2u^2\frac{d^2u}{d\theta^2} \quad \text{since} \quad \theta' = hu^2.$$

Substituting from these equalities into (6.4.6) and recalling that $r=1/u$ yields

$$-m\left(h^2u^2\frac{d^2u}{d\theta^2}+\frac{1}{u}h^2u^4\right) = f(1/u),$$

and dividing through by $-mh^2u^2$ yields (6.4.1).

Eqn. (6.4.7) has the following geometrical interpretation, which is known as Kepler's Second Law (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kepler.html).

Theorem 6.4.1

The position vector of an object moving under a central force sweeps out equal areas in equal times; more precisely, if $\theta(t_1) \le \theta(t_2)$ then the $($signed$)$ area of the sector

$$\{(x,y) = (r\cos\theta, r\sin\theta) \ : \ 0 \le r \le r(\theta),\ \theta(t_1) \le \theta(t_2)\}$$

(Figure 6.4.3) is given by

$$A = \frac{h(t_2 - t_1)}{2},$$

where $h = r^2\theta'$, which we have shown to be constant.

Figure 6.4.3.

Proof Recall from calculus that the area of the shaded sector in Figure 6.4.3 is

$$A = \frac{1}{2}\int_{\theta(t_1)}^{\theta(t_2)} r^2(\theta)\,d\theta,$$

where $r = r(\theta)$ is the polar representation of the orbit. Making the change of variable $\theta = \theta(t)$ yields

$$A = \frac{1}{2}\int_{t_1}^{t_2} r^2(\theta(t))\theta'(t)\,dt. \tag{6.4.9}$$

But (6.4.7) and (6.4.9) imply that

$$A = \frac{1}{2}\int_{t_1}^{t_2} h\,dt = \frac{h(t_2 - t_1)}{2},$$

which completes the proof.

Motion Under an Inverse Square Law Force

In the special case where $f(r) = -mk/r^2 = -mku^2$, so $\mathbf{F}$ can be interpreted as a gravitational force, (6.4.1) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}. \tag{6.4.10}$$

The general solution of the complementary equation

$$\frac{d^2u}{d\theta^2} + u = 0$$

can be written in amplitude–phase form as

$$u = A\cos(\theta - \phi),$$

where $A \geq 0$ and ϕ is a phase angle. Since $u_p = k/h^2$ is a particular solution of (6.4.10), the general solution of (6.4.10) is

$$u = A\cos(\theta - \phi) + \frac{k}{h^2};$$

hence, the orbit is given by

$$r = \left(A\cos(\theta - \phi) + \frac{k}{h^2}\right)^{-1},$$

which we rewrite as

$$r = \frac{\rho}{1 + e\cos(\theta - \phi)}, \tag{6.4.11}$$

where

$$\rho = \frac{h^2}{k} \quad \text{and} \quad e = A\rho.$$

A curve satisfying (6.4.11) is a conic section with a focus at the origin (Exercise 1). The nonnegative constant e is the eccentricity of the orbit, which is an ellipse if $e < 1$ ellipse (a circle if $e = 0$), a parabola if $e = 1$, or a hyperbola if $e > 1$.

Figure 6.4.4.

If the orbit is an ellipse, then the minimum and maximum values of r are

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Figure 6.4.4 shows a typical elliptic orbit. The point P on the orbit where $r = r_{\min}$ is the perigee and the point A where $r = r_{\max}$ is the apogee.

For example, Earth's orbit around the Sun is approximately an ellipse with $e \approx .017$, $r_{\min} \approx 91 \times 10^6$ miles, and $r_{\max} \approx 95 \times 10^6$ miles. Halley's comet has a very elongated approximately elliptical orbit around the sun, with $e \approx .967$, $r_{\min} \approx 55 \times 10^6$ miles, and $r_{\max} \approx 33 \times 10^8$ miles. Some comets (the nonrecurring type) have parabolic or hyperbolic orbits.

6.4 Exercises

1. Find the equation of the curve

$$r = \frac{\rho}{1 + e\cos(\theta - \phi)} \tag{A}$$

in terms of $(X, Y) = (r\cos(\theta - \phi), r\sin(\theta - \phi))$, which are rectangular coordinates with respect to the axes shown in Figure 6.4.5. Use your results to verify that (A) is the equation of an ellipse if $0 < e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$. If $e < 1$, leave your answer in the form

$$\frac{(X - X_0)^2}{a^2} + \frac{(Y - Y_0)^2}{b^2} = 1,$$

and show that the area of the ellipse is

$$A = \frac{\pi\rho^2}{(1 - e^2)^{3/2}}.$$

Then use Theorem 6.4.1 to show that the time required for the object to traverse the entire orbit is

$$T = \frac{2\pi\rho^2}{h(1 - e^2)^{3/2}}.$$

(This is Kepler's third law (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kepler.html); T is called the period of the orbit.)

Figure 6.4.5.

Show answer

2. Suppose an object with mass m moves in the xy-plane under the central force

$$\mathbf{F}(r,\theta) = -\frac{mk}{r^2}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),$$

where k is a positive constant. As we shown, the orbit of the object is given by

$$r = \frac{\rho}{1 + e\cos(\theta - \phi)}.$$

Determine ρ, e, and ϕ in terms of the initial conditions

$$r(0) = r_0, \quad r'(0) = r_0', \;\text{ and }\; \theta(0) = \theta_0, \quad \theta'(0) = \theta_0'.$$

Assume that the initial position and velocity vectors are not collinear.

Show answer

3. Suppose we wish to put a satellite with mass m into an elliptical orbit around Earth. Assume that the only force acting on the object is Earth's gravity, given by

$$\mathbf{F}(r,\theta) = -mg\left(\frac{R^2}{r^2}\right)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),$$

where R is Earth's radius, g is the acceleration due to gravity at Earth's surface, and r and θ are polar coordinates in the plane of the orbit, with the origin at Earth's center.

a. Find the eccentricity required to make the aphelion and perihelion distances equal to $R\gamma_1$ and $R\gamma_2$, respectively, where $1 < \gamma_1 < \gamma_2$.

b. Find the initial conditions

$$r(0) = r_0, \quad r'(0) = r_0', \;\text{ and }\; \theta(0) = \theta_0, \quad \theta'(0) = \theta_0'$$

required to make the initial point the perigee, and the motion along the orbit in the direction of increasing θ. *Hint: Use the results of Exercise 2.*

Show answer

4. An object with mass m moves in a spiral orbit $r = c\theta^2$ under a central force

$$\mathbf{F}(r,\theta) = f(r)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).$$

Find f.

Show answer

5. An object with mass m moves in the orbit $r = r_0 e^{\gamma\theta}$ under a central force

$$\mathbf{F}(r,\theta) = f(r)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).$$

Find f.

Show answer

6. Suppose an object with mass m moves under the central force

$$\mathbf{F}(r,\theta) = -\frac{mk}{r^3}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),$$

with

$$r(0) = r_0, \quad r'(0) = r_0', \ \text{ and } \ \theta(0) = \theta_0, \quad \theta'(0) = \theta_0',$$

where $h = r_0^2\theta_0' \neq 0$.

a. Set up a second order initial value problem for $u = 1/r$ as a function of θ.

b. Determine $r = r(\theta)$ if (i) $h^2 < k$; (ii) $h^2 = k$; (iii) $h^2 > k$.

Show answer

Chapter 6 Applications of Linear Second Order Equations

6.1 Spring Problems I

1 $y=$ \dst$3\cos 4\sqrt{6}t-\frac{1}{2\sqrt{6}}\sin 4\sqrt{6}t$ ft

2 $y=$ \dst$-\frac{1}{4}\cos 8\sqrt{5}t-\frac{1}{4\sqrt{5}}\sin 8\sqrt{5}t$ ft

3 $y=1.5\cos 14\sqrt{10}t$ cm

4 $y=$ \dst$\frac{1}{4}\cos 8t-\frac{1}{16}\sin 8t$ ft; $R=$ \dst$\frac{\sqrt{17}}{16}$ ft; $\omega_0=8$ rad/s; $T=\pi/4$ s;

ϕ

$\approx -.245 \text{ rad}$

$\approx -14.04^\circ$;

5 $y=$ \dst$10\cos 14t+\frac{25}{14}\sin 14t$ cm; $R=$ \dst$\frac{5}{14}\sqrt{809}$ cm; $\omega_0=14$ rad/s; $T=\pi/7$ s;

ϕ

$\approx .177$

rad

$\approx 10.12^\circ$

6 $y=$ \dst$-\frac{1}{4}\cos\sqrt{70}\,t+\frac{2}{\sqrt{70}}\sin\sqrt{70}\,t$ m; $R=$ \dst$\frac{1}{4}\sqrt{\frac{67}{35}}$ m $\omega_0=\sqrt{70}$ rad/s;

T

$=2\pi/\sqrt{70}$

s; ϕ

$\approx 2.38 \text{ rad}$

$\approx 136.28^\circ$

7 $y=$ \dst$\frac{2}{3}\cos 16t-\frac{1}{4}\sin 16t \text{ ft}$

8 $y=$ \dst$\frac{1}{2}\cos 8t-\frac{3}{8}\sin 8t$ ft

9 $.72 \text{ m}$

10 $y=$ \dst$\frac{1}{3}\sin t+\frac{1}{2}\cos 2t+\frac{5}{6}\sin 2t$ ft

11 $y=$ \dst$\frac{16}{5}\left(4\sin\frac{t}{4}-\sin t\right)$

12 $y =$ \dst$-\frac{1}{16}\sin 8t + \frac{1}{3}\cos 4\sqrt{2}t - \frac{1}{8\sqrt{2}}\sin 4\sqrt{2}t$

13 $y =$ \dst$-t\cos 8t - \frac{1}{6}\cos 8t + \frac{1}{8}\sin 8t$ ft

14 $T = 4\sqrt{2}$ s

15 $\omega = 8$ rad/s $y = -$\dst$\frac{t}{16}(-\cos 8t + 2\sin 8t) + \frac{1}{128}\sin 8t$ ft

16 \dst$\omega = 4\sqrt{6}\ \mathrm{rad/s}; \quad y = -\frac{t}{\sqrt{6}}\left[\frac{8}{3}\cos 4\sqrt{6}t + 4\sin 4\sqrt{6}t\right] + \frac{1}{9}\sin 4\sqrt{6}t$ ft

17 $y =$ \dst$\frac{t}{2}\cos 2t - \frac{t}{4}\sin 2t + 3\cos 2t + 2\sin 2t$ m

18 $y =$ \dst$y_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t;\ R = \frac{1}{\omega_0}\sqrt{(\omega_0 y_0)^2 + (v_0)^2}$;

$\cos\phi$
$=$ \dst$\frac{y_0\omega_0}{\sqrt{(\omega_0 y_0)^2+(v_0)^2}}$;
$\sin\phi$
$=$ \dst$\frac{v_0}{\sqrt{(\omega_0 y_0)^2+(v_0)^2}}$

19 The object with the longer period weighs four times as much as the other.

20 $T_2 = \sqrt{2}T_1$, where T_1 is the period of the smaller object.

21 $k_1 = 9k_2$, where k_1 is the spring constant of the system with the shorter period.

6.2 Spring Problems II

1 $y =$ \dst$\frac{e^{-2t}}{2}(3\cos 2t - \sin 2t)$ ft; $\sqrt{\frac{5}{2}}e^{-2t}$ ft

2 $y =$ \dst$-e^{-t}\left(3\cos 3t + \frac{1}{3}\sin 3t\right)$ ft $\frac{\sqrt{82}}{3}e^{-t}$ ft

3 $y =$ \dst$e^{-16t}\left(\frac{1}{4} + 10t\right)$ ft

4 $y =$ \dst$-\frac{e^{-3t}}{4}(5\cos t + 63\sin t)$ ft

5 $0 \leq c < 8$ lb-sec/ft

6 $y =$ \dst$\frac{1}{2}e^{-3t}\left(\cos\sqrt{91}t + \frac{11}{\sqrt{91}}\sin\sqrt{91}t\right)$ ft

7 $y = -$\dst$\frac{e^{-4t}}{3}(2 + 8t)$ ft

8 $y =$ \dst$e^{-10t}\left(9\cos 4\sqrt{6}t + \frac{45}{2\sqrt{6}}\sin 4\sqrt{6}t\right)$ cm

9 \dst$y = e^{-3t/2}\left(\frac{3}{2}\cos\frac{\sqrt{41}}{2}t + \frac{9}{2\sqrt{41}}\sin\frac{\sqrt{41}}{2}t\right)$ ft

10 $y =$ \dst$e^{-\frac{3}{2}t}\left(\frac{1}{2}\cos\frac{\sqrt{119}}{2}t - \frac{9}{2\sqrt{119}}\sin\frac{\sqrt{119}}{2}t\right)$ ft

11 \dst$y = e^{-8t}\left(\frac{1}{4}\cos 8\sqrt{2}t - \frac{1}{4\sqrt{2}}\sin 8\sqrt{2}t\right)$ ft

12 \dst$y = e^{-t}\left(-\frac{1}{3}\cos 3\sqrt{11}t + \frac{14}{9\sqrt{11}}\sin 3\sqrt{11}t\right)$ ft

13 \dst$y_p = \frac{22}{61}\cos 2t + \frac{2}{61}\sin 2t$ ft

14 $y = -$\dst$\frac{2}{3}(e^{-8t} - 2e^{-4t})$

15 $y =$ \dst$e^{-2t}\left(\frac{1}{10}\cos 4t - \frac{1}{5}\sin 4t\right)$ m

16 $y = e^{-3t}(10\cos t - 70\sin t)$ cm

17 \dst$y_p = -\frac{2}{15}\cos 3t + \frac{1}{15}\sin 3t$ ft

18 $y_p =$ \dst$\frac{11}{100}\cos 4t + \frac{27}{100}\sin 4t$ cm

19 \dst$y_p = \frac{42}{73}\cos t + \frac{39}{73}\sin t$ ft

20 $y =$ \dst$-\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$ m

21 $\backslash\mathrm{dst}y_p = \frac{1}{c\omega_0}(-\beta\cos\omega_0 t + \alpha\sin\omega_0 t)$

24 $y = \backslash\mathrm{dst}e^{-ct/2m}\left(y_0\cos\omega_1 t + \frac{1}{\omega_1}(v_0 + \frac{cy_0}{2m})\sin\omega_1 t\right)$

25 $y = \backslash\mathrm{dst}\frac{r_2 y_0 - v_0}{r_2 - r_1}e^{r_1 t} + \frac{v_0 - r_1 y_0}{r_2 - r_1}e^{r_2 t}$

26 $y = e^{r_1 t}\backslash\mathrm{dst}(y_0 + (v_0 - r_1 y_0)t)$

6.3 The RLC Circuit

1 $I = e^{-15t}\backslash\mathrm{dst}\left(2\cos 5\sqrt{15}t - \frac{6}{\sqrt{31}}\sin 5\sqrt{31}t\right)$

2 $I = e^{-20t}(2\cos 40t - 101\sin 40t)$

3 $I = -\backslash\mathrm{dst}\frac{200}{3}e^{-10t}\sin 30t$

4 $I = -10e^{-30t}(\cos 40t + 18\sin 40t)$

5 $I = -e^{-40t}(2\cos 30t - 86\sin 30t)$

6 $I_p = \backslash\mathrm{dst} - \frac{1}{3}(\cos 10t + 2\sin 10t)$

7 $I_p = \backslash\mathrm{dst}\frac{20}{37}(\cos 25t - 6\sin 25t)$

8 $I_p = \backslash\mathrm{dst}\frac{3}{13}(8\cos 50t - \sin 50t)$

9 $I_p = \backslash\mathrm{dst}\frac{20}{123}(17\sin 100t - 11\cos 100t)$

10 $I_p = -\backslash\mathrm{dst}\frac{45}{52}(\cos 30t + 8\sin 30t)$

12 $\omega_0 = 1/\sqrt{LC}$ maximum amplitude $= \sqrt{U^2 + V^2}/R$

6.4 Motion Under a Central Force

1 If $e = 1$, then $Y^2 = \rho(\rho - 2X)$; if $e \neq 1$ $\dst\left(X + \frac{e\rho}{1-e^2}\right)^2 + \dst\frac{Y^2}{1-e^2} = \dst\frac{\rho^2}{(1-e^2)^2}$ if ;

> e
> < 1
> let
> X_0
> $= -\dst\frac{e\rho}{1-e^2}$,
> a
> $= \dst\frac{\rho}{1-e^2}$,
> b
> $= \dst\frac{\rho}{\sqrt{1-e^2}}$.

2 Let $h=r_0^2\theta_0'$; then $\rho=\displaystyle\frac{h^2}{k},\ e=\displaystyle\left[\left(\frac{\rho}{r_0}-1\right)^2+\left(\frac{\rho r_0'}{h}\right)^2\right]^{1/2}$. If $e=0$, then

> θ_0 is undefined, but also irrelevant if $e \neq 0$ then $\phi = \theta_0 - \alpha$, where $-\pi \leq \alpha < \pi$, $\cos\alpha = \displaystyle\frac{1}{e}\left(\frac{\rho}{r_0}-1\right)$ and $\sin\alpha = \displaystyle\frac{\rho r_0'}{eh}$.

3 **(a)** $e=\displaystyle\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}$ **(b)** $r_0=R\gamma_1,\ r_0'=0,\ \theta_0$ arbitrary, $\theta_0'=\displaystyle\left[\frac{2g\gamma_2}{R\gamma_1^3(\gamma_1+\gamma_2)}\right]^{1/2}$

4 $f(r)=-mh^2\displaystyle\left(\frac{6c}{r^4}+\frac{1}{r^3}\right)$

5 $f(r)=-\displaystyle\frac{mh^2(\gamma^2+1)}{r^3}$

<u>6</u> (a) $\dst \frac{d^2u}{d\theta^2} + \left(1 - \frac{k}{h^2}\right) u = 0,\ u(\theta_0) = \dst \frac{1}{r_0},\ \dst \frac{du(\theta_0)}{d\theta} = -\dst \frac{r_0'}{h}$. **(b)** with $\gamma =$

> $\left|1 - \dst \frac{k}{h^2}\right|^{1/2}$:
> (i)
> r
> $= r_0 \left(\cosh \gamma(\theta - \theta_0) - \dst \frac{r_0 r_0'}{\gamma h} \sinh \gamma(\theta - \theta_0)\right)^{-1}$
> (ii)
> r
> $= r_0 \left(1 - \dst \frac{r_0 r_0'}{h} (\theta - \theta_0)\right)^{-1}$;
> (iii)
> r
> $= r_0 \left(\cos \gamma(\theta - \theta_0) - \dst \frac{r_0 r_0'}{\gamma h} \sin \gamma(\theta - \theta_0)\right)^{-1}$