4 Applications of First Order Equations

4 Applications of First Order Equations

IN THIS CHAPTER we consider applications of first order differential equations.

SECTION 4.1 begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other other factors for example, a radioactive susbstance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

SECTION 4.2 deals with applications of Newton’s law of cooling and with mixing problems.

SECTION 4.3 discusses applications to elementary mechanics involving Newton’s second law of motion. The problems considered include motion under the influence of gravity in a resistive medium, and determining the initial velocity required to launch a satellite.

SECTION 4.4 deals with methods for dealing with a type of second order equation that often arises in applications of Newton’s second law of motion, by reformulating it as first order equation with a different independent variable. Although the method doesn’t usually lead to an explicit solution of the given equation, it does provide valuable insights into the behavior of the solutions.

SECTION 4.5 deals with applications of differential equations to curves.

129

Answers to selected exercises in this chapter

4.1 Growth and Decay

Since the applications in this section deal with functions of time, we’ll denote the independent variable by t. If Q is a function of t, Q' will denote the derivative of Q with respect to t; thus,

$$Q' = \frac{dQ}{dt}.$$

Exponential Growth and Decay

One of the most common mathematical models for a physical process is the exponential model, where it’s assumed that the rate of change of a quantity Q is proportional to Q; thus

$$Q' = aQ, \tag{4.1.1}$$

where a is the constant of proportionality.

From Example 3, the general solution of (4.1.1) is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \tag{4.1.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation grows exponentially if $a > 0$, or decays exponentially if $a < 0$ (Figure 4.1.1).

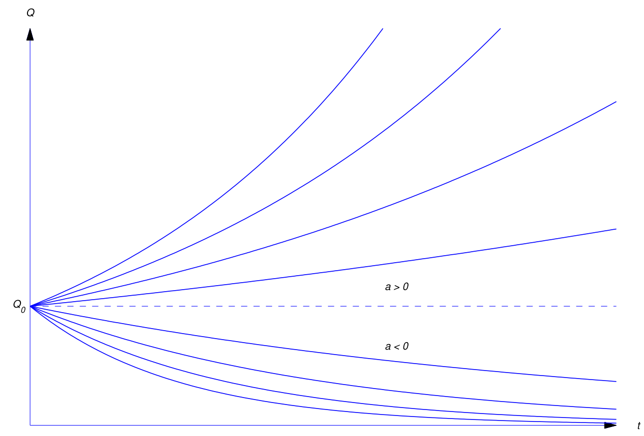

Figure 4.1.1. *Exponential growth and decay*

Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time t satisfies (4.1.1), where a is a negative constant whose value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant a by $-k$, where k is a positive number that we'll call the decay constant of the material. Thus, (4.1.1) becomes

$$Q' = -kQ.$$

If the mass of the material present at $t = t_0$ is Q_0, the mass present at time t is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From (4.1.2) with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \tag{4.1.3}$$

The half–life τ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \tag{4.1.4}$$

From (4.1.3) with $t = \tau + t_0$, (4.1.4) is equivalent to

$$Q_0 e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln\frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k}\ln 2. \tag{4.1.5}$$

(Figure 4.1.2). The half-life is independent of t_0 and Q_0, since it's determined by the properties of material, not by the amount of the material present at any particular time.

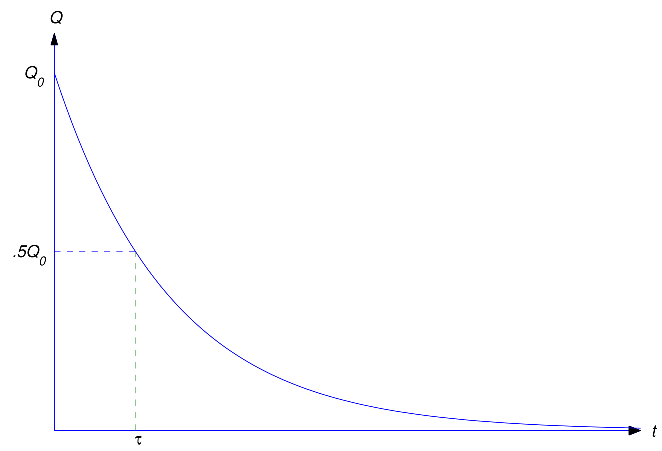

Figure 4.1.2. *Half-life of a radioactive substance*

Example 4.1.1

A radioactive substance has a half-life of 1620 years.

a. If its mass is now 4 g (grams), how much will be left 810 years from now?

b. Find the time t_1 when 1.5 g of the substance remain.

Solution (a) From (4.1.3) with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \tag{4.1.6}$$

where we determine k from (4.1.5), with τ= 1620 years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in (4.1.6) yields

$$Q = 4e^{-(t \ln 2)/1620}. \tag{4.1.7}$$

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810 \ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2} \text{ g.} \end{aligned}$$

Solution (b) Setting $t = t_1$ in (4.1.7) and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln \frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620\frac{\ln 8/3}{\ln 2} \approx 2292.4 \text{ years.}$$

Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r. To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is compounded. If the interest is compounded annually, the value of the account is multiplied by $1+r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1+r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1+r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0\left(1+\frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1+r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0\left(1+\frac{r}{n}\right)^{nt}. \tag{4.1.8}$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 4.1.1 shows the effect of increasing the number of compoundings over $t=5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table 4.1.1. *Table The effect of compound interest*

n (number of compoundings per year)	\dst $\$100\left(1+\frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years)
1	\$133.82
2	\$134.39
4	\$134.68
8	\$134.83
364	\$134.98

You can see from Table 4.1.1 that the value of the account after 5 years is an increasing function of n. Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't ; then competing banks can attract savers by compounding often. The ultimate step in this direction is to compound continuously, by which we mean that $n\to\infty$ in (4.1.8). Since we know from calculus that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r,$$

this yields

$$\begin{aligned} Q(t) \quad \backslash\mathrm{dst} &= \lim_{n\to\infty} Q_0\left(1+\tfrac{r}{n}\right)^{nt} = Q_0\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^n\right]^t \\ &= Q_0e^{rt}. \end{aligned}$$

Observe that $Q = Q_0e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

Example 4.1.2

If \$150 is deposited in a bank that pays $5\frac{1}{2}$% annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it's necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

Example 4.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}$% annual interest compounded continuously. How much must we deposit in the account?

Solution The value of the account at time t is

$$Q(t) = Q_0e^{.055t}. \tag{4.1.9}$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000 = Q_0e^{.55}, \tag{4.1.10}$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in (4.1.9). Solving (4.1.10) for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Mixed Growth and Decay

Example 4.1.4

A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

a. Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t.

b. Find $\lim_{t\to\infty} Q(t)$.

Solution (a) Here

$$Q' = \text{ rate of increase of } Q - \text{ rate of decrease of } Q.$$

The rate of increase is the constant a. Since Q is radioactive with decay constant k, the rate of decrease is kQ. Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \tag{4.1.11}$$

Since e^{-kt} is a solution of the complementary equation, the solutions of (4.1.11) are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right) e^{-kt}. \tag{4.1.12}$$

Solution (b) Since $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$, so from (4.1.12)

$$\lim_{t\to\infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k, and not on Q_0. We say that a/k is the steady state value of Q. From (4.1.12) we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (Figure 4.1.3).

Figure 4.1.3. *$Q(t)$ approaches the steady state value \dst$\frac{a}{k}$ as $t \to \infty$*

Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of carbon dating, devised by the American chemist and Nobel Prize Winner W.S. Libby (http://www.nobelprize.org/nobel_prizes/chemistry/laureates/1960/libby-lecture.pdf).

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R, has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R. However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

Example 4.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t, the number of years since death occurred. This yields

$$t = -5570 \frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = .42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570 \frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570 \frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it’s reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

A Savings Program

> **Example 4.1.6**
>
> A person opens a savings account with an initial deposit of $1000 and subsequently deposits$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution Observe that Q isn’t continuous, since there are 52 discrete deposits per year of $50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate o$
2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + .06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \tag{4.1.13}$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We’ll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of (4.1.13) are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06} e^{-.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \tag{4.1.14}$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{.06},$$

and substituting this into (4.1.14) yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1), \tag{4.1.15}$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits. ∎

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by (4.1.15) (See Exercise 21.). The follwing table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Unsupported use of \hfill

4.1 Exercises

1. The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t > 0$ if $Q(0) = 20$ g.

Show answer

2. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

Show answer

3. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

Show answer

4. A tree contains a known percentage p_0 of a radioactive substance with half-life τ. When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1, how long has the tree been dead?

Show answer

5. If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?

Show answer

6. Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2.

Show answer

7. A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k = .1(\mathrm{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time t, find $\lim_{t\to\infty} Q(t)$.

Show answer

8. A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?

Show answer

9. At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

Show answer

10. A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.

a. How long will it take for the value of the account to triple?

b. What is Q_0 if the value of the account after 10 years is $100,000 dollars?

Show answer

11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t.

a. Find $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at $t = 0$.

b. Find $\lim_{t\to\infty} Q(t)$.

Show answer

12. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?

Show answer

13. A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0) = V_0$, find V at any time t. How long will it take for V to increase to $100V_0$?

Show answer

14. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

Show answer

15. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time t. Find $W(t)$ and $\lim_{t\to\infty} W(t)$ if $W(0) = 1$.

Show answer

16. A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to 1/10 of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t, and find $\lim_{t\to\infty} S(t)$.

Show answer

17. A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

Show answer

18. A person deposits $25,000 inabankthatpays5yearinterest, compoundedcontinuously. Thepersoncontinuouslywithdrawsfromtheaccountattherateof$750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.

Show answer

19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth W of the fortune as a function of t if it was $1 million 6 monthsagoandis$4 million today.

Show answer

20. Let $p = p(t)$ be the quantity of a product present at time t. The product is manufactured continuously at a rate proportional to p, with proportionality constant 1/2, and it's consumed continuously at a rate proportional to p^2, with proportionality constant 1/8. Find $p(t)$ if $p(0) = 100$.

Show answer

21.

a. In the situation of Example 4.1.6 find the exact value $P(t)$ of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT

At time t the initial $\$1000$ has been on deposit for t years. There have been $52t$ deposits of $\$50$ each. The first $\$50$ has been on deposit for $t - 1/52$ years, the second for $t - 2/52$ years $\cdots$ in general, the jth $\$50$ has been on deposit for $t - j/52$ years $(1 \le j \le 52t)$. Find the present value of each $\$50$ deposit assuming 6% interest compounded continuously, and use the formula

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad (x \neq 1)$$

to find their total value.

b. Let

$$p(t) = \frac{Q(t) - P(t)}{P(t)}$$

be the relative error after t years. Find

$$p(\infty) = \lim_{t\to\infty} p(t).$$

Show answer

22. A homebuyer borrows P_0 dollars at an annual interest rate r, agreeing to repay the loan with equal monthly payments of M dollars per month over N years.

a. Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t > 0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See Example 4.1.6 .)

b. Solve the equation derived in **(a)**.

c. Use the result of **(b)** to determine an approximate value for M assuming that each year has exactly 12 months of equal length.

d. It can be shown that the exact value of M is given by

$$M = \frac{rP_0}{12}\left(1 - (1 + r/12)^{-12N}\right)^{-1}.$$

Compare the value of M obtained from the answer in **(c)** to the exact value if (i) $P_0 = \$50,000$, $r = 7\frac{1}{2}$%, $N = 20$ (ii) $P_0 = \$150,000$, $r = 9.0$%, $N = 30$.

Show answer

23. Assume that the homebuyer of Exercise 22 elects to repay the loan continuously at the rate of αM dollars per month, where α is a constant greater than 1. (This is called accelerated payment.)

a. Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.

b. Suppose $P_0 = \$50,000$, $r = 8$%, and $N = 15$. Compute the savings realized by accelerated payments with $\alpha = 1.05, 1.10$, and 1.15.

Show answer

24. A benefactor wishes to establish a trust fund to pay a researcher's salary for T years. The salary is to start at S_0 dollars per year and increase at a fractional rate of a per year. Find the amount of money P_0 that the benefactor must deposit in a trust fund paying interest at a rate r per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

Show answer

25. L A radioactive substance with decay constant k is produced at the rate of

$$\frac{at}{1 + btQ(t)}$$

units of mass per unit time, where a and b are positive constants and $Q(t)$ is the mass of the substance present at time t; thus, the rate of production is small at the start and tends to slow when Q is large.

a. Set up a differential equation for Q.

b. Choose your own positive values for a, b, k, and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \to \infty$. (Be precise, expressing your conclusions in terms of a, b, k. However, no proof is required.)

26. L Follow the instructions of Exercise 25, assuming that the substance is produced at the rate of $at/(1 + bt(Q(t))^2)$ units of mass per unit of time.

27. L Follow the instructions of Exercise 25, assuming that the substance is produced at the rate of $at/(1 + bt)$ units of mass per unit of time.

4.2 Cooling and Mixing

Newton's Law of Cooling

Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature $T_m(t)$, the rate of change of T at time t is proportional to $T(t) - T_m(t)$; thus, T satisfies a differential equation of the form

$$T' = -k(T - T_m). \tag{4.2.1}$$

Here $k > 0$, since the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. We'll call k the temperature decay constant of the medium.

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m. This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it's reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it's a huge cauldron of molten metal. (For more on this see Exercise 17.)

To solve (4.2.1), we rewrite it as

$$T' + kT = kT_m.$$

Since e^{-kt} is a solution of the complementary equation, the solutions of this equation are of the form $T = ue^{-kt}$, where $u'e^{-kt} = kT_m$, so $u' = kT_m e^{kt}$. Hence,

$$u = T_m e^{kt} + c,$$

so

$$T = ue^{-kt} = T_m + ce^{-kt}.$$

If $T(0) = T_0$, setting $t = 0$ here yields $c = T_0 - T_m$, so

$$T = T_m + (T_0 - T_m)e^{-kt}. \tag{4.2.2}$$

Note that $T - T_m$ decays exponentially, with decay constant k.

Example 4.2.1

A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is its temperature after 8 minutes?

Solution Here $T_0 = 400$ and $T_m = 25$, so (4.2.2) becomes

$$T = 25 + 375e^{-kt}. \tag{4.2.3}$$

We determine k from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4} \ln \frac{7}{15} = \frac{1}{4} \ln \frac{15}{7}.$$

Substituting this into (4.2.3) yields

$$T = 25 + 375e^{-\frac{t}{4} \ln \frac{15}{7}}$$

(Figure 4.2.1). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) \quad & \backslash\text{dst} = 25 + 375e^{-2 \ln \frac{15}{7}} \\ & \backslash\text{dst} = 25 + 375 \left(\tfrac{7}{15}\right)^2 \approx 107^\circ\text{C}. \end{aligned}$$

Figure 4.2.1. $T = 25 + 375e^{-(t/4) \ln 15/7}$

Example 4.2.2

An object with temperature 72°F is placed outside, where the temperature is -20°F. At 11:05 the temperature of the object is 60°F and at 11:07 its temperature is 50°F. At what time was the object placed outside?

Solution Let $T(t)$ be the temperature of the object at time t. For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time τ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into (4.2.2) yields

$$T = -20 + \big(60 - (-20)\big)e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \tag{4.2.4}$$

We obtain k from the stated condition that the temperature of the object is 50°F at 11:07. Since 11:07 is $t = 2$ on our time scale, we can determine k by substituting $T = 50$ and $t = 2$ into (4.2.4) to obtain

$$50 = -20 + 80e^{-2k}$$

(Figure 4.2.2); hence,

$$e^{-2k} = \frac{70}{80} = \frac{7}{8}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{2}\ln\frac{7}{8} = \frac{1}{2}\ln\frac{8}{7}.$$

Substituting this into (4.2.4) yields

$$T = -20 + 80e^{-\frac{t}{2}\ln\frac{8}{7}},$$

and the condition $T(\tau) = 72$ implies that

$$72 = -20 + 80e^{-\frac{\tau}{2}\ln\frac{8}{7}};$$

hence,

$$e^{-\frac{\tau}{2}\ln\frac{8}{7}} = \frac{92}{80} = \frac{23}{20}.$$

Taking logarithms and solving for τ yields

$$\tau = -\frac{2\ln\frac{23}{20}}{\ln\frac{8}{7}} \approx -2.09 \text{ min}.$$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

Figure 4.2.2. $T = -20 + 80e^{-\frac{t}{2}\ln\frac{8}{7}}$

Mixing Problems

In the next two examples a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. This is an example of a mixing problem. To construct a tractable mathematical model for mixing problems we assume in our examples (and most exercises) that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Exercises 22 and 23 deal with situations where this isn't so, but the distribution of salt becomes approximately uniform as $t \to \infty$.

Example 4.2.3

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (Figure 4.2.3).

a. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.

b. Find $\lim_{t\to\infty} Q(t)$.

Figure 4.2.3. *A mixing problem*

Solution (a) To find a differential equation for Q, we must use the given information to derive an expression for Q'. But Q' is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if rate in denotes the rate at which salt enters the tank and rate out denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \tag{4.2.5}$$

The rate in is

$$\left(\frac{1}{2}\ \text{lb/gal}\right) \times (4\ \text{gal/min}) = 2\ \text{lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we're removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t, the rate out at any time t is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned}
\text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\
&= (\text{lb/gal}) \times (\text{gal/min}) \\
&= \backslash\mathrm{dst}\tfrac{Q(t)}{600} \times 4 = \backslash\mathrm{dst}\tfrac{Q(t)}{150}.
\end{aligned}$$

We can now write (4.2.5) as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \tag{4.2.6}$$

(Figure 4.2.4). Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

Solution (b) From (4.2.6), we see that that $\lim_{t\to\infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

Example 4.2.4

A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure 4.2.5). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Figure 4.2.4. $Q = 300 - 260e^{-t/150}$

Figure 4.2.5. *Another mixing problem*

Solution We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we're adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \le t \le 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t, where $0 \le t \le 150$. As in Example 4.2.3,

$$Q' = \text{rate in} - \text{rate out}. \tag{4.2.7}$$

The rate in is

$$\left(\frac{1}{4}\text{ g/liter}\right) \times (4\text{ liters/min}) = 1\text{ g/min}. \tag{4.2.8}$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ liters in the tank at time t, the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t+200} = \frac{1}{t+100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t,

$$\text{rate out} = \frac{Q(t)}{t+100}. \tag{4.2.9}$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned}\text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) = (\text{g/liter}) \times (\text{liters/min}) \\ &= \backslash\text{dst}\frac{Q(t)}{2t+200} \times 2 = \backslash\text{dst}\frac{Q(t)}{t+100}.\end{aligned}$$

Substituting (4.2.8) and (4.2.9) into (4.2.7) yields

$$Q' = 1 - \frac{Q}{t+100}, \quad \text{so} \quad Q' + \frac{1}{t+100}Q = 1. \tag{4.2.10}$$

By separation of variables, $1/(t+100)$ is a solution of the complementary equation, so the solutions of (4.2.10) are of the form

$$Q = \frac{u}{t+100}, \quad \text{where} \quad \frac{u'}{t+100} = 1, \quad \text{so} \quad u' = t + 100.$$

Hence,

$$u = \frac{(t+100)^2}{2} + c. \tag{4.2.11}$$

Since $Q(0) = 10$ and $u = (t+100)Q$, (4.2.11) implies that

$$(100)(10) = \frac{(100)^2}{2} + c,$$

so

$$c = 100(10) - \frac{(100)^2}{2} = -4000$$

and therefore

$$u = \frac{(t+100)^2}{2} - 4000.$$

Hence,

$$Q = \frac{u}{t+200} = \frac{t+100}{2} - \frac{4000}{t+100}.$$

Now let $K(t)$ be the concentration of salt at time t. Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$$

(Figure 4.2.6).

Figure 4.2.6. $K(t) = \dst\frac{1}{4} - \frac{2000}{(t+100)^2}$

4.2 Exercises

1. A thermometer is moved from a room where the temperature is 70°F to a freezer where the temperature is $12^\circ F$. After 30 seconds the thermometer reads 40°F. What does it read after 2 minutes?

Show answer

2. A fluid initially at 100°C is placed outside on a day when the temperature is -10°C, and the temperature of the fluid drops 20°C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.

Show answer

3. At 12:00 PM a thermometer reading 10°F is placed in a room where the temperature is 70°F. It reads 56° when it's placed outside, where the temperature is 5°F, at 12:03. What does it read at 12:05 PM?

Show answer

4. A thermometer initially reading 212°F is placed in a room where the temperature is 70°F. After 2 minutes the thermometer reads 125°F.

a. What does the thermometer read after 4 minutes?

b. When will the thermometer read 72°F?

c. When will the thermometer read 69°F?

Show answer

5. An object with initial temperature 150°C is placed outside, where the temperature is 35°C. Its temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively.

a. At what time was the object placed outside?

b. When will its temperature be 40°C?

Show answer

6. An object is placed in a room where the temperature is 20°C. The temperature of the object drops by 5°C in 4 minutes and by 7°C in 8 minutes. What was the temperature of the object when it was initially placed in the room?

Show answer

7. A cup of boiling water is placed outside at 1:00 PM. One minute later the temperature of the water is 152°F. After another minute its temperature is 112° F. What is the outside temperature?

Show answer

8. A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gal/min, and the resulting solution dranes out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

Show answer

9. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with 1/2 pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

Show answer

10. A tank initially contains 100 liters of a salt solution with a concentration of .1 g/liter. A solution with a salt concentration of .3 g/liter is added to the tank at 5 liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of t.

Show answer

11. A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with 1/4 pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank as it's about to overflow.

Show answer

12. Suppose water is added to a tank at 10 gal/min, but leaks out at the rate of 1/5 gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?

Show answer

13. A chemical reaction in a laboratory with volume V (in ft^3) produces q_1 ft^3/min of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\overline{c}$, but harmless at concentrations $\le \overline{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of q_2 ft^3/min and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration c_0 is at a safe level, find the smallest value of q_2 required to maintain safe conditions in the laboratory for all time.

Show answer

14. A 1200-gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t > 0$ prior to overflow.

Show answer

15. Tank T_1 initially contain 50 gallons of pure water. Starting at $t_0 = 0$, water that contains 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2, which initially contains 50 gallons of pure water. Also starting at $t_0 = 0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into T_2 at the rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min.

a. Find a differential equation for the quantity $Q(t)$ of salt in tank T_2 at time $t > 0$.

b. Solve the equation derived in **(a)** to determine $Q(t)$.

c. Find $\lim_{t\to\infty} Q(t)$.

Show answer

16. Suppose an object with initial temperature T_0 is placed in a sealed container, which is in turn placed in a medium with temperature T_m. Let the initial temperature of the container be S_0. Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)

a. Assuming that the container and the medium have distinct temperature decay constants k and k_m respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.

b. Assuming that the container and the medium have the same temperature decay constant k, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.

c. Find $\lim_{\cdot t\to\infty} S(t)$ and $\lim_{t\to\infty} T(t)$.

Show answer

17. In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium, respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by Newton's law of cooling,

$$T' = -k(T - T_m). \tag{A}$$

We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and that the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0. \tag{B}$$

a. Equation (A) involves two unknown functions T and T_m. Use (A) and (B) to derive a differential equation involving only T.

b. Find $T(t)$ and $T_m(t)$ for $t > 0$.

c. Find $\lim_{t\to\infty} T(t)$ and $\lim_{t\to\infty} T_m(t)$.

Show answer

18. Control mechanisms allow fluid to flow into a tank at a rate proportional to the volume V of fluid in the tank, and to flow out at a rate proportional to V^2. Suppose $V(0) = V_0$ and the constants of proportionality are a and b, respectively. Find $V(t)$ for $t > 0$ and find $\lim_{t\to\infty} V(t)$.

Show answer

19. Identical tanks T_1 and T_2 initially contain W gallons each of pure water. Starting at $t_0 = 0$, a salt solution with constant concentration c is pumped into T_1 at r gal/min and drained from T_1 into T_2 at the same rate. The resulting mixture in T_2 is also drained at the same rate. Find the concentrations $c_1(t)$ and $c_2(t)$ in tanks T_1 and T_2 for $t > 0$.

Show answer

20. An infinite sequence of identical tanks T_1, T_2, …, T_n, …, initially contain W gallons each of pure water. They are hooked together so that fluid drains from T_n into T_{n+1} $(n = 1, 2, \cdots)$. A salt solution is circulated through the tanks so that it enters and leaves each tank at the constant rate of r gal/min. The solution has a concentration of c pounds of salt per gallon when it enters T_1.

a. Find the concentration $c_n(t)$ in tank T_n for $t > 0$.

b. Find $\lim_{t\to\infty} c_n(t)$ for each n.

Show answer

21. Tanks T_1 and T_2 have capacities W_1 and W_2 liters, respectively. Initially they are both full of dye solutions with concentrations c_1 and c_2 grams per liter. Starting at $t_0 = 0$, the solution from T_1 is pumped into T_2 at a rate of r liters per minute, and the solution from T_2 is pumped into T_1 at the same rate.

a. Find the concentrations $c_1(t)$ and $c_2(t)$ of the dye in T_1 and T_2 for $t > 0$.

b. Find $\lim_{t\to\infty} c_1(t)$ and $\lim_{t\to\infty} c_2(t)$.

Show answer

22. L Consider the mixing problem of Example 4.2.3, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{150} Q = 2$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} Q(t)$?.

b. Use numerical methods to confirm your guess in the these cases:

$$\text{\parti}\ \ a(t) = t/(1+t) \quad \text{\partii}\ \ a(t) = 1 - e^{-t^2} \quad \text{\partiii}\ \ a(t) = 1 - \sin(e^{-t}).$$

23. L Consider the mixing problem of Example 4.2.4 in a tank with infinite capacity, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{t+100} Q = 1$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Let $K(t)$ be the concentration of salt at time t. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} K(t)$?

b. Use numerical methods to confirm your guess in the these cases:

$$\text{\parti}\ \ a(t) = t/(1+t) \quad \text{\partii}\ \ a(t) = 1 - e^{-t^2} \quad \text{\partiii}\ \ a(t) = 1 + \sin(e^{-t}).$$

4.3 Elementary Mechanics

Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F. Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t, and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t. Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t. Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \tag{4.3.1}$$

Units

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 1 shows the other units. Consistent with (4.3.1), the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)$/s^2$ to one unit of mass.

	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

Table 1.

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg, where g is called the acceleration due to gravity. (To be completely accurate, g should be called the magnitude of the acceleration due to gravity at Earth's surface.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British)}. \end{aligned}$$

In general, the force F in (4.3.1) may depend upon t, y, and y'. Since $a = y''$, (4.3.1) can be written in the form

$$my'' = F(t, y, y'), \tag{4.3.2}$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which (4.3.2) can be recast as a first order equation. This is possible if F does not depend on y, so (4.3.2) is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v:

$$mv' = F(t, v). \tag{4.3.3}$$

Solving this equation yields v as a function of t. If we know $y(t_0)$ for some time t_0, we can integrate v to obtain y as a function of t.

Equations of the form (4.3.3) occur in problems involving motion through a resisting medium.

Motion Through a Resisting Medium Under Constant Gravitational Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \to \infty$. We call this limit the terminal velocity.

> **Example 4.3.1**
>
> An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v.) Find the velocity of the object as a function of t, and find the terminal velocity. Assume that the initial velocity is v_0.

Solution The total force acting on the object is

$$F = -mg + F_1, \tag{4.3.4}$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \le 0$), the resisting force is upward (Figure 4.3.1**(a)**), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \geq 0$), the resisting force is downward (Figure 4.3.1**(b)**), so

$$F_1 = -k|v| = -kv.$$

Thus, (4.3.4) can be written as

$$F = -mg - kv, \tag{4.3.5}$$

regardless of the sign of the velocity.

Figure 4.3.1. *Resistive forces*

From Newton's second law of motion,

$$F = ma = mv',$$

so (4.3.5) yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \tag{4.3.6}$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of (4.3.6) are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \tag{4.3.7}$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and (4.3.7) becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \to \infty$ here shows that the terminal velocity is

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (Figure 4.3.2).

Figure 4.3.2. *Solutions of* $mv' = -mg - kv$

Example 4.3.2

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t, and find its terminal velocity.

Solution Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \tag{4.3.8}$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in (4.3.8) yields

$$60 = -320 + c,$$

so $c = 380$, and (4.3.8) becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t\to\infty} v(t) = -320 \text{ ft/s}.$$

Example 4.3.3

A 10 kg mass is given an initial velocity $v_0 \le 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t, and find the terminal velocity.

Solution Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \tag{4.3.9}$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2 \text{ N-s}^2/\text{m}^2$. Since $m = 10$ and $g = 9.8$, (4.3.9) becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \tag{4.3.10}$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \geq 0$. If $v_0 \neq -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49} v' = \frac{1}{5}, \tag{4.3.11}$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v-7)(v+7)} = \frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]. \tag{4.3.12}$$

Substituting (4.3.12) into (4.3.11) yields

$$\frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right] v' = \frac{1}{5},$$

so

$$\left[\frac{1}{v-7} - \frac{1}{v+7}\right] v' = \frac{14}{5}.$$

Integrating this yields

$$\ln|v-7| - \ln|v+7| = 14t/5 + k.$$

Therefore

$$\left|\frac{v-7}{v+7}\right| = e^k e^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v-7)/(v+7)$ can't change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7} = ce^{14t/5}, \tag{4.3.13}$$

which is an implicit solution of (4.3.10). Solving this for v yields

$$v = -7\frac{c + e^{-14t/5}}{c - e^{-14t/5}}. \tag{4.3.14}$$

Since $v(0) = v_0$, it (4.3.13) implies that

$$c = \frac{v_0 - 7}{v_0 + 7}.$$

Substituting this into (4.3.14) and simplifying yields

$$v = -7\frac{v_0(1 + e^{-14t/5}) - 7(1 - e^{-14t/5})}{v_0(1 - e^{-14t/5}) - 7(1 + e^{-14t/5})}.$$

Since $v_0 \leq 0$, v is defined and negative for all $t > 0$. The terminal velocity is

$$\lim_{t\to\infty} v(t) = -7 \text{ m/s},$$

independent of v_0. More generally, it can be shown (Exercise 11) that if v is any solution of (4.3.9) such that $v_0 \leq 0$ then

$$\lim_{t\to\infty} v(t) = -\sqrt{\frac{mg}{k}}$$

(Figure 4.3.3).

Figure 4.3.3. Solutions of $mv' = -mg + kv^2,\ v(0) = v_0 \le 0$

Example 4.3.4

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \le t < T$ and $v(T) = 0$. Although the mass of the object and our assumptions concerning the forces acting on it are the same as those in Example 3, (4.3.10) does not apply here, since the resisting force is negative if $v > 0$; therefore, we replace (4.3.10) by

$$10v' = -98 - 2v^2. \tag{4.3.15}$$

Separating variables yields

$$\frac{5}{v^2+49}v' = -1,$$

and integrating this yields

$$\frac{5}{7}\tan^{-1}\frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan\theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7}\tan^{-1}\frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7}\tan^{-1}\frac{v}{7} = -t + \frac{5}{7}\tan^{-1}\frac{v_0}{7}, \quad 0 \le t \le T. \tag{4.3.16}$$

Solving this for v yields

$$v = 7\tan\left(-\frac{7t}{5} + \tan^{-1}\frac{v_0}{7}\right). \tag{4.3.17}$$

Using the identity

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1}\theta) = \theta$, we can simplify (4.3.17) to

$$v = 7\frac{v_0 - 7\tan(7t/5)}{7 + v_0\tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, (4.3.16) implies that

$$-T + \frac{5}{7}\tan^{-1}\frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7}\tan^{-1}\frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0, the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 4.3.4 shows graphs of v over $[0, T]$ for various values of v_0.

Figure 4.3.4. *Solutions of (4.3.15) for various $v_0 > 0$*

Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth, where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude y above Earth is

$$F = -\frac{K}{(y+R)^2}, \tag{4.3.18}$$

where R is Earth's radius (Figure 4.3.5).

Figure 4.3.5. *Escape velocity*

Since $F = -mg$ when $y = 0$, setting $y = 0$ in (4.3.18) yields

$$-mg = -\frac{K}{R^2};$$

therefore $K = mgR^2$ and (4.3.18) can be written more specifically as

$$F = -\frac{mgR^2}{(y+R)^2}. \tag{4.3.19}$$

From Newton's second law of motion,

$$F = m\frac{d^2y}{dt^2},$$

so (4.3.19) implies that

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y+R)^2}. \tag{4.3.20}$$

We'll show that there's a number v_e, called the escape velocity, with these properties:

1. If $v_0 \geq v_e$ then $v(t) > 0$ for all $t > 0$, and the vehicle continues to climb for all $t > 0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t\to\infty} y(t) = \infty$ in this case? For a proof, see Exercise 20.)
2. If $v_0 < v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

Since (4.3.20) is second order, we can't solve it by methods discussed so far. However, we're concerned with v rather than y, and v is easier to find. Since $v = y'$ the chain rule implies that

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting this into (4.3.20) yields the first order separable equation

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}. \tag{4.3.21}$$

When $t = 0$, the velocity is v_0 and the altitude is h. Therefore we can obtain v as a function of y by solving the initial value problem

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(h) = v_0.$$

Integrating (4.3.21) with respect to y yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \tag{4.3.22}$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so (4.3.22) becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right). \tag{4.3.23}$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

the parenthetical expression in (4.3.23) is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity v_e. We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left(\frac{2gR^2}{h+R}\right)^{1/2}. \tag{4.3.24}$$

If (4.3.24) holds then the parenthetical expression in (4.3.23) is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

4.3 Exercises

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg. In exercises involving vertical motion take the upward direction to be positive.

1. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.

   Show answer

2. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality k. Find k, given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t. Assume that she starts from rest.

   Show answer

3. A boat weighs 64,000 lb. Its propellor produces a constant thrust of 50,000 lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

   Show answer

4. A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with .5 N for every m/s of speed. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.

   Show answer

5. A stone weighing 1/2 lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to speed, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.

   Show answer

6. A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k = 1 \text{ lb-s}^2/ft^2$. Also find its terminal velocity.

   Show answer

7. A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is -128 ft/s.

   Show answer

8. An object with mass m moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y = y(t)$ be the altitude of the object at time t, with $y(0) = y_0$. Use the results of Example 4.3.1 to show that

$$y(t) = y_0 + \frac{m}{k}(v_0 - v - gt).$$

9. An object with mass m is launched vertically upward with initial velocity v_0 from Earth's surface ($y_0 = 0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time T when the object attains its maximum altitude y_m. Then use the result of Exercise 8 to find y_m.

   Show answer

10. An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is 1 lb when $|v| = 4 \text{ ft/s}$. Find v for $t > 0$, and find its terminal velocity.

    Show answer

11. An object with mass m is given an initial velocity $v_0 \leq 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t > 0$, and find its terminal velocity.

    Show answer

12. An object with mass m is launched vertically upward with initial velocity v_0 in a medium that exerts a resistive force with magnitude proportional to the square of the speed.

    a. Find the time T when the object reaches its maximum altitude.

    b. Use the result of Exercise 11 to find the velocity of the object for $t > T$.

    Show answer

13. L An object with mass m is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force of the form $a|v|/(1+|v|)$, where a is positive constant.

a. Set up a differential equation for the speed of the object.

b. Use your favorite numerical method to solve the equation you found in **(a)**, to convince yourself that there's a unique number a_0 such that $\lim_{t\to\infty} s(t) = \infty$ if $a \le a_0$ and $\lim_{t\to\infty} s(t)$ exists (finite) if $a > a_0$. (We say that a_0 is the bifurcation value of a.) Try to find a_0 and $\lim_{t\to\infty} s(t)$ in the case where $a > a_0$. *Hint: See Exercise* $???$.

Show answer

14. An object of mass m falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and f is strictly increasing and differentiable on $(0, \infty)$.

a. Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \ge 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).

b. Show that if $\lim_{s\to\infty} f(s) \le mg$ then $\lim_{t\to\infty} s(t) = \infty$.

c. Show that if $\lim_{s\to\infty} f(s) > mg$ then $\lim_{t\to\infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$. *Hint: Use Theorem 2.3.1*.

Show answer

15. A 100-g mass with initial velocity $v_0 \le 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is $.1$ N if the speed is 3 m/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.

b. Use Exercise 14(c) to determine the terminal velocity of the object.

c. C To confirm your answer to **(b)**, use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 1]$ (seconds) of the initial value problem of **(a)**, with initial values $v_0 = 0$, -2, -4, …, -12. Present your results in graphical form similar to Figure 4.3.3.

Show answer

16. A 64-lb object with initial velocity $v_0 \le 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.

b. Use Exercise 14(c) to determine the terminal velocity of the object.

c. C To confirm your answer to **(b)**, use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 4]$ (seconds) of the initial value problem of **(a)**, with initial values $v_0 = 0$, -5, -10, …, -30. Present your results in graphical form similar to Figure 4.3.3.

Show answer

In Exercises 17-20, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

Show answer

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s². Find the escape velocity in miles/s.

Show answer

19. a. Show that Eqn. (4.3.23) can be rewritten as

$$v^2 = \frac{h-y}{y+R} v_e^2 + v_0^2.$$

b. Show that if $v_0 = \rho v_e$ with $0 \le \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1-\rho^2}.$$

c. By requiring that $v(y_m) = 0$, use Eqn. (4.3.22) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1-\rho^2)(y_m - y)}{y+R} \right]^{1/2},$$

where y_m and ρ are as defined in **(b)** and $y \ge h$.

d. Deduce from **(c)** that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

20. In the situation considered in the discussion of escape velocity, show that $\lim_{t\to\infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

Hint: Use a proof by contradiction. Assume that there's a number y_m such that $y(t) \le y_m$ for all $t > 0$. Deduce from this that there's positive number α such that $y''(t) \le -\alpha$ for all $t \ge 0$. Show that this contradicts the assumption that $v(t) > 0$ for all $t > 0$.

Show answer

4.4 Autonomous Second Order Equations

A second order differential equation that can be written as

$$y'' = F(y, y') \tag{4.4.1}$$

where F is independent of t, is said to be autonomous. An autonomous second order equation can be converted into a first order equation relating $v = y'$ and y. If we let $v = y'$, (4.4.1) becomes

$$v' = F(y, v). \tag{4.4.2}$$

Since

$$v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}, \tag{4.4.3}$$

(4.4.2) can be rewritten as

$$v\frac{dv}{dy} = F(y, v). \tag{4.4.4}$$

The integral curves of (4.4.4) can be plotted in the (y, v) plane, which is called the Poincaré phase plane (http://www-history.mcs.st-and.ac.uk/Mathematicians/Poincare.html) of (4.4.1). If y is a solution of (4.4.1) then $y = y(t), v = y'(t)$ is a parametric equation for an integral curve of (4.4.4). We'll call these integral curves trajectories of (4.4.1), and we'll call (4.4.4) the phase plane equivalent of (4.4.1).

In this section we'll consider autonomous equations that can be written as

$$y'' + q(y, y')y' + p(y) = 0. \tag{4.4.5}$$

Equations of this form often arise in applications of Newton's second law of motion. For example, suppose y is the displacement of a moving object with mass m. It's reasonable to think of two types of time-independent forces acting on the object. One type - such as gravity - depends only on position. We could write such a force as $-mp(y)$. The second type - such as atmospheric resistance or friction - may depend on position and velocity. (Forces that depend on velocity are called damping forces.) We write this force as $-mq(y, y')y'$, where $q(y, y')$ is usually a positive function and we've put the factor y' outside to make it explicit that the force is in the direction opposing the motion. In this case Newton's, second law of motion leads to (4.4.5).

The phase plane equivalent of (4.4.5) is

$$v\frac{dv}{dy} + q(y, v)v + p(y) = 0. \tag{4.4.6}$$

Some statements that we'll be making about the properties of (4.4.5) and (4.4.6) are intuitively reasonable, but difficult to prove. Therefore our presentation in this section will be informal: we'll just say things without proof, all of which are true if we assume that $p = p(y)$ is continuously differentiable for all y and $q = q(y, v)$ is continuously differentiable for all (y, v). We begin with the following statements:

- **Statement 1.** If y_0 and v_0 are arbitrary real numbers then $(???)$ has a unique solution on $(-\infty, \infty)$ such that $y(0) = y_0$ and $y'(0) = v_0$.
- **Statement 2.)** If $y = y(t)$ is a solution of $(???)$ and τ is any constant then $y_1 = y(t - \tau)$ is also a solution of $(???)$, and y and y_1 have the same trajectory.
- **Statement 3.** If two solutions y and y_1 of $(???)$ have the same trajectory then $y_1(t) = y(t - \tau)$ for some constant τ.
- **Statement 4.** Distinct trajectories of $(???)$ can't intersect; that is, if two trajectories of $(???)$ intersect, they are identical.
- **Statement 5.** If the trajectory of a solution of (4.4.5) is a closed curve then $(y(t), v(t))$ traverses the trajectory in a finite time T, and the solution is periodic with period T; that is, $y(t + T) = y(t)$ for all t in $(-\infty, \infty)$.

If $\overline{y}$ is a constant such that $p(\overline{y}) = 0$ then $y \equiv \overline{y}$ is a constant solution of (4.4.5). We say that $\overline{y}$ is an equilibrium of (4.4.5) and $(\overline{y}, 0)$ is a critical point of the phase plane equivalent equation (4.4.6). We say that the equilibrium and the critical point are stable if, for any given $\epsilon > 0$ no matter how small, there's a $\delta > 0$, sufficiently small, such that if

$$\sqrt{(y_0 - \overline{y})^2 + v_0^2} < \delta$$

then the solution of the initial value problem

$$y'' + q(y, y')y' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = v_0$$

satisfies the inequality

$$\sqrt{(y(t) - \overline{y})^2 + (v(t))^2} < \epsilon$$

for all $t > 0$. Figure 4.4.1 illustrates the geometrical interpretation of this definition in the Poincaré phase plane: if (y_0, v_0) is in the smaller shaded circle (with radius δ), then $(y(t), v(t))$ must be in in the larger circle (with radius ϵ) for all $t > 0$.

Figure 4.4.1. *Stability: if (y_0, v_0) is in the smaller circle then $(y(t), v(t))$ is in the larger circle for all $t > 0$*

If an equilibrium and the associated critical point are not stable, we say they are unstable. To see if you really understand what stable means, try to give a direct definition of unstable (Exercise 22). We'll illustrate both definitions in the following examples.

The Undamped Case

We'll begin with the case where $q \equiv 0$, so (4.4.5) reduces to

$$y'' + p(y) = 0. \tag{4.4.7}$$

We say that this equation - as well as any physical situation that it may model - is undamped. The phase plane equivalent of (4.4.7) is the separable equation

$$v\frac{dv}{dy} + p(y) = 0.$$

Integrating this yields

$$\frac{v^2}{2} + P(y) = c, \tag{4.4.8}$$

where c is a constant of integration and $P(y) = \int p(y)\,dy$ is an antiderivative of p.

If (4.4.7) is the equation of motion of an object of mass m, then $mv^2/2$ is the kinetic energy and $mP(y)$ is the potential energy of the object; thus, (4.4.8) says that the total energy of the object remains constant, or is conserved. In particular, if a trajectory passes through a given point (y_0, v_0) then

$$c = \frac{v_0^2}{2} + P(y_0).$$

Example 4.4.1

Consider an object with mass m suspended from a spring and moving vertically. Let y be the displacement of the object from the position it occupies when suspended at rest from the spring (Figure 4.4.2).

Figure 4.4.2. ***(a)*** $y > 0$ ***(b)*** $y = 0$ ***(c)*** $y < 0$

Assume that if the length of the spring is changed by an amount ΔL (positive or negative), then the spring exerts an opposing force with magnitude $k|\Delta L|$, where k is a positive constant. In Section 6.1 it will be shown that if the mass of the spring is negligible compared to m and no other forces act on the object then Newton's second law of motion implies that

$$my'' = -ky, \tag{4.4.9}$$

which can be written in the form (4.4.7) with $p(y) = ky/m$. This equation can be solved easily by a method that we'll study in Section 5.2, but that method isn't available here. Instead, we'll consider the phase plane equivalent of (4.4.9).

From (4.4.3), we can rewrite (4.4.9) as the separable equation

$$mv\frac{dv}{dy} = -ky.$$

Integrating this yields

$$\frac{mv^2}{2} = -\frac{ky^2}{2} + c,$$

which implies that

$$mv^2 + ky^2 = \rho \tag{4.4.10}$$

($\rho = 2c$). This defines an ellipse in the Poincaré phase plane (Figure 4.4.3).

Figure 4.4.3. *Trajectories of* $my'' + ky = 0$

We can identify ρ by setting $t = 0$ in (4.4.10); thus, $\rho = mv_0^2 + ky_0^2$, where $y_0 = y(0)$ and $v_0 = v(0)$. To determine the maximum and minimum values of y we set $v = 0$ in (4.4.10); thus,

$$y_{\max} = R \quad \text{and} \quad y_{\min} = -R, \quad \text{with } R = \sqrt{\frac{\rho}{k}}. \tag{4.4.11}$$

Equation (4.4.9) has exactly one equilibrium, $\overline{y} = 0$, and it's stable. You can see intuitively why this is so: if the object is displaced in either direction from equilibrium, the spring tries to bring it back.

In this case we can find y explicitly as a function of t. (Don't expect this to happen in more complicated problems!) If $v > 0$ on an interval I, (4.4.10) implies that

$$\frac{dy}{dt} = v = \sqrt{\frac{\rho - ky^2}{m}}$$

on I. This is equivalent to

$$\frac{\sqrt{k}}{\sqrt{\rho - ky^2}}\frac{dy}{dt} = \omega_0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{4.4.12}$$

Since

$$\int \frac{\sqrt{k}\,dy}{\sqrt{\rho - ky^2}} = \sin^{-1}\left(\sqrt{\frac{k}{\rho}}y\right) + c = \sin^{-1}\left(\frac{y}{R}\right) + c$$

(see (4.4.11)), (4.4.12) implies that that there's a constant ϕ such that

$$\sin^{-1}\left(\frac{y}{R}\right) = \omega_0 t + \phi$$

or

$$y = R\sin(\omega_0 t + \phi)$$

for all t in I. Although we obtained this function by assuming that $v > 0$, you can easily verify that y satisfies (4.4.9) for all values of t. Thus, the displacement varies periodically between $-R$ and R, with period $T = 2\pi/\omega_0$ (Figure 4.4.4). (If you've taken a course in elementary mechanics you may recognize this as simple harmonic motion.)

Figure 4.4.4. $y = R\sin(\omega_0 t + \phi)$

Example 4.4.2

Now we consider the motion of a pendulum with mass m, attached to the end of a weightless rod with length L that rotates on a frictionless axle (Figure 4.4.5). We assume that there's no air resistance.

Let y be the angle measured from the rest position (vertically downward) of the pendulum, as shown in Figure 4.4.5. Newton's second law of motion says that the product of m and the tangential acceleration equals the tangential component of the gravitational force; therefore, from Figure 4.4.5,

$$mLy'' = -mg\sin y,$$

or

$$y'' = -\frac{g}{L}\sin y. \tag{4.4.13}$$

Figure 4.4.5. *The undamped pendulum*

Since $\sin n\pi = 0$ if n is any integer, (4.4.13) has infinitely many equilibria $\overline{y}_n = n\pi$. If n is even, the mass is directly below the axle (Figure 4.4.6 **(a)**) and gravity opposes any deviation from the equilibrium. However, if n is odd, the mass is directly above the axle (Figure 4.4.6 **(b)**) and gravity increases any deviation from the equilibrium. Therefore we conclude on physical grounds that $\overline{y}_{2m} = 2m\pi$ is stable and $\overline{y}_{2m+1} = (2m+1)\pi$ is unstable.

Figure 4.4.6. (a) *Stable equilibrium* ***(b)*** *Unstable equilibrium*

The phase plane equivalent of (4.4.13) is

$$v\frac{dv}{dy} = -\frac{g}{L}\sin y,$$

where $v = y'$ is the angular velocity of the pendulum. Integrating this yields

$$\frac{v^2}{2} = \frac{g}{L}\cos y + c. \tag{4.4.14}$$

If $v = v_0$ when $y = 0$, then

$$c = \frac{v_0^2}{2} - \frac{g}{L},$$

so (4.4.14) becomes

$$\frac{v^2}{2} = \frac{v_0^2}{2} - \frac{g}{L}(1 - \cos y) = \frac{v_0^2}{2} - \frac{2g}{L}\sin^2\frac{y}{2},$$

which is equivalent to

$$v^2 = v_0^2 - v_c^2 \sin^2\frac{y}{2}, \tag{4.4.15}$$

where

$$v_c = 2\sqrt{\frac{g}{L}}.$$

The curves defined by (4.4.15) are the trajectories of (4.4.13). They are periodic with period 2π in y, which isn't surprising, since if $y = y(t)$ is a solution of (4.4.13) then so is $y_n = y(t) + 2n\pi$ for any integer n. Figure 4.4.7 shows trajectories over the interval $[-\pi, \pi]$. From (4.4.15), you can see that if $|v_0| > v_c$ then v is nonzero for all t, which means that the object whirls in the same direction forever, as in Figure 4.4.8. The trajectories associated with this whirling motion are above the upper dashed curve and below the lower dashed curve in Figure 4.4.7. You can also see from (4.4.15) that if $0 < |v_0| < v_c$,then $v = 0$ when $y = \pm y_{\max}$, where

$$y_{\max} = 2\sin^{-1}(|v_0|/v_c).$$

In this case the pendulum oscillates periodically between $-y_{\max}$ and $y_{\max}$, as shown in Figure 4.4.9. The trajectories associated with this kind of motion are the ovals between the dashed curves in Figure 4.4.7. It can be shown (see Exercise 21 for a partial proof) that the period of the oscillation is

$$T = 8\int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2\theta}}. \tag{4.4.16}$$

Although this integral can't be evaluated in terms of familiar elementary functions, you can see that it's finite if $|v_0| < v_c$.

The dashed curves in Figure 4.4.7 contain four trajectories. The critical points $(\pi, 0)$ and $(-\pi, 0)$ are the trajectories of the unstable equilibrium solutions $\overline{y} = \pm\pi$. The upper dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = v_c$. If y is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = \pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = -\pi.$$

The lower dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = -v_c$. If y is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = -\pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = \pi.$$

Consistent with this, the integral (4.4.16) diverges to ∞ if $v_0 = \pm v_c$. (Exercise 21) .

Since the dashed curves separate trajectories of whirling solutions from trajectories of oscillating solutions, each of these curves is called a separatrix.

In general, if (4.4.7) has both stable and unstable equilibria then the separatrices are the curves given by (4.4.8) that pass through unstable critical points. Thus, if $(\overline{y}, 0)$ is an unstable critical point, then

$$\frac{v^2}{2} + P(y) = P(\overline{y}) \tag{4.4.17}$$

defines a separatrix passing through $(\overline{y}, 0)$.

Figure 4.4.7. *Trajectories of the undamped pendulum*

Figure 4.4.8. *The whirling undamped pendulum*

Figure 4.4.9. *The oscillating undamped pendulum*

Stability and Instability Conditions for $y'' + p(y) = 0$

It can be shown (Exercise 23) that an equilibrium $\overline{y}$ of an undamped equation

$$y'' + p(y) = 0 \tag{4.4.18}$$

is stable if there's an open interval (a, b) containing $\overline{y}$ such that

$$p(y) < 0 \;\text{ if }\; a < y < \overline{y} \;\text{ and }\; p(y) > 0 \;\text{ if }\; \overline{y} < y < b. \tag{4.4.19}$$

If we regard $p(y)$ as a force acting on a unit mass, (4.4.19) means that the force resists all sufficiently small displacements from $\overline{y}$.

We've already seen examples illustrating this principle. The equation (4.4.9) for the undamped spring-mass system is of the form (4.4.18) with $p(y) = ky/m$, which has only the stable equilibrium $\overline{y} = 0$. In this case (4.4.19) holds with $a = -\infty$ and $b = \infty$. The equation (4.4.13) for the undamped pendulum is of the form (4.4.18) with $p(y) = (g/L)\sin y$. We've seen that $\overline{y} = 2m\pi$ is a stable equilibrium if m is an integer. In this case

$$p(y) = \sin y < 0 \;\text{ if }\; (2m-1)\pi < y < 2m\pi$$

and

$$p(y) > 0 \;\text{ if }\; 2m\pi < y < (2m+1)\pi.$$

It can also be shown (Exercise 24) that $\overline{y}$ is unstable if there's a $b > \overline{y}$ such that

$$p(y) < 0 \;\text{ if }\; \overline{y} < y < b \tag{4.4.20}$$

or an $a < \overline{y}$ such that

$$p(y) > 0 \;\text{ if }\; a < y < \overline{y}. \tag{4.4.21}$$

If we regard $p(y)$ as a force acting on a unit mass, (4.4.20) means that the force tends to increase all sufficiently small positive displacements from $\overline{y}$, while (4.4.21) means that the force tends to increase the magnitude of all sufficiently small negative displacements from $\overline{y}$.

The undamped pendulum also illustrates this principle. We've seen that $\overline{y} = (2m+1)\pi$ is an unstable equilibrium if m is an integer. In this case

$$\sin y < 0 \;\text{ if }\; (2m+1)\pi < y < (2m+2)\pi,$$

so (4.4.20) holds with $b = (2m+2)\pi$, and

$$\sin y > 0 \;\text{ if }\; 2m\pi < y < (2m+1)\pi,$$

so (4.4.21) holds with $a = 2m\pi$.

Example 4.4.3

The equation

$$y'' + y(y-1) = 0 \tag{4.4.22}$$

is of the form (4.4.18) with $p(y) = y(y-1)$. Therefore $\overline{y} = 0$ and $\overline{y} = 1$ are the equilibria of (4.4.22). Since

Missing dimension or its units for \hspace

$\bar{y} = 0$ is unstable and $\bar{y} = 1$ is stable.

The phase plane equivalent of (4.4.22) is the separable equation

$$v\frac{dv}{dy} + y(y-1) = 0.$$

Integrating yields

$$\frac{v^2}{2} + \frac{y^3}{3} - \frac{y^2}{2} = C,$$

which we rewrite as

$$v^2 + \frac{1}{3}y^2(2y-3) = c \tag{4.4.23}$$

after renaming the constant of integration. These are the trajectories of (4.4.22). If y is any solution of (4.4.22), the point $(y(t), v(t))$ moves along the trajectory of y in the direction of increasing y in the upper half plane ($v = y' > 0$), or in the direction of decreasing y in the lower half plane ($v = y' < 0$).

Figure 4.4.10 shows typical trajectories. The dashed curve through the critical point $(0,0)$, obtained by setting $c = 0$ in (4.4.23), separates the y-v plane into regions that contain different kinds of trajectories; again, we call this curve a separatrix. Trajectories in the region bounded by the closed loop **(b)** are closed curves, so solutions associated with them are periodic. Solutions associated with other trajectories are not periodic. If y is any such solution with trajectory not on the separatrix, then

$$\begin{array}{rclrcl}\backslash\text{dst}\lim_{t\to\infty} y(t) & = & -\infty, & \backslash\text{dst}\lim_{t\to-\infty} y(t) & = & -\infty,\\ \backslash\text{dst}\lim_{t\to\infty} v(t) & = & -\infty, & \backslash\text{dst}\lim_{t\to-\infty} v(t) & = & \infty.\end{array}$$

Figure 4.4.10. *Trajectories of* $y'' + y(y-1) = 0$

The separatrix contains four trajectories of (4.4.22). One is the point $(0,0)$, the trajectory of the equilibrium $\bar{y} = 0$. Since distinct trajectories can't intersect, the segments of the separatrix marked **(a)**, **(b)**, and **(c)** – which don't include $(0,0)$ – are distinct trajectories, none of which can be traversed in finite time. Solutions with these trajectories have the following asymptotic behavior:

$$\begin{array}{rclrclc}\backslash\text{dst}\lim_{t\to\infty} y(t) & = & 0, & \backslash\text{dst}\lim_{t\to-\infty} y(t) & = & -\infty, & \\ \backslash\text{dst}\lim_{t\to\infty} v(t) & = & 0, & \backslash\text{dst}\lim_{t\to-\infty} v(t) & = & \infty & (\text{on } \backslash\text{part}\,\text{a})\\ \backslash\text{dst}\lim_{t\to\infty} y(t) & = & 0, & \backslash\text{dst}\lim_{t\to-\infty} y(t) & = & 0, & \\ \backslash\text{dst}\lim_{t\to\infty} v(t) & = & 0, & \backslash\text{dst}\lim_{t\to-\infty} v(t) & = & 0 & (\text{on } \backslash\text{part}\,\text{b})\\ \backslash\text{dst}\lim_{t\to\infty} y(t) & = & -\infty, & \backslash\text{dst}\lim_{t\to-\infty} y(t) & = & 0, & \\ \backslash\text{dst}\lim_{t\to\infty} v(t) & = & -\infty, & \backslash\text{dst}\lim_{t\to-\infty} v(t) & = & 0 & (\text{on } \backslash\text{part}\,\text{c}).\end{array} .$$

The Damped Case

The phase plane equivalent of the damped autonomous equation

$$y'' + q(y,y')y' + p(y) = 0 \tag{4.4.24}$$

is

$$v\frac{dv}{dy} + q(y,v)v + p(y) = 0.$$

This equation isn't separable, so we can't solve it for v in terms of y, as we did in the undamped case, and conservation of energy doesn't hold. (For example, energy expended in overcoming friction is lost.) However, we can study the qualitative behavior of its solutions by rewriting it as

$$\frac{dv}{dy} = -q(y,v) - \frac{p(y)}{v} \tag{4.4.25}$$

and considering the direction fields for this equation. In the following examples we'll also be showing computer generated trajectories of this equation, obtained by numerical methods. The exercises call for similar computations. The methods discussed in Chapter 3 are not suitable for this task, since $p(y)/v$ in (4.4.25) is undefined on the y axis of the Poincaré phase plane. Therefore we're forced to apply numerical methods briefly discussed in Section 10.1 to the system

$$\begin{aligned} y' &= v \\ v' &= -q(y,v)v - p(y), \end{aligned}$$

which is equivalent to (4.4.24) in the sense defined in Section 10.1. Fortunately, most differential equation software packages enable you to do this painlessly.

In the text we'll confine ourselves to the case where q is constant, so (4.4.24) and (4.4.25) reduce to

$$y'' + cy' + p(y) = 0 \tag{4.4.26}$$

and

$$\frac{dv}{dy} = -c - \frac{p(y)}{v}.$$

(We'll consider more general equations in the exercises.) The constant c is called the damping constant. In situations where (4.4.26) is the equation of motion of an object, c is positive; however, there are situations where c may be negative.

The Damped Spring-Mass System

Earlier we considered the spring - mass system under the assumption that the only forces acting on the object were gravity and the spring's resistance to changes in its length. Now we'll assume that some mechanism (for example, friction in the spring or atmospheric resistance) opposes the motion of the object with a force proportional to its velocity. In Section 6.1 it will be shown that in this case Newton's second law of motion implies that

$$my'' + cy' + ky = 0, \tag{4.4.27}$$

where $c > 0$ is the damping constant. Again, this equation can be solved easily by a method that we'll study in Section 5.2, but that method isn't available here. Instead, we'll consider its phase plane equivalent, which can be written in the form (4.4.25) as

$$\frac{dv}{dy} = -\frac{c}{m} - \frac{ky}{mv}. \tag{4.4.28}$$

(A minor note: the c in (4.4.26) actually corresponds to c/m in this equation.) Figure 4.4.11 shows a typical direction field for an equation of this form. Recalling that motion along a trajectory must be in the direction of increasing y in the upper half plane ($v > 0$) and in the direction of decreasing y in the lower half plane ($v < 0$), you can infer that all trajectories approach the origin in clockwise fashion. To confirm this, Figure 4.4.12 shows the same direction field with some trajectories filled in. All the trajectories shown there correspond to solutions of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

where

$$mv_0^2 + ky_0^2 = \rho \quad \text{(a positive constant)};$$

thus, if there were no damping ($c = 0$), all the solutions would have the same dashed elliptic trajectory, shown in Figure 4.4.14.

Figure 4.4.11. *A typical direction field for $my'' + cy' + ky = 0$ with $0 < c < c_1$*

Figure 4.4.12. *Figure 4.4.11 with some trajectories added*

Solutions corresponding to the trajectories in Figure 4.4.12 cross the y-axis infinitely many times. The corresponding solutions are said to be oscillatory (Figure 4.4.13) It is shown in Section 6.2 that there's a number c_1 such that if $0 \le c < c_1$ then all solutions of (4.4.27) are oscillatory, while if $c \ge c_1$, no solutions of (4.4.27) have this property. (In fact, no solution not identically zero can have more than two zeros in this case.) Figure 4.4.14 shows a direction field and some integral curves for (4.4.28) in this case.

Figure 4.4.13. *An oscillatory solution of $my'' + cy' + ky = 0$*

EXAMPLE 4.4.5 *(The Damped Pendulum)*

Now we return to the pendulum. If we assume that some mechanism (for example, friction in the axle or atmospheric resistance) opposes the motion of the pendulum with a force proportional to its angular velocity, Newton's second law of motion implies that

$$mLy'' = -cy' - mg\sin y, \tag{4.4.29}$$

where $c > 0$ is the damping constant. (Again, a minor note: the c in (4.4.26) actually corresponds to c/mL in this equation.) To plot a direction field for (4.4.29) we write its phase plane equivalent as

$$\frac{dv}{dy} = -\frac{c}{mL} - \frac{g}{Lv}\sin y.$$

Figure 4.4.15 shows trajectories of four solutions of (4.4.29), all satisfying $y(0) = 0$. For each $m = 0, 1, 2, 3$, imparting the initial velocity $v(0) = v_m$ causes the pendulum to make m complete revolutions and then settle into decaying oscillation about the stable equilibrium $\bar{y} = 2m\pi$.

Figure 4.4.14. *A typical direction field for $my'' + cy' + ky = 0$ with $c > c_1$*

Figure 4.4.15. *Four trajectories of the damped pendulum*

4.4 Exercises

In Exercises 1–4 find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and plot some trajectories. *Hint: Use Eqn.* $(???)$ *to obtain the equations of the trajectories.*

1. C/G $y'' + y^3 = 0$

   Show answer

2. C/G $y'' + y^2 = 0$

   Show answer

3. C/G $y'' + y|y| = 0$

   Show answer

4. C/G $y'' + ye^{-y} = 0$

   Show answer

In Exercises 5–8 find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and find the equations of the separatrices (that is, the curves through the unstable equilibria). Plot the separatrices and some trajectories in each of the regions of Poincaré plane determined by them. Hint: Use Eqn. $(???)$ *to determine the separatrices.*

5. C/G $y'' - y^3 + 4y = 0$

   Show answer

6. C/G $y'' + y^3 - 4y = 0$

   Show answer

7. C/G $y'' + y(y^2 - 1)(y^2 - 4) = 0$

   Show answer

8. C/G $y'' + y(y - 2)(y - 1)(y + 2) = 0$

   Show answer

In Exercises 9–12 plot some trajectories of the given equation for various values (positive, negative, zero) of the parameter a. Find the equilibria of the equation and classify them as stable or unstable. Explain why the phase plane plots corresponding to positive and negative values of a differ so markedly. Can you think of a reason why zero deserves to be called the critical value of a?

9. L $y'' + y^2 - a = 0$

Show answer

10. L $y'' + y^3 - ay = 0$

Show answer

11. L $y'' - y^3 + ay = 0$

Show answer

12. L $y'' + y - ay^3 = 0$

Show answer

In Exercises 13-18 plot trajectories of the given equation for $c = 0$ and small nonzero (positive and negative) values of c to observe the effects of damping.

13. L $y'' + cy' + y^3 = 0$

14. L $y'' + cy' - y = 0$

15. L $y'' + cy' + y^3 = 0$

16. L $y'' + cy' + y^2 = 0$

17. L $y'' + cy' + y|y| = 0$

18. L $y'' + y(y-1) + cy = 0$

19. L The van der Pol equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Van_der_Pol.html)

$$y'' - \mu(1-y^2)y' + y = 0, \tag{A}$$

where μ is a positive constant and y is electrical current (Section 6.3), arises in the study of an electrical circuit whose resistive properties depend upon the current. The damping term
$-\mu(1-y^2)y'$ works to reduce $|y|$ if $|y| < 1$ or to increase $|y|$ if $|y| > 1$. It can be shown that van der Pol's equation has exactly one closed trajectory, which is called a limit cycle. Trajectories inside the limit cycle spiral outward to it, while trajectories outside the limit cycle spiral inward to it (Figure 4.4.16). Use your favorite differential equations software to verify this for $\mu = .5, 1. 1.5, 2$. Use a grid with $-4 < y < 4$ and $-4 < v < 4$.

Figure 4.4.16. *Trajectories of van der Pol's equation*

20. L Rayleigh's equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Rayleigh.html),

$$y'' - \mu(1-(y')^2/3)y' + y = 0$$

also has a limit cycle. Follow the directions of Exercise 19 for this equation.

21. In connection with Eqn (4.4.15), suppose $y(0) = 0$ and $y'(0) = v_0$, where $0 < v_0 < v_c$.

a. Let T_1 be the time required for y to increase from zero to $y_{\max} = 2\sin^{-1}(v_0/v_c)$. Show that

$$\frac{dy}{dt} = \sqrt{v_0^2 - v_c^2 \sin^2 y/2}, \quad 0 \le t < T_1. \tag{A}$$

b. Separate variables in (A) and show that

$$T_1 = \int_0^{y_{\max}} \frac{du}{\sqrt{v_0^2 - v_c^2 \sin^2 u/2}} \tag{B}$$

c. Substitute $\sin u/2 = (v_0/v_c)\sin\theta$ in (B) to obtain

$$T_1 = 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2\theta}}. \tag{C}$$

d. Conclude from symmetry that the time required for $(y(t), v(t))$ to traverse the trajectory

$$v^2 = v_0^2 - v_c^2 \sin^2 y/2$$

is $T = 4T_1$, and that consequently $y(t+T) = y(t)$ and $v(t+T) = v(t)$; that is, the oscillation is periodic with period T.

e. Show that if $v_0 = v_c$, the integral in (C) is improper and diverges to ∞. Conclude from this that $y(t) < \pi$ for all t and $\lim_{t\to\infty} y(t) = \pi$.

22. Give a direct definition of an unstable equilibrium of $y'' + p(y) = 0$.

Show answer

23. Let p be continuous for all y and $p(0)=0$. Suppose there's a positive number ρ such that $p(y)>0$ if $0<y\le\rho$ and $p(y)<0$ if $-\rho\le y<0$. For $0<r\le\rho$ let

$$\alpha(r)=\min\left\{\int_0^r p(x)\,dx,\ \int_{-r}^0 |p(x)|\,dx\right\}\quad\text{and}\quad \beta(r)=\max\left\{\int_0^r p(x)\,dx,\ \int_{-r}^0 |p(x)|\,dx\right\}.$$

Let y be the solution of the initial value problem

$$y''+p(y)=0,\quad y(0)=v_0,\quad y'(0)=v_0,$$

and define $c(y_0,v_0)=v_0^2+2\int_0^{y_0}p(x)\,dx$.

a. Show that

$$0<c(y_0,v_0)<v_0^2+2\beta(|y_0|)\quad\text{if}\quad 0<|y_0|\le\rho.$$

b. Show that

$$v^2+2\int_0^y p(x)\,dx=c(y_0,v_0),\quad t>0.$$

c. Conclude from **(b)** that if $c(y_0,v_0)<2\alpha(r)$ then $|y|<r,\ t>0$.

d. Given $\epsilon>0$, let $\delta>0$ be chosen so that

$$\delta^2+2\beta(\delta)<\max\left\{\epsilon^2/2,2\alpha(\epsilon/\sqrt{2})\right\}.$$

Show that if $\sqrt{y_0^2+v_0^2}<\delta$ then $\sqrt{y^2+v^2}<\epsilon$ for $t>0$, which implies that $\overline{y}=0$ is a stable equilibrium of $y''+p(y)=0$.

e. Now let p be continuous for all y and $p(\overline{y})=0$, where $\overline{y}$ is not necessarily zero. Suppose there's a positive number ρ such that $p(y)>0$ if $\overline{y}<y\le\overline{y}+\rho$ and $p(y)<0$ if $\overline{y}-\rho\le y<\overline{y}$. Show that $\overline{y}$ is a stable equilibrium of $y''+p(y)=0$.

24. Let p be continuous for all y.

a. Suppose $p(0)=0$ and there's a positive number ρ such that $p(y)<0$ if $0<y\le\rho$. Let ϵ be any number such that $0<\epsilon<\rho$. Show that if y is the solution of the initial value problem

$$y''+p(y)=0,\quad y(0)=y_0,\quad y'(0)=0$$

with $0<y_0<\epsilon$, then $y(t)\ge\epsilon$ for some $t>0$. Conclude that $\overline{y}=0$ is an unstable equilibrium of $y''+p(y)=0$. *Hint: Let $k=\min_{y_0\le x\le\epsilon}(-p(x))$, which is positive. Show that if $y(t)<\epsilon$ for $0\le t<T$ then $kT^2<2(\epsilon-y_0)$.*

b. Now let $p(\overline{y})=0$, where $\overline{y}$ isn't necessarily zero. Suppose there's a positive number ρ such that $p(y)<0$ if $\overline{y}<y\le\overline{y}+\rho$. Show that $\overline{y}$ is an unstable equilibrium of $y''+p(y)=0$.

c. Modify your proofs of **(a)** and **(b)** to show that if there's a positive number ρ such that $p(y)>0$ if $\overline{y}-\rho\le y<\overline{y}$, then $\overline{y}$ is an unstable equilibrium of $y''+p(y)=0$.

4.5 Applications to Curves

One-Parameter Families of Curves

We begin with two examples of families of curves generated by varying a parameter over a set of real numbers.

Example 4.5.1

For each value of the parameter c, the equation

$$y-cx^2=0 \tag{4.5.1}$$

defines a curve in the xy-plane. If $c\ne0$, the curve is a parabola through the origin, opening upward if $c>0$ or downward if $c<0$. If $c=0$, the curve is the x axis (Figure 4.5.1).

Figure 4.5.1. *A family of curves defined by* $y - cx^2 = 0$

Example 4.5.2

For each value of the parameter c the equation

$$y = x + c \tag{4.5.2}$$

defines a line with slope 1(Figure 4.5.2).

Figure 4.5.2. *A family of lines defined by* $y = x + c$

Figure 4.5.3. *A family of circles defined by* $x^2 + y^2 - c^2 = 0$

Definition 4.5.1

An equation that can be written in the form

$$H(x, y, c) = 0 \tag{4.5.3}$$

is said to define a one-parameter family of curves if, for each value of c in in some nonempty set of real numbers, the set of points (x, y) that satisfy (4.5.3) forms a curve in the xy-plane.

Equations (4.5.1) and (4.5.2) define one–parameter families of curves. (Although (4.5.2) isn't in the form (4.5.3), it can be written in this form as $y - x - c = 0$.)

Example 4.5.3

If $c > 0$, the graph of the equation

$$x^2 + y^2 - c = 0 \tag{4.5.4}$$

is a circle with center at $(0, 0)$ and radius $\sqrt{c}$. If $c = 0$, the graph is the single point $(0, 0)$. (We don't regard a single point as a curve.) If $c < 0$, the equation has no graph. Hence, (4.5.4) defines a one–parameter family of curves for positive values of c. This family consists of all circles centered at $(0, 0)$ (Figure 4.5.3).

Example 4.5.4

The equation

$$x^2 + y^2 + c^2 = 0$$

does not define a one-parameter family of curves, since no (x, y) satisfies the equation if $c \neq 0$, and only the single point $(0, 0)$ satisfies it if $c = 0$. ∎

Recall from Section 1.2 that the graph of a solution of a differential equation is called an integral curve of the equation. Solving a first order differential equation usually produces a one–parameter family of integral curves of the equation. Here we are interested in the converse problem: given a one–parameter family of curves, is there a first order differential equation for which every member of the family is an integral curve. This suggests the next definition.

Definition 4.5.2

If every curve in a one-parameter family defined by the equation

$$H(x, y, c) = 0 \tag{4.5.5}$$

is an integral curve of the first order differential equation

$$F(x, y, y') = 0, \tag{4.5.6}$$

then (4.5.6) is said to be a differential equation for the family.

To find a differential equation for a one–parameter family we differentiate its defining equation (4.5.5) implicitly with respect to x, to obtain

$$H_x(x, y, c) + H_y(x, y, c)y' = 0. \tag{4.5.7}$$

If this equation doesn't, then it's a differential equation for the family. If it does contain c, it may be possible to obtain a differential equation for the family by eliminating c between (4.5.5) and (4.5.7).

Example 4.5.5

Find a differential equation for the family of curves defined by

$$y = cx^2. \tag{4.5.8}$$

Solution Differentiating (4.5.8) with respect to x yields

$$y' = 2cx.$$

Therefore $c = y'/2x$, and substituting this into (4.5.8) yields

$$y = \frac{xy'}{2}$$

as a differential equation for the family of curves defined by (4.5.8). The graph of any function of the form $y = cx^2$ is an integral curve of this equation. ∎

The next example shows that members of a given family of curves may be obtained by joining integral curves for more than one differential equation.

Example 4.5.6

a. Try to find a differential equation for the family of lines tangent to the parabola $y = x^2$.
b. Find two tangent lines to the parabola $y = x^2$ that pass through $(2, 3)$, and find the points of tangency.

Solution (a) The equation of the line through a given point (x_0, y_0) with slope m is

$$y = y_0 + m(x - x_0). \tag{4.5.9}$$

If (x_0, y_0) is on the parabola, then $y_0 = x_0^2$ and the slope of the tangent line through (x_0, x_0^2) is $m = 2x_0$; hence, (4.5.9) becomes

$$y = x_0^2 + 2x_0(x - x_0),$$

or, equivalently,

$$y = -x_0^2 + 2x_0x. \tag{4.5.10}$$

Here x_0 plays the role of the constant c in Definition 4.5.1; that is, varying x_0 over $(-\infty, \infty)$ produces the family of tangent lines to the parabola $y = x^2$.

Differentiating (4.5.10) with respect to x yields $y' = 2x_0$.. We can express x_0 in terms of x and y by rewriting (4.5.10) as

$$x_0^2 - 2x_0x + y = 0$$

and using the quadratic formula to obtain

$$x_0 = x \pm \sqrt{x^2 - y}. \tag{4.5.11}$$

We must choose the plus sign in (4.5.11) if $x < x_0$ and the minus sign if $x > x_0$; thus,

$$x_0 = \left(x + \sqrt{x^2 - y}\right) \text{ if } x < x_0$$

and

$$x_0 = \left(x - \sqrt{x^2 - y}\right) \text{ if } x > x_0.$$

Since $y' = 2x_0$, this implies that

$$y' = 2\left(x + \sqrt{x^2 - y}\right), \quad \text{if} \quad x < x_0 \tag{4.5.12}$$

and

$$y' = 2\left(x - \sqrt{x^2 - y}\right), \quad \text{if} \quad x > x_0. \tag{4.5.13}$$

Neither (4.5.12) nor (4.5.13) is a differential equation for the family of tangent lines to the parabola $y = x^2$. However, if each tangent line is regarded as consisting of two tangent half lines joined at the point of tangency, (4.5.12) is a differential equation for the family of tangent half lines on which x is less than the abscissa of the point of tangency (Figure 4.5.4**(a)**), while (4.5.13) is a differential equation for the family of tangent half lines on which x is greater than this abscissa (Figure 4.5.4(b)). The parabola $y = x^2$ is also an integral curve of both (4.5.12) and (4.5.13).

Figure 4.5.4.

Solution (b) From (4.5.10) the point $(x, y) = (2, 3)$ is on the tangent line through (x_0, x_0^2) if and only if

$$3 = -x_0^2 + 4x_0,$$

which is equivalent to

$$x_0^2 - 4x_0 + 3 = (x_0 - 3)(x_0 - 1) = 0.$$

Letting $x_0 = 3$ in (4.5.10) shows that $(2, 3)$ is on the line

$$y = -9 + 6x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (3, 9)$, as shown in Figure 4.5.5

Letting $x_0 = 1$ in (4.5.10) shows that $(2, 3)$ is on the line

$$y = -1 + 2x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (1, 1)$, as shown in Figure 4.5.5.

Figure 4.5.5.

Geometric Problems

We now consider some geometric problems that can be solved by means of differential equations.

Example 4.5.7

Find curves $y = y(x)$ such that every point $(x_0, y(x_0))$ on the curve is the midpoint of the line segment with endpoints on the coordinate axes and tangent to the curve at $(x_0, y(x_0))$ (Figure 4.5.6).

Solution The equation of the line tangent to the curve at $P = (x_0, y(x_0)$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

If we denote the x and y intercepts of the tangent line by x_I and y_I (Figure 4.5.6), then

$$0 = y(x_0) + y'(x_0)(x_I - x_0) \tag{4.5.14}$$

and

$$y_I = y(x_0) - y'(x_0)x_0. \tag{4.5.15}$$

From Figure 4.5.6, P is the midpoint of the line segment connecting $(x_I, 0)$ and $(0, y_I)$ if and only if $x_I = 2x_0$ and $y_I = 2y(x_0)$. Substituting the first of these conditions into (4.5.14) or the second into (4.5.15) yields

$$y(x_0) + y'(x_0)x_0 = 0.$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + xy' = 0,$$

which can be rewritten as

$$(xy)' = 0.$$

Integrating yields $xy = c$, or

$$y = \frac{c}{x}.$$

If $c = 0$ this curve is the line $y = 0$, which does not satisfy the geometric requirements imposed by the problem; thus, $c \neq 0$, and the solutions define a family of hyperbolas (Figure 4.5.7).

Figure 4.5.6.

Figure 4.5.7.

Example 4.5.8

Find curves $y = y(x)$ such that the tangent line to the curve at any point $(x_0, y(x_0))$ intersects the x-axis at $(x_0^2, 0)$. Figure 4.5.8 illustrates the situation in the case where the curve is in the first quadrant and $0 < x < 1$.

Figure 4.5.8.

Figure 4.5.9.

Solution The equation of the line tangent to the curve at $(x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

Since $(x_0^2, 0)$ is on the tangent line,

$$0 = y(x_0) + y'(x_0)(x_0^2 - x_0).$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + y'(x^2 - x) = 0.$$

Therefor

$$\frac{y'}{y} = -\frac{1}{x^2 - x} = -\frac{1}{x(x-1)} = \frac{1}{x} - \frac{1}{x-1},$$

so

$$\ln|y| = \ln|x| - \ln|x-1| + k = \ln\left|\frac{x}{x-1}\right| + k,$$

and

$$y = \frac{cx}{x-1}.$$

If $c = 0$, the graph of this function is the x-axis. If $c \neq 0$, it's a hyperbola with vertical asymptote $x = 1$ and horizontal asymptote $y = c$. Figure 4.5.9 shows the graphs for $c \neq 0$.

Orthogonal Trajectories

Two curves C_1 and C_2 are said to be orthogonal at a point of intersection (x_0, y_0) if they have perpendicular tangents at (x_0, y_0). (Figure 4.5.10). A curve is said to be an orthogonal trajectory of a given family of curves if it's orthogonal to every curve in the family. For example, every line through the origin is an orthogonal trajectory of the family of circles centered at the origin. Conversely, any such circle is an orthogonal trajectory of the family of lines through the origin (Figure 4.5.11).

Figure 4.5.10. *Curves orthogonal at a point of intersection*

Figure 4.5.11. *Orthogonal families of circles and lines*

Orthogonal trajectories occur in many physical applications. For example, if $u = u(x, y)$ is the temperature at a point (x, y), the curves defined by

$$u(x, y) = c \tag{4.5.16}$$

are called isothermal curves. The orthogonal trajectories of this family are called heat-flow lines, because at any given point the direction of maximum heat flow is perpendicular to the isothermal through the point. If u represents the potential energy of an object moving under a force that depends upon (x, y), the curves (4.5.16) are called equipotentials, and the orthogonal trajectories are called lines of force.

From analytic geometry we know that two nonvertical lines L_1 and L_2 with slopes m_1 and m_2, respectively, are perpendicular if and only if $m_2 = -1/m_1$; therefore, the integral curves of the differential equation

$$y' = -\frac{1}{f(x, y)}$$

are orthogonal trajectories of the integral curves of the differential equation

$$y' = f(x, y),$$

because at any point (x_0, y_0) where curves from the two families intersect the slopes of the respective tangent lines are

$$m_1 = f(x_0, y_0) \quad \text{and} \quad m_2 = -\frac{1}{f(x_0, y_0)}.$$

This suggests a method for finding orthogonal trajectories of a family of integral curves of a first order equation.

Finding Orthogonal Trajectories

Step 1. Find a differential equation

$$y' = f(x, y)$$

for the given family.

Step 2. Solve the differential equation

$$y' = -\frac{1}{f(x, y)}$$

to find the orthogonal trajectories.

Example 4.5.9

Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = c^2 \quad (c > 0). \tag{4.5.17}$$

Solution To find a differential equation for the family of circles we differentiate (4.5.17) implicitly with respect to x to obtain

$$2x + 2yy' = 0,$$

or

$$y' = -\frac{x}{y}.$$

Therefore the integral curves of

$$y' = \frac{y}{x}$$

are orthogonal trajectories of the given family. We leave it to you to verify that the general solution of this equation is

$$y = kx,$$

where k is an arbitrary constant. This is the equation of a nonvertical line through $(0, 0)$. The y axis is also an orthogonal trajectory of the given family. Therefore every line through the origin is an orthogonal trajectory of the given family (4.5.17) (Figure 4.5.11). This is consistent with the theorem of plane geometry which states that a diameter of a circle and a tangent line to the circle at the end of the diameter are perpendicular.

Example 4.5.10

Find the orthogonal trajectories of the family of hyperbolas

$$xy = c \quad (c \neq 0) \tag{4.5.18}$$

(Figure 4.5.7).

Solution Differentiating (4.5.18) implicitly with respect to x yields

$$y + xy' = 0,$$

or

$$y' = -\frac{y}{x};$$

thus, the integral curves of

$$y' = \frac{x}{y}$$

are orthogonal trajectories of the given family. Separating variables yields

$$y'y = x$$

and integrating yields

$$y^2 - x^2 = k,$$

which is the equation of a hyperbola if $k \neq 0$, or of the lines $y = x$ and $y = -x$ if $k = 0$ (Figure 4.5.12).

Figure 4.5.12. *Orthogonal trajectories of the hyperbolas* $xy = c$

Example 4.5.11

Find the orthogonal trajectories of the family of circles defined by

$$(x-c)^2 + y^2 = c^2 \quad (c \neq 0). \tag{4.5.19}$$

These circles are centered on the x-axis and tangent to the y-axis (Figure 4.5.13**(a)**).

Solution Multiplying out the left side of (4.5.19) yields

$$x^2 - 2cx + y^2 = 0, \tag{4.5.20}$$

and differentiating this implicitly with respect to x yields

$$2(x-c) + 2yy' = 0. \tag{4.5.21}$$

From (4.5.20),

$$c = \frac{x^2+y^2}{2x},$$

so

$$x - c = x - \frac{x^2+y^2}{2x} = \frac{x^2-y^2}{2x}.$$

Substituting this into (4.5.21) and solving for y' yields

$$y' = \frac{y^2-x^2}{2xy}. \tag{4.5.22}$$

The curves defined by (4.5.19) are integral curves of (4.5.22), and the integral curves of

$$y' = \frac{2xy}{x^2-y^2}$$

are orthogonal trajectories of the family (4.5.19). This is a homogeneous nonlinear equation, which we studied in Section 2.4. Substituting $y = ux$ yields

$$u'x + u = \frac{2x(ux)}{x^2 - (ux)^2} = \frac{2u}{1 - u^2},$$

so

$$u'x = \frac{2u}{1 - u^2} - u = \frac{u(u^2 + 1)}{1 - u^2},$$

Separating variables yields

$$\frac{1 - u^2}{u(u^2 + 1)} u' = \frac{1}{x},$$

or, equivalently,

$$\left[\frac{1}{u} - \frac{2u}{u^2 + 1}\right] u' = \frac{1}{x}.$$

Therefore

$$\ln|u| - \ln(u^2 + 1) = \ln|x| + k.$$

By substituting $u = y/x$, we see that

$$\ln|y| - \ln|x| - \ln(x^2 + y^2) + \ln(x^2) = \ln|x| + k,$$

which, since $\ln(x^2) = 2\ln|x|$, is equivalent to

$$\ln|y| - \ln(x^2 + y^2) = k,$$

or

$$|y| = e^k(x^2 + y^2).$$

To see what these curves are we rewrite this equation as

$$x^2 + |y|^2 - e^{-k}|y| = 0$$

and complete the square to obtain

$$x^2 + (|y| - e^{-k}/2)^2 = (e^{-k}/2)^2.$$

This can be rewritten as

$$x^2 + (y - h)^2 = h^2,$$

where

$$h = \begin{cases} \text{\textbackslash dst}\,\frac{e^{-k}}{2} & \text{if } y \geq 0, \\ -\text{\textbackslash dst}\,\frac{e^{-k}}{2} & \text{if } y \leq 0. \end{cases}$$

Thus, the orthogonal trajectories are circles centered on the y axis and tangent to the x axis (Figure 4.5.13(b)). The circles for which $h > 0$ are above the x-axis, while those for which $h < 0$ are below.

Figure 4.5.13. (a) *The circles* $(x - c)^2 + y^2 = c^2$ ***(b)*** *The circles* $x^2 + (y - h)^2 = h^2$

4.5 Exercises

In Exercises 1–8 find a first order differential equation for the given family of curves.

1. $y(x^2+y^2)=c$

   Show answer

2. $e^{xy}=cy$

   Show answer

3. $\ln|xy|=c(x^2+y^2)$

   Show answer

4. $y=x^{1/2}+cx$

   Show answer

5. $y=e^{x^2}+ce^{-x^2}$

   Show answer

6. \dst$y=x^3+\frac{c}{x}$

   Show answer

7. $y=\sin x+ce^x$

   Show answer

8. $y=e^x+c(1+x^2)$

   Show answer

9. Show that the family of circles

$$(x-x_0)^2+y^2=1,\ -\infty<x_0<\infty,$$

   can be obtained by joining integral curves of two first order differential equations. More specifically, find differential equations for the families of semicircles

$$(x-x_0)^2+y^2=1,\ x_0<x<x_0+1,\ -\infty<x_0<\infty,$$

$$(x-x_0)^2+y^2=1,\ x_0-1<x<x_0,\ -\infty<x_0<\infty.$$

10. Suppose f and g are differentiable for all x. Find a differential equation for the family of functions $y=f+cg$ (c=constant).

    Show answer

In Exercises 11–13 find a first order differential equation for the given family of curves.

11. Lines through a given point (x_0, y_0).

Show answer

12. Circles through $(-1, 0)$ and $(1, 0)$.

Show answer

13. Circles through $(0, 0)$ and $(0, 2)$.

Show answer

14. Use the method Example 4.5.6**(a)** to find the equations of lines through the given points tangent to the parabola $y = x^2$. Also, find the points of tangency.

(a) $(5, 9)$	**(b)** $(6, 11)$	**(c)** $(-6, 20)$	**(d)** $(-3, 5)$

Show answer

15. a. Show that the equation of the line tangent to the circle

$$x^2 + y^2 = 1 \tag{A}$$

at a point (x_0, y_0) on the circle is

$$y = \dst\frac{1 - x_0 x}{y_0} \quad \text{if} \quad x_0 \neq \pm 1. \tag{B}$$

b. Show that if y' is the slope of a nonvertical tangent line to the circle (A) and (x, y) is a point on the tangent line then

$$(y')^2(x^2 - 1) - 2xyy' + y^2 - 1 = 0. \tag{C}$$

c. Show that the segment of the tangent line (B) on which $(x - x_0)/y_0 > 0$ is an integral curve of the differential equation

$$y' = \frac{xy - \sqrt{x^2 + y^2 - 1}}{x^2 - 1}, \tag{D}$$

while the segment on which $(x - x_0)/y_0 < 0$ is an integral curve of the differential equation

$$y' = \frac{xy + \sqrt{x^2 + y^2 - 1}}{x^2 - 1}. \tag{E}$$

Hint: Use the quadratic formula to solve (C) *for* y'*. Then substitute (B) for* y *and choose the* $\pm$ *sign in the quadratic formula so that the resulting expression for* y' *reduces to the known slope* $y' = -x_0/y_0$*.*

d. Show that the upper and lower semicircles of (A) are also integral curves of (D) and (E).

e. Find the equations of two lines through (5,5) tangent to the circle (A), and find the points of tangency.

Show answer

16. a. Show that the equation of the line tangent to the parabola

$$x = y^2 \tag{A}$$

at a point $(x_0, y_0) \neq (0,0)$ on the parabola is

$$y = \dst\frac{y_0}{2} + \dst\frac{x}{2y_0}. \tag{B}$$

b. Show that if y' is the slope of a nonvertical tangent line to the parabola (A) and (x, y) is a point on the tangent line then

$$4x^2(y')^2 - 4xyy' + x = 0. \tag{C}$$

c. Show that the segment of the tangent line defined in **(a)** on which $x > x_0$ is an integral curve of the differential equation

$$y' = \frac{y + \sqrt{y^2 - x}}{2x}, \tag{D}$$

while the segment on which $x < x_0$ is an integral curve of the differential equation

$$y' = \frac{y - \sqrt{y^2 - x}}{2x}, \tag{E}$$

Hint: Use the quadratic formula to solve (C) *for* y'*. Then substitute (B) for* y *and choose the* $\pm$ *sign in the quadratic formula so that the resulting expression for* y' *reduces to the known slope* $y' = \dst\frac{1}{2y_0}$.

d. Show that the upper and lower halves of the parabola (A), given by $y = \sqrt{x}$ and $y = -\sqrt{x}$ for $x > 0$, are also integral curves of (D) and (E).

17. Use the results of Exercise 16 to find the equations of two lines tangent to the parabola $x = y^2$ and passing through the given point. Also find the points of tangency.

(a) $(-5, 2)$	**(b)** $(-4, 0)$	**(c)** $(7, 4)$	**(d)** $(5, -3)$

Show answer

18. Find a curve $y = y(x)$ through (1,2) such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $\dst x_I = x_0/2$.

Show answer

19. Find all curves $y = y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0^3$.

Show answer

20. Find all curves $y = y(x)$ such that the tangent to the curve at any point passes through a given point (x_1, y_1).

Show answer

21. Find a curve $y = y(x)$ through $(1, -1)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = x_0^3$.

Show answer

22. Find all curves $y = y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = x_0$.

Show answer

23. Find a curve $y = y(x)$ through $(0, 2)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0 + 1$.

Show answer

24. Find a curve $y = y(x)$ through $(2, 1)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = 2y(x_0)$.

Show answer

In Exercises 25–29 find the orthogonal trajectories of the given family of curves.

25. $x^2 + 2y^2 = c^2$

Show answer

26. $x^2 + 4xy + y^2 = c$

Show answer

27. $y = ce^{2x}$

Show answer

28. $xye^{x^2} = c$

Show answer

29. \dst$y = \frac{ce^x}{x}$

Show answer

30. Find a curve through $(-1, 3)$ orthogonal to every parabola of the form

$$y = 1 + cx^2$$

that it intersects. Which of these parabolas does the desired curve intersect?

Show answer

31. Show that the orthogonal trajectories of

$$x^2 + 2axy + y^2 = c$$

satisfy

$$|y - x|^{a+1}|y + x|^{a-1} = k.$$

32. If lines L and L_1 intersect at (x_0, y_0) and α is the smallest angle through which L must be rotated counterclockwise about (x_0, y_0) to bring it into coincidence with L_1, we say that α is the angle from L to L_1; thus, $0 \le \alpha < \pi$. If L and L_1 are tangents to curves C and C_1, respectively, that intersect at (x_0, y_0), we say that C_1 intersects C at the angle α. Use the identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

to show that if C and C_1 are intersecting integral curves of

$$y' = f(x, y) \quad \text{and} \quad y' = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha} \quad \left(\alpha \ne \frac{\pi}{2}\right),$$

respectively, then C_1 intersects C at the angle α.

33. Use the result of Exercise 32 to find a family of curves that intersect every nonvertical line through the origin at the angle $\alpha = \pi/4$.

Show answer

34. Use the result of Exercise 32 to find a family of curves that intersect every circle centered at the origin at a given angle $\alpha \ne \pi/2$.

Show answer

Chapter 4 Applications of First Order Equations

4.1 Growth and Decay

1 $Q = 20e^{-(t\ln 2)/3200}$ g

2 $\frac{2\ln 10}{\ln 2}$ days

3 \dst $\tau = 10\frac{\ln 2}{\ln 4/3}$ minutes

4 \dst $\tau\frac{\ln(p_0/p_1)}{\ln 2}$

5 \dst $\frac{t_p}{t_q} = \frac{\ln p}{\ln q}$

6 \dst $k = \frac{1}{t_2 - t_1}\ln\frac{Q_1}{Q_2}$

7 20 g

8 \dst $\frac{50\ln 2}{3}$ yrs

9 \dst $\frac{25}{2}\ln 2$%

10 **(a)** $= 20\ln 3$ yr **(b)**. $Q_0 = 100000e^{-.5}$

11 **(a)** $Q(t) = 5000 - 4750e^{-t/10}$ **(b)** 5000 lbs

12 \dst $\frac{1}{25}$ yrs;

13 $V = V_0 e^{t\ln 10/2}$ 4 hours

14 \dst $\frac{1500\ln\frac{4}{3}}{\ln 2}$ $\text{yrs};\ 2^{-4/3}Q_0$

15 $W(t) = 20 - 19e^{-t/20}$; $\lim_{t\to\infty} W(t) = 20$ ounces

16 $S(t) = 10(1 + e^{-t/10});\ \lim_{t\to\infty} S(t) = 10$ g

17 10 gallons

18 $V(t) = 15000 + 10000e^{t/20}$

19 $W(t) = 4\times 10^6(t+1)^2$ dollars t years from now

20 \dst $p = \frac{100}{25 - 24e^{-t/2}}$

21 **(a)** \dst $P(t) = 1000e^{.06t} + 50\frac{e^{.06t} - 1}{e^{.06/52} - 1}$ **(b)** 5.64×10^{-4}

22 **(a)** $P' = rP - 12M$ **(b)** \dst $P = \frac{12M}{r}(1 - e^{rt}) + P_0e^{rt}$ **(c)** \dst $M \approx \frac{rP_0}{12(1 - e^{-rN})}$

for **(ii)** approximate $M = \$1206.05$, exact $M = \$1206.93$.

(d) For
(i)
approximate
M
$= \$402.25$,
exact
M
$= \$402.80$

23 **(a)** $T(\alpha) = \dst -\frac{1}{r}\ln\left(1-(1-e^{-rN})/\alpha\right)$ years

(b) $T(1.05) = 13.69$ yrs, $S(1.05) = \$3579.94$ $T(1.10) = 12.61$ yrs,
$S(\alpha)$
$= \dst\frac{P_0}{(1-e^{-rN})}\left[rN+\alpha\ln\left(1-(1-e^{-rN})/\alpha\right)\right]$

$S(1.10)$
$= \$6476.63$
$T(1.15)$
$= 11.70$
yrs,
$S(1.15)$
$= \$8874.98.$

24 $P_0 = \begin{cases} \dst\frac{S_0(1-e^{(a-r)T})}{r-a} \text{ if } a \ne r, \\ S_0T \text{ if } a = r. \end{cases}$

4.2 Cooling and Mixing

1 $\approx 15.15^{\circ}$F

2 \dst$T = -10 + 110e^{-t \ln \frac{11}{9}}$

3 $\approx 24.33^{\circ}$F

4 **(a)** 91.30°F **(b)** 8.99 minutes after being placed outside **(c)** never

5 **(a)** 12:11:32 **(b)** 12:47:33

6 \dst$(85/3)^{\circ}C$

7 32°F

8 \dst$Q(t) = 40(1 - e^{-3t/40})$

9 \dst$Q(t) = 30 - 20e^{-t/10}$

10 \dst$K(t) = .3 - .2e^{-t/20}$

11 $Q(50) = 47.5$ (pounds)

12 50 gallons

13 $\min q_2 = q_1/\overline{c}$

14 \dst$Q = t + 300 - \frac{234 \times 10^5}{(t+300)^2}, \quad 0 \le t \le 300$

15 **(a)** \dst$Q' + \frac{2}{25}Q = 6 - 2e^{-t/25}$ **(b)** $Q = 75 - 50e^{-t/25} - 25e^{-2t/25}$ **(c)** 75

16 **(a)** $T = T_m + (T_0 - T_m)e^{-kt} +$ \dst$\frac{k(S_0 - T_m)}{(k - k_m)}\left(e^{-k_m t} - e^{-kt}\right)$

(b) T
$= T_m$
$+ k(S_0$
$- T_m)te^{-kt}$
$+ (T_0$
$- T_m)e^{-kt}$
(c)
$\lim_{t\to\infty} T(t)$
$= \lim_{t\to\infty} S(t)$
$= T_m$

17 **(a)** $T' =$ \dst$-k\left(1 + \frac{a}{a_m}\right)T + k\left(T_{m0} + \frac{a}{a_m}T_0\right)$ **(b)** \dst$T = \frac{aT_0 + a_m T_{m0}}{a + a_m} + \frac{a_m(T_0 - T_{m0})}{a + a_m}e^{-k(1+a/a_m)t}$,

\dst$T_m = \frac{aT_0 + a_m T_{m0}}{a + a_m} + \frac{a(T_{m0} - T_0)}{a + a_m}e^{-k(1+a/a_m)t}$;
(c)
$\lim_{t\to\infty} T(t)$
$= \lim_{t\to\infty} T_m(t)$
$=$ \dst$\frac{aT_0 + a_m T_{m0}}{a + a_m}$

18 $V =$ \dst$\frac{a}{b}$\dst$\frac{V_0}{V_0 - (V_0 - a/b)e^{-at}}, \qquad \lim_{t\to\infty} V(t) = a/b$

19 $c_1 = c\left(1 - e^{-rt/W}\right)$, $c_2 = c$\dst$\left(1 - e^{-rt/W} - \frac{r}{W}te^{-rt/W}\right)$.

20 **(a)** \dst$c_n = c\left(1 - e^{-rt/W}\sum_{j=0}^{n-1}\frac{1}{j!}\left(\frac{rt}{W}\right)^j\right)$ **(b)** c **(c)** 0

21 Let $c_\infty = \displaystyle\frac{c_1W_1+c_2W_2}{W_1+W_2}$, $\alpha = \displaystyle\frac{c_2W_2^2-c_1W_1^2}{W_1+W_2}$, and $\beta = \displaystyle\frac{W_1+W_2}{W_1W_2}$. Then:

(b) $\lim_{t\to\infty} c_1(t) = \lim_{t\to\infty} c_2(t) = c_\infty$

(a)
$c_1(t)$
$= c_\infty$
$+ \displaystyle\frac{\alpha}{W_1} e^{-r\beta t}$,
$c_2(t)$
$= c_\infty$
$- \displaystyle\frac{\alpha}{W_2} e^{-r\beta t}$

4.3 Elementary Mechanics

1 $v = -\displaystyle\frac{384}{5}\left(1-e^{-5t/12}\right)$; $-\displaystyle\frac{384}{5}$ ft/s

2 $k = 12; \quad v = -16(1-e^{-2t})$

3 $v = 25(1-e^{-t})$;25 ft/s

4 $v = 20 - 27e^{-t/40}$

5 ≈ 17.10 ft

6 $v = -\displaystyle\frac{40(13+3e^{-4t/5})}{13-3e^{-4t/5}}$; -40 ft/s

7 $v = -128(1-e^{-t/4})$

9 $T = \displaystyle\frac{m}{k}\ln\left(1+\frac{v_0k}{mg}\right); \quad y_m = y_0 + \displaystyle\frac{m}{k}\left[v_0 - \frac{mg}{k}\ln\left(1+\frac{v_0k}{mg}\right)\right]$

10 $v = -\displaystyle\frac{64(1-e^{-t})}{1+e^{-t}}$; -64 ft/s

11 $v = \alpha\displaystyle\frac{v_0(1+e^{-\beta t})-\alpha(1-e^{-\beta t})}{\alpha(1+e^{-\beta t})-v_0(1-e^{-\beta t})}; \quad -\alpha$, where $\alpha = \displaystyle\sqrt{\frac{mg}{k}}$ and $\beta = 2\displaystyle\sqrt{\frac{kg}{m}}$.

12 $\displaystyle T = \sqrt{\frac{m}{kg}}\tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)$ $v = -\displaystyle\sqrt{\frac{mg}{k}}; \frac{1-e^{-2\sqrt{\frac{gk}{m}}(t-T)}}{1+e^{-2\sqrt{\frac{gk}{m}}(t-T)}}$

13 $s' = mg - \displaystyle\frac{as}{s+1}$; $a_0 = mg.$

14 **(a)** $ms' = mg - f(s)$

15 **(a)** $v' = -9.8 + v^4/81$ **(b)** $v_T \approx -5.308$ m/s

16 **(a)** $v' = -32 + 8\sqrt{|v|};\ v_T = -16$ ft/s **(b)** From Exercise 4.3. 14**(c)**, v_T is the negative

number
such
that
-32
$+ 8\sqrt{|v_T|}$
$= 0$;
thus,
v_T
$= -16$
ft/s.

17 ≈ 6.76 miles/s

18 ≈ 1.47 miles/s

20 $\alpha = \displaystyle\frac{gR^2}{(y_m+R)^2}$

4.4 Autonomous Second Order Equations

1 $\overline{y}=0$ is a stable equilibrium; trajectories are $v^2+\dst\frac{y^4}{4}=c$

2 $\overline{y}=0$ is an unstable equilibrium; trajectories are $v^2+\dst\frac{2y^3}{3}=c$

3 $\overline{y}=0$ is a stable equilibrium; trajectories are $v^2+\dst\frac{2|y|^3}{3}=c$

4 $\overline{y}=0$ is a stable equilibrium; trajectories are $v^2-e^{-y}(y+1)=c$

5 equilibria: 0 (stable) and $-2,2$ (unstable); trajectories: $2v^2-y^4+8y^2=c$;

separatrix:
$2v^2-y^4$
$+8y^2$
$=16$

6 equilibria: 0 (unstable) and $-2,2$ (stable); trajectories: $2v^2+y^4-8y^2=c$;

separatrix:
$2v^2+y^4$
$-8y^2$
$=0$

7 equilibria: $0,-2,2$ (stable), $-1,1$ (unstable); trajectories:

$6v^2$
$+y^2(2y^4$
$-15y^2$
$+24)$
$=c$;
separatrix:
$6v^2$
$+y^2(2y^4$
$-15y^2$
$+24)$
$=11$

8 equilibria: $0,2$ (stable) and $-2,1$ (unstable);

separatrices: $30v^2+y^2(12y^3-15y^2-80y+120)=496$ and
trajectories:
$30v^2$
$+y^2(12y^3$
$-15y^2$
$-80y$
$+120)$
$=c$;

$30v^2$
$+y^2(12y^3$
$-15y^2$
$-80y$
$+120)$
$=37$

9 No equilibria if $a < 0$; 0 is unstable if $a = 0$; $\sqrt{a}$ is stable and

*

$-\sqrt{a}$ is unstable if $a > 0$.

10 0 is a stable equilibrium if $a \le 0$; $-\sqrt{a}$ and $\sqrt{a}$ are stable and 0 is unstable if $a > 0$.

11 0 is unstable if $a \le 0$; $-\sqrt{a}$ and $\sqrt{a}$ are unstable and 0 is stable if $a > 0$.

12 0 is stable if $a \le 0$; 0 is stable and $-\sqrt{a}$ and $\sqrt{a}$ are unstable if $a \le 0$.

22 An equilibrium solution $\overline{y}$ of $y'' + p(y) = 0$ is unstable if there's an $\epsilon > 0$

> such that, for every $\delta > 0$, there's a solution of (A) with $\sqrt{(y(0) - \overline{y})^2 + v^2(0)} < \delta$, but $\sqrt{(y(t) - \overline{y})^2 + v^2(t)} \ge \epsilon$ for some $t > 0$.

4.5 Applications to Curves

1 \dst$y' = -\frac{2xy}{x^2+3y^2}$

2 $y' = -$\dst$\frac{y^2}{(xy-1)}$

3 $y' = -$\dst$\frac{y(x^2+y^2-2x^2\ln|xy|)}{x(x^2+y^2-2y^2\ln|xy|)}$.

4 \dst$xy' - y = -\frac{x^{1/2}}{2}$

5 $y' + 2xy = 4xe^{x^2}$

6 $xy' + y = 4x^3$

7 $y' - y = \cos x - \sin x$

8 $(1+x^2)y' - 2xy = (1-x)^2e^x$

10 $y'g - yg' = f'g - fg'$.

11 $(x - x_0)y' = y - y_0$

12 $y'(y^2 - x^2 + 1) + 2xy = 0$

13 $2x(y-1)y' - y^2 + x^2 + 2y = 0$

14 **(a)** $y = -81 + 18x, (9, 81) \quad y = -1 + 2x, (1, 1)$

(c) $y = -100 - 20x, (-10, 100) \quad y = -4 - 4x, (-2, 4)$

(b) $y = -121 + 22x, (11, 121) \quad y = -1 + 2x, (1, 1)$

(d) $y = -25 - 10x, (-5, 25) \quad y = -1 - 2x, (-1, 1)$

15 **(e)** $y =$ \dst$\frac{5+3x}{4}$, $(-3/5, 4/5)$ $y = -$\dst$\frac{5-4x}{3}$, $(4/5, -3/5)$

17 **(a)** \dst$y = -\frac{1}{2}(1+x), (1, -1); \quad y = \frac{5}{2} + \frac{x}{10}, (25, 5)$

(c) \dst$y = \frac{1}{2}(1+x), (1, 1) \quad y = \frac{7}{2} + \frac{x}{14}, (49, 7)$

(b)
\dst$y = \frac{1}{4}(4+x), (4, 2) \quad y = -\frac{1}{4}(4+x), (4, -2)$;

(d)
\dst$y = -\frac{1}{2}(1+x), (1, -1) \quad y = -\frac{5}{2} - \frac{x}{10}, (25, -5)$

18 $y = 2x^2$

19 \dst$y = \frac{cx}{\sqrt{|x^2-1|}}$

20 $y = y_1 + c(x - x_1)$

21 \dst$y = -\frac{x^3}{2} - \frac{x}{2}$

22 $y=-x\ln|x|+cx$

23 \dst$y=\sqrt{2x+4}$

24 \dst$y=\sqrt{x^2-3}$

25 $y=kx^2$

26 $(y-x)^3(y+x)=k$

27 $y^2=-x+k$

28 \dst$y^2=-\frac{1}{2}\ln(1+2x^2)+k$

29 $y^2=-2x-\ln(x-1)^2+k$

30 \dst$y=1+\sqrt{\frac{9-x^2}{2}};$ those with $c>0$

33 \dst$\tan^{-1}\frac{y}{x}-\frac{1}{2}\ln(x^2+y^2)=k$

34 \dst$\frac{1}{2}\ln(x^2+y^2)+(\tan\alpha)\tan^{-1}\frac{y}{x}=k$