3 Numerical Methods

3 Numerical Methods

In this chapter we study numerical methods for solving a first order differential equation

$$y' = f(x, y).$$

SECTION 3.1 deals with Euler's method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html), which is really too crude to be of much use in practical applications. However, its simplicity allows for an introduction to the ideas required to understand the better methods discussed in the other two sections.

SECTION 3.2 discusses improvements on Euler's method.

SECTION 3.3 deals with the Runge (http://www-history.mcs.st-and.ac.uk/Mathematicians/Runge.html)-Kutta (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kutta.html) method, perhaps the most widely used method for numerical solution of differential equations.

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Answers to selected exercises in this chapter

3.1 Euler's Method

If an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{3.1.1}$$

can't be solved analytically, it's necessary to resort to numerical methods to obtain useful approximations to a solution of (3.1.1). We'll consider such methods in this chapter.

We're interested in computing approximate values of the solution of (3.1.1) at equally spaced points x_0, x_1, …, $x_n = b$ in an interval $[x_0, b]$. Thus,

$$x_i = x_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$h = \frac{b - x_0}{n}.$$

We'll denote the approximate values of the solution at these points by y_0, y_1, …, y_n; thus, y_i is an approximation to $y(x_i)$. We'll call

$$e_i = y(x_i) - y_i$$

the error at the ith step. Because of the initial condition $y(x_0) = y_0$, we'll always have $e_0 = 0$. However, in general $e_i \neq 0$ if $i > 0$.

We encounter two sources of error in applying a numerical method to solve an initial value problem:

- The formulas defining the method are based on some sort of approximation. Errors due to the inaccuracy of the approximation are called truncation errors.
- Computers do arithmetic with a fixed number of digits, and therefore make errors in evaluating the formulas defining the numerical methods. Errors due to the computer's inability to do exact arithmetic are called roundoff errors.

Since a careful analysis of roundoff error is beyond the scope of this book, we'll consider only truncation errors.

Euler's Method

The simplest numerical method for solving (3.1.1) is Euler's method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html). This method is so crude that it is seldom used in practice; however, its simplicity makes it useful for illustrative purposes.

Euler's method is based on the assumption that the tangent line to the integral curve of (3.1.1) at $(x_i, y(x_i))$ approximates the integral curve over the interval $[x_i, x_{i+1}]$. Since the slope of the integral curve of (3.1.1) at $(x_i, y(x_i))$ is $y'(x_i) = f(x_i, y(x_i))$, the equation of the tangent line to the integral curve at $(x_i, y(x_i))$ is

$$y = y(x_i) + f(x_i, y(x_i))(x - x_i). \tag{3.1.2}$$

Setting $x = x_{i+1} = x_i + h$ in (3.1.2) yields

$$y_{i+1} = y(x_i) + hf(x_i, y(x_i)) \tag{3.1.3}$$

as an approximation to $y(x_{i+1})$. Since $y(x_0) = y_0$ is known, we can use (3.1.3) with $i = 0$ to compute

$$y_1 = y_0 + hf(x_0, y_0).$$

However, setting $i = 1$ in (3.1.3) yields

$$y_2 = y(x_1) + hf(x_1, y(x_1)),$$

which isn't useful, since we don't know $y(x_1)$. Therefore we replace $y(x_1)$ by its approximate value y_1 and redefine

$$y_2 = y_1 + hf(x_1, y_1).$$

Having computed y_2, we can compute

$$y_3 = y_2 + hf(x_2, y_2).$$

In general, Euler's method starts with the known value $y(x_0) = y_0$ and computes y_1, y_2, …, y_n successively by with the formula

$$y_{i+1} = y_i + hf(x_i, y_i), \quad 0 \le i \le n-1. \tag{3.1.4}$$

The next example illustrates the computational procedure indicated in Euler's method.

Example 3.1.1

Use Euler's method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \tag{3.1.5}$$

at $x = 0.1, 0.2, 0.3$.

Solution We rewrite (3.1.5) as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.1.1), with

$$f(x, y) = -2y + x^3 e^{-2x},\ x_0 = 0,\ \text{and}\ y_0 = 1.$$

Euler's method yields

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We've written the details of these computations to ensure that you understand the procedure. However, in the rest of the examples as well as the exercises in this chapter, we'll assume that you can use a programmable calculator or a computer to carry out the necessary computations.

Examples Illustrating The Error in Euler's Method

Example 3.1.2

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, $\ldots$, 1.0. Compare these approximate values with the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4), \tag{3.1.6}$$

which can be obtained by the method of Section 2.1. (Verify.)

Solution Table 3.1.1 shows the values of the exact solution (3.1.6) at the specified points, and the approximate values of the solution at these points obtained by Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$. In examining this table, keep in mind that the approximate values in the column corresponding to $h = .05$ are actually the results of 20 steps with Euler's method. We haven't listed the estimates of the solution obtained for $x = 0.05$, 0.15, $\ldots$, since there's nothing to compare them with in the column corresponding to $h = 0.1$. Similarly, the approximate values in the column corresponding to $h = 0.025$ are actually the results of 40 steps with Euler's method.

Table 3.1.1. *Numerical solution of* $y' + 2y = x^3 e^{-2x},\ y(0) = 1$*, by Euler's method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Ex
0.0	1.000000000	1.000000000	1.000000000	1.000000
0.1	0.800000000	0.810005655	0.814518349	0.818751
0.2	0.640081873	0.656266437	0.663635953	0.670588
0.3	0.512601754	0.532290981	0.541339495	0.549922
0.4	0.411563195	0.432887056	0.442774766	0.452204
0.5	0.332126261	0.353785015	0.363915597	0.373627
0.6	0.270299502	0.291404256	0.301359885	0.310952
0.7	0.222745397	0.242707257	0.252202935	0.261398
0.8	0.186654593	0.205105754	0.213956311	0.222570
0.9	0.159660776	0.176396883	0.184492463	0.192412
1.0	0.139778910	0.154715925	0.162003293	0.169169

You can see from Table 3.1.1 that decreasing the step size improves the accuracy of Euler's method. For example,

$$y_{\text{exact}}(1) - y_{\text{approx}}(1) \approx \begin{cases} .0293 \text{ with } h = 0.1, \\ .0144 \text{ with } h = 0.05, \\ .0071 \text{ with } h = 0.025. \end{cases}$$

Based on this scanty evidence, you might guess that the error in approximating the exact solution at a fixed value of x by Euler's method is roughly halved when the step size is halved. You can find more evidence to support this conjecture by examining Table 3.1.2, which lists the approximate values of $y_{\text{exact}} - y_{\text{approx}}$ at $x = 0.1$, 0.2, $\ldots$, 1.0.

Table 3.1.2. *Errors in approximate solutions of* $y' + 2y = x^3 e^{-2x},\ y(0) = 1$*, obtained by Euler's method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0187	0.0087	0.0042
0.2	0.0305	0.0143	0.0069
0.3	0.0373	0.0176	0.0085
0.4	0.0406	0.0193	0.0094
0.5	0.0415	0.0198	0.0097
0.6	0.0406	0.0195	0.0095
0.7	0.0386	0.0186	0.0091
0.8	0.0359	0.0174	0.0086
0.9	0.0327	0.0160	0.0079
1.0	0.0293	0.0144	0.0071

Example 3.1.3

Tables 3.1.3 and 3.1.4 show analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1, \tag{3.1.7}$$

except in this case we can't solve (3.1.7) exactly. The results in the "Exact" column were obtained by using a more accurate numerical method known as the Runge (http://www-history.mcs.st-and.ac.uk/Mathematicians/Runge.html)-Kutta (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kutta.html) method with a small step size. They are exact to eight decimal places. ∎

Since we think it's important in evaluating the accuracy of the numerical methods that we'll be studying in this chapter, we often include a column listing values of the exact solution of the initial value problem, even if the directions in the example or exercise don't specifically call for it. If quotation marks are included in the heading, the values were obtained by applying the Runge-Kutta method in a way that's explained in Section 3.3. If quotation marks are not included, the values were obtained from a known formula for the solution. In either case, the values are exact to eight places to the right of the decimal point.

Table 3.1.3. *Numerical solution of* $y' = -2y^2 + xy + x^2,\ y(0) = 1$, *by Euler's method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exa
0.0	1.000000000	1.000000000	1.000000000	1.000000
0.1	0.800000000	0.821375000	0.829977007	0.837584
0.2	0.681000000	0.707795377	0.719226253	0.729641
0.3	0.605867800	0.633776590	0.646115227	0.657580
0.4	0.559628676	0.587454526	0.600045701	0.611901
0.5	0.535376972	0.562906169	0.575556391	0.587575
0.6	0.529820120	0.557143535	0.569824171	0.581942
0.7	0.541467455	0.568716935	0.581435423	0.593629
0.8	0.569732776	0.596951988	0.609684903	0.621907
0.9	0.614392311	0.641457729	0.654110862	0.666250
1.0	0.675192037	0.701764495	0.714151626	0.726015

Table 3.1.4. *Errors in approximate solutions of* $y' = -2y^2 + xy + x^2,\ y(0) = 1$, *obtained by Euler's method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0376	0.0162	0.0076
0.2	0.0486	0.0218	0.0104
0.3	0.0517	0.0238	0.0115
0.4	0.0523	0.0244	0.0119
0.5	0.0522	0.0247	0.0121
0.6	0.0521	0.0248	0.0121
0.7	0.0522	0.0249	0.0122
0.8	0.0522	0.0250	0.0122
0.9	0.0519	0.0248	0.0121
1.0	0.0508	0.0243	0.0119

Truncation Error in Euler's Method

Consistent with the results indicated in Tables 3.1.1–3.1.4, we'll now show that under reasonable assumptions on f there's a constant K such that the error in approximating the solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

at a given point $b > x_0$ by Euler's method with step size $h = (b - x_0)/n$ satisfies the inequality

$$|y(b) - y_n| \le Kh,$$

where K is a constant independent of n.

There are two sources of error (not counting roundoff) in Euler's method:

1. The error committed in approximating the integral curve by the tangent line (3.1.2) over the interval $[x_i, x_{i+1}]$.
2. The error committed in replacing $y(x_i)$ by y_i in (3.1.2) and using (3.1.4) rather than (3.1.2) to compute y_{i+1}.

Euler's method assumes that y_{i+1} defined in (3.1.2) is an approximation to $y(x_{i+1})$. We call the error in this approximation the local truncation error at the ith step, and denote it by T_i; thus,

$$T_i = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)). \tag{3.1.8}$$

We'll now use Taylor's theorem (http://www-history.mcs.st-and.ac.uk/Mathematicians/Taylor.html) to estimate T_i, assuming for simplicity that f, f_x, and f_y are continuous and bounded for all (x, y). Then y'' exists and is bounded on $[x_0, b]$. To see this, we differentiate

$$y'(x) = f(x, y(x))$$

to obtain

$$\begin{aligned} y''(x) &= f_x(x, y(x)) + f_y(x, y(x))y'(x) \\ &= f_x(x, y(x)) + f_y(x, y(x))f(x, y(x)). \end{aligned}$$

Since we assumed that f, f_x and f_y are bounded, there's a constant M such that

$$|f_x(x, y(x)) + f_y(x, y(x))f(x, y(x))| \le M, \quad x_0 < x < b,$$

which implies that

$$|y''(x)| \le M, \quad x_0 < x < b. \tag{3.1.9}$$

Since $x_{i+1} = x_i + h$, Taylor's theorem implies that

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\tilde{x}_i),$$

where $\tilde{x}_i$ is some number between x_i and x_{i+1}. Since $y'(x_i) = f(x_i, y(x_i))$ this can be written as

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2}y''(\tilde{x}_i),$$

or, equivalently,

$$y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)) = \frac{h^2}{2}y''(\tilde{x}_i).$$

Comparing this with (3.1.8) shows that

$$T_i = \frac{h^2}{2}y''(\tilde{x}_i).$$

Recalling (3.1.9), we can establish the bound

$$|T_i| \le \frac{Mh^2}{2}, \quad 1 \le i \le n. \tag{3.1.10}$$

Although it may be difficult to determine the constant M, what is important is that there's an M such that (3.1.10) holds. We say that the local truncation error of Euler's method is of order h^2, which we write as $O(h^2)$.

Note that the magnitude of the local truncation error in Euler's method is determined by the second derivative y'' of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y''|$ is large, or smaller where $|y''|$ is small.

Since the local truncation error for Euler's method is $O(h^2)$, it's reasonable to expect that halving h reduces the local truncation error by a factor of 4. This is true, but halving the step size also requires twice as many steps to approximate the solution at a given point. To analyze the overall effect of truncation error in Euler's method, it's useful to derive an equation relating the errors

$$e_{i+1} = y(x_{i+1}) - y_{i+1} \quad \text{and} \quad e_i = y(x_i) - y_i.$$

To this end, recall that

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + T_i \tag{3.1.11}$$

and

$$y_{i+1} = y_i + hf(x_i, y_i). \tag{3.1.12}$$

Subtracting (3.1.12) from (3.1.11) yields

$$e_{i+1} = e_i + h\left[f(x_i, y(x_i)) - f(x_i, y_i)\right] + T_i. \tag{3.1.13}$$

The last term on the right is the local truncation error at the ith step. The other terms reflect the way errors made at previous steps affect e_{i+1}. Since $|T_i| \le Mh^2/2$, we see from (3.1.13) that

$$|e_{i+1}| \le |e_i| + h|f(x_i, y(x_i)) - f(x_i, y_i)| + \frac{Mh^2}{2}. \tag{3.1.14}$$

Since we assumed that f_y is continuous and bounded, the mean value theorem implies that

$$f(x_i, y(x_i)) - f(x_i, y_i) = f_y(x_i, y_i^*)(y(x_i) - y_i) = f_y(x_i, y_i^*)e_i,$$

where y_i^* is between y_i and $y(x_i)$. Therefore

$$|f(x_i, y(x_i)) - f(x_i, y_i)| \le R|e_i|$$

for some constant R. From this and (3.1.14),

$$|e_{i+1}| \le (1 + Rh)|e_i| + \frac{Mh^2}{2}, \quad 0 \le i \le n - 1. \tag{3.1.15}$$

For convenience, let $C = 1 + Rh$. Since $e_0 = y(x_0) - y_0 = 0$, applying (3.1.15) repeatedly yields

$$\begin{aligned}
|e_1| &\le \frac{Mh^2}{2} \\
|e_2| &\le C|e_1| + \frac{Mh^2}{2} \le (1 + C)\frac{Mh^2}{2} \\
|e_3| &\le C|e_2| + \frac{Mh^2}{2} \le (1 + C + C^2)\frac{Mh^2}{2} \\
&\vdots \\
|e_n| &\le C|e_{n-1}| + \frac{Mh^2}{2} \le (1 + C + \cdots + C^{n-1})\frac{Mh^2}{2}.
\end{aligned} \tag{3.1.16}$$

Recalling the formula for the sum of a geometric series, we see that

$$1 + C + \cdots + C^{n-1} = \frac{1 - C^n}{1 - C} = \frac{(1 + Rh)^n - 1}{Rh}$$

(since $C = 1 + Rh$). From this and (3.1.16),

$$|y(b) - y_n| = |e_n| \le \frac{(1 + Rh)^n - 1}{R} \frac{Mh}{2}. \tag{3.1.17}$$

Since Taylor's theorem implies that

$$1 + Rh < e^{Rh}$$

(verify),

$$(1 + Rh)^n < e^{nRh} = e^{R(b - x_0)} \quad (\text{since } nh = b - x_0).$$

This and (3.1.17) imply that

$$|y(b) - y_n| \le Kh, \tag{3.1.18}$$

with

$$K = M \frac{e^{R(b - x_0)} - 1}{2R}.$$

Because of (3.1.18) we say that the global truncation error of Euler's method is of order (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html) h, which we write as $O(h)$.

Semilinear Equations and Variation of Parameters

An equation that can be written in the form

$$y' + p(x)y = h(x, y) \tag{3.1.19}$$

with $p \not\equiv 0$ is said to be semilinear. (Of course, (3.1.19) is linear if h is independent of y.) One way to apply Euler's method to an initial value problem

$$y' + p(x)y = h(x, y), \quad y(x_0) = y_0 \tag{3.1.20}$$

for (3.1.19) is to think of it as

$$y' = f(x, y), \quad y(x_0) = y_0,$$

where

$$f(x, y) = -p(x)y + h(x, y).$$

However, we can also start by applying variation of parameters to (3.1.20), as in Sections 2.1 and 2.4; thus, we write the solution of (3.1.20) as $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation $y' + p(x)y = 0$. Then $y = uy_1$ is a solution of (3.1.20) if and only if u is a solution of the initial value problem

$$u' = h(x, uy_1(x))/y_1(x), \quad u(x_0) = y(x_0)/y_1(x_0). \tag{3.1.21}$$

We can apply Euler's method to obtain approximate values u_0, u_1, …, u_n of this initial value problem, and then take

$$y_i = u_i y_1(x_i)$$

as approximate values of the solution of (3.1.20). We'll call this procedure the Euler semilinear method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html).

The next two examples show that the Euler and Euler semilinear methods may yield drastically different results.

Example 3.1.4

In Example 2.1.7 we had to leave the solution of the initial value problem

$$y' - 2xy = 1, \quad y(0) = 3 \tag{3.1.22}$$

in the form

$$y = e^{x^2}\left(3 + \int_0^x e^{-t^2} dt\right) \tag{3.1.23}$$

because it was impossible to evaluate this integral exactly in terms of elementary functions. Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of (3.1.22) at $x = 0$, 0.2, 0.4, 0.6, …, 2.0 by **(a)** Euler's method; **(b)** the Euler semilinear method.

Solution (a) Rewriting (3.1.22) as

$$y' = 1 + 2xy, \quad y(0) = 3 \tag{3.1.24}$$

and applying Euler's method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.1.5. Because of the large differences between the estimates obtained for the three values of h, it would be clear that these results are useless even if the "exact" values were not included in the table.

***Table 3.1.5.** Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, with Euler's method.*

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	
0.0	3.000000000	3.000000000	3.000000000	3
0.2	3.200000000	3.262000000	3.294348537	3
0.4	3.656000000	3.802028800	3.881421103	3
0.6	4.440960000	4.726810214	4.888870783	5
0.8	5.706790400	6.249191282	6.570796235	6
1.0	7.732963328	8.771893026	9.419105620	10
1.2	11.026148659	13.064051391	14.405772067	16
1.4	16.518700016	20.637273893	23.522935872	27
1.6	25.969172024	34.570423758	41.033441257	50
1.8	42.789442120	61.382165543	76.491018246	98
2.0	73.797840446	115.440048291	152.363866569	211

It's easy to see why Euler's method yields such poor results. Recall that the constant M in (3.1.10) – which plays an important role in determining the local truncation error in Euler's method – must be an upper bound for the values of the second derivative y'' of the solution of the initial value problem (3.1.22) on $(0, 2)$. The problem is that y'' assumes very large values on this interval. To see this, we differentiate (3.1.24) to obtain

$$y''(x) = 2y(x) + 2xy'(x) = 2y(x) + 2x(1 + 2xy(x)) = 2(1 + 2x^2)y(x) + 2x,$$

where the second equality follows again from (3.1.24). Since (3.1.23) implies that $y(x) > 3e^{x^2}$ if $x > 0$,

$$y''(x) > 6(1 + 2x^2)e^{x^2} + 2x, \quad x > 0.$$

For example, letting $x = 2$ shows that $y''(2) > 2952$.

Solution (b) Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the Euler semilinear method to (3.1.22), with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.1.6 are clearly better than those obtained by Euler's method.

***Table 3.1.6.** Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the Euler semilinear method.*

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	
0.0	3.000000000	3.000000000	3.000000000	
0.2	3.330594477	3.329558853	3.328788889	
0.4	3.980734157	3.974067628	3.970230415	
0.6	5.106360231	5.087705244	5.077622723	
0.8	7.021003417	6.980190891	6.958779586	
1.0	10.350076600	10.269170824	10.227464299	1
1.2	16.381180092	16.226146390	16.147129067	1
1.4	27.890003380	27.592026085	27.441292235	2
1.6	51.183323262	50.594503863	50.298106659	5
1.8	101.424397595	100.206659076	99.595562766	9
2.0	217.301032800	214.631041938	213.293582978	21

We can't give a general procedure for determining in advance whether Euler's method or the semilinear Euler method will produce better results for a given semilinear initial value problem (3.1.19). As a rule of thumb, the Euler semilinear method will yield better results than Euler's method if $|u''|$ is small on $[x_0, b]$, while Euler's method yields better results if $|u''|$ is large on $[x_0, b]$. In many cases the results obtained by the two methods don't differ appreciably. However, we propose the an intuitive way to decide which is the better method: Try both methods with multiple step sizes, as we did in Example 3.1.4, and accept the results obtained by the method for which the approximations change less as the step size decreases.

Example 3.1.5

Applying Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' - 2y = \frac{x}{1 + y^2}, \quad y(1) = 7 \tag{3.1.25}$$

on $[1, 2]$ yields the results in Table 3.1.7. Applying the Euler semilinear method with

$$y = ue^{2x} \quad \text{and} \quad u' = \frac{xe^{-2x}}{1 + u^2e^{4x}}, \quad u(1) = 7e^{-2}$$

yields the results in Table 3.1.8. Since the latter are clearly less dependent on step size than the former, we conclude that the Euler semilinear method is better than Euler’s method for (3.1.25). This conclusion is supported by comparing the approximate results obtained by the two methods with the “exact” values of the solution.

Table 3.1.7. *Numerical solution of $y' - 2y = x/(1+y^2),\ y(1) = 7$, by Euler's method.*

x	$h=0.1$	$h=0.05$	$h=0.025$	
1.0	7.000000000	7.000000000	7.000000000	7.000
1.1	8.402000000	8.471970569	8.510493955	8.551
1.2	10.083936450	10.252570169	10.346014101	10.446
1.3	12.101892354	12.406719381	12.576720827	12.760
1.4	14.523152445	15.012952416	15.287872104	15.586
1.5	17.428443554	18.166277405	18.583079406	19.037
1.6	20.914624471	21.981638487	22.588266217	23.253
1.7	25.097914310	26.598105180	27.456479695	28.401
1.8	30.117766627	32.183941340	33.373738944	34.690
1.9	36.141518172	38.942738252	40.566143158	42.371
2.0	43.369967155	47.120835251	49.308511126	51.752

Table 3.1.8. *Numerical solution of $y' - 2y = x/(1+y^2),\ y(1) = 7$, by the Euler semilinear method.*

x	$h=0.1$	$h=0.05$	$h=0.025$	
1.0	7.000000000	7.000000000	7.000000000	7.000
1.1	8.552262113	8.551993978	8.551867007	8.551
1.2	10.447568674	10.447038547	10.446787646	10.446
1.3	12.762019799	12.761221313	12.760843543	12.760
1.4	15.588535141	15.587448600	15.586934680	15.586
1.5	19.040580614	19.039172241	19.038506211	19.037
1.6	23.256721636	23.254942517	23.254101253	23.253
1.7	28.406184597	28.403969107	28.402921581	28.401
1.8	34.695649222	34.692912768	34.691618979	34.690
1.9	42.377544138	42.374180090	42.372589624	42.371
2.0	51.760178446	51.756054133	51.754104262	51.752

Example 3.1.6

Applying Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' + 3x^2y = 1 + y^2, \quad y(2) = 2 \tag{3.1.26}$$

on $[2, 3]$ yields the results in Table 3.1.9. Applying the Euler semilinear method with

$$y = ue^{-x^3} \quad \text{and} \quad u' = e^{x^3}(1 + u^2e^{-2x^3}), \quad u(2) = 2e^8$$

yields the results in Table 3.1.10. Noting the close agreement among the three columns of Table 3.1.9 (at least for larger values of x) and the lack of any such agreement among the columns of Table 3.1.10, we conclude that Euler's method is better than the Euler semilinear method for (3.1.26). Comparing the results with the exact values supports this conclusion.

Table 3.1.9. *Numerical solution of* $y' + 3x^2y = 1 + y^2, \quad y(2) = 2$, *by Euler's method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exa
2.0	2.000000000	2.000000000	2.000000000	2.0000000
2.1	0.100000000	0.493231250	0.609611171	0.7011629
2.2	0.068700000	0.122879586	0.180113445	0.2369868
2.3	0.069419569	0.070670890	0.083934459	0.1038157
2.4	0.059732621	0.061338956	0.063337561	0.0683907
2.5	0.056871451	0.056002363	0.056249670	0.0572810
2.6	0.050560917	0.051465256	0.051517501	0.0517116
2.7	0.048279018	0.047484716	0.047514202	0.0475641
2.8	0.042925892	0.043967002	0.043989239	0.0440144
2.9	0.042148458	0.040839683	0.040857109	0.0408753
3.0	0.035985548	0.038044692	0.038058536	0.0380728

Table 3.1.10. *Numerical solution of* $y' + 3x^2y = 1 + y^2, \quad y(2) = 2$, *by the Euler semilinear method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exa
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = .0$
2.0	2.000000000	2.000000000	2.000000000	2.0000000
2.1	0.708426286	0.702568171	0.701214274	0.7011629
2.2	0.214501852	0.222599468	0.228942240	0.2369868
2.3	0.069861436	0.083620494	0.092852806	0.1038157
2.4	0.032487396	0.047079261	0.056825805	0.0683907
2.5	0.021895559	0.036030018	0.045683801	0.0572810
2.6	0.017332058	0.030750181	0.040189920	0.0517116
2.7	0.014271492	0.026931911	0.036134674	0.0475641
2.8	0.011819555	0.023720670	0.032679767	0.0440144
2.9	0.009776792	0.020925522	0.029636506	0.0408753
3.0	0.008065020	0.018472302	0.026931099	0.0380728

In the next two sections we'll study other numerical methods for solving initial value problems, called the improved Euler method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html), the midpoint method, Heun's method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Heun.html) and the Runge (http://www-history.mcs.st-and.ac.uk/Mat hematicians/Runge.html)-Kutta method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kutta.html). If the initial value problem is semilinear as in (3.1.19), we also have the option of using variation of parameters and then applying the given numerical method to the initial value problem (3.1.21) for u. By analogy with the terminology used here, we'll call the resulting procedure the improved Euler semilinear method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html), the *midpoint semilinear method*, Heun's semilinear method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Heun.html) or the Runge (http://www-history.mcs.st-and.ac.uk/Mathematicians/Runge.html) -Kutta semilinear method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kutta.html), as the case may be.

3.1 Exercises

You may want to save the results of these exercises, sincewe'll revisit in the next two sections. In Exercises 1–5 use Euler's method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point wher the initial condition is imposed and $i = 1, 2, 3$. The purpose of these exercises is to familiarize you with the computational procedure of Euler's method.

1. C $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$

Show answer

2. C $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$

Show answer

3. C $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$

Show answer

4. C $y' = $ \dst $\frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$

Show answer

5. C $y' + x^2y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

Show answer

6. C Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.1.1.

Show answer

7. C Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.1.1.

Show answer

8. C Use Euler's method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0, 1.05, 1.10, 1.15, \ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}$$

obtained in Example 2.4.3. Present your results in a table like Table 3.1.1.

Show answer

9. C In Example 2.2.3 it was shown that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \ldots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of (x, y) appearing in the first table.

Show answer

10. C You can see from Example 2.5.1 that

$$x^4y^3 + x^2y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3y^3 + 2xy^5 + 2y}{3x^4y^2 + 5x^2y^4 + 2x}, \quad y(1) = 1. \tag{A}$$

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0,\ 1.1,\ 1.2,\ 1.3,\ \ldots,\ 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4y^3 + x^2y^5 + 2xy - 4$$

for each value of (x, y) appearing in the first table.

Show answer

11. C Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

Missing dimension or its units for \hspace

at $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \ldots,\ 1.0$.

Show answer

12. C Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

Missing dimension or its units for \hspace

at $x = 1.0,\ 1.1,\ 1.2,\ 1.3,\ \ldots,\ 2.0$.

Show answer

13. C Use Euler's method and the Euler semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{-3x}, \quad y(0) = 6$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution $y = e^{-3x}(7x + 6)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

Show answer

The linear initial value problems in Exercises 14–19 can't be solved exactly in terms of known elementary functions. In each exercise, use Euler's method and the Euler semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. C $y' - 2y =$ \dst $\frac{1}{1+x^2}, \quad y(2) = 2; \quad h = 0.1, 0.05, 0.025$ on $[2, 3]$

Show answer

15. C $y' + 2xy = x^2, \quad y(0) = 3$ (Exercise 2.1. 38); $h = 0.2, 0.1, 0.05$ on $[0, 2]$

Show answer

16. C \dst $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2;$ (Exercise 2.1. 39); $h = 0.2, 0.1, 0.05$ on $[1, 3]$

Show answer

17. C \dst $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0;$ (Exercise 2.1. 40); $h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$

Show answer

18. C \dst $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1;$ (Exercise 2.1. 41); $h = 0.2, 0.1, 0.05$ on $[0, 2]$

Show answer

19. C $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2;$ (Exercise 2.1. 42); $h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$

Show answer

In Exercises 20–22, use Euler's method and the Euler semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. C $y' + 3y = xy^2(y+1), \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025$ on $[0,1]$

Show answer

21. C $\dst y' - 4y = \frac{x}{y^2(y+1)}, \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025$ on $[0,1]$

Show answer

22. C $\dst y' + 2y = \frac{x^2}{1+y^2}, \quad y(2) = 1; \quad h = 0.1, 0.05, 0.025$ on $[2,3]$

Show answer

23. Numerical Quadrature. The fundamental theorem of calculus says that if f is continuous on a closed interval $[a, b]$ then it has an antiderivative F such that $F'(x) = f(x)$ on $[a, b]$ and

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{A}$$

This solves the problem of evaluating a definite integral if the integrand f has an antiderivative that can be found and evaluated easily. However, if f doesn't have this property, (A) doesn't provide a useful way to evaluate the definite integral. In this case we must resort to approximate methods. There's a class of such methods called numerical quadrature, where the approximation takes the form

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} c_i f(x_i), \tag{B}$$

where $a = x_0 < x_1 < \cdots < x_n = b$ are suitably chosen points and c_0, c_1, …, c_n are suitably chosen constants. We call (B) a quadrature formula.

a. Derive the quadrature formula

$$\int_a^b f(x)\,dx \approx h \sum_{i=0}^{n-1} f(a + ih) \quad (\text{where } h = (b-a)/n) \tag{C}$$

by applying Euler's method to the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

b. The quadrature formula (C) is sometimes called the left rectangle rule. Draw a figure that justifies this terminology.

c. L For several choices of a, b, and A, apply (C) to $f(x) = A$ with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.

d. L For several choices of a, b, A, and B, apply (C) to $f(x) = A + Bx$ with $n = 10$, 20, 40, 80, 160, 320. Compare your results with the exact answers and explain what you find.

3.2 The Improved Euler Method and Related Methods

In Section 3.1 we saw that the global truncation error of Euler's method is $O(h)$, which would seem to imply that we can achieve arbitrarily accurate results with Euler's method by simply choosing the step size sufficiently small. However, this isn't a good idea, for two reasons. First, after a certain point decreasing the step size will increase roundoff errors to the point where the accuracy will deteriorate rather than improve. The second and more important reason is that in most applications of numerical methods to an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{3.2.1}$$

the expensive part of the computation is the evaluation of f. Therefore we want methods that give good results for a given number of such evaluations. This is what motivates us to look for numerical methods better than Euler's.

To clarify this point, suppose we want to approximate the value of e by applying Euler's method to the initial value problem

$$y' = y, \quad y(0) = 1, \quad (\text{with solution } y = e^x)$$

on $[0, 1]$, with $h = 1/12$, $1/24$, and $1/48$, respectively. Since each step in Euler's method requires one evaluation of f, the number of evaluations of f in each of these attempts is $n = 12$, 24, and 48, respectively. In each case we accept y_n as an approximation to e. The second column of Table 3.2.1 shows the results. The first column of the table indicates the number of evaluations of f required to obtain the approximation, and the last column contains the value of e rounded to ten significant figures.

In this section we'll study the improved Euler method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html), which requires two evaluations of f at each step. We've used this method with $h = 1/6$, $1/12$, and $1/24$. The required number of evaluations of f were 12, 24, and 48, as in the three applications of Euler's method; however, you can see from the third column of Table 3.2.1 that the approximation to e obtained by the improved Euler method with only 12 evaluations of f is better than the approximation obtained by Euler's method with 48 evaluations.

In Section 3.1 we'll study the Runge (http://www-history.mcs.st-and.ac.uk/Mathematicians/Runge.html)- Kutta method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kutta.html), which requires four evaluations of f at each step. We've used this method with $h = 1/3$, $1/6$, and $1/12$. The required number of evaluations of f were again 12, 24, and 48, as in the three applications of Euler's method and the improved Euler method; however, you can see from the fourth column of Table 3.2.1 that the approximation to e obtained by the Runge-Kutta method with only 12 evaluations of f is better than the approximation obtained by the improved Euler method with 48 evaluations.

Table 3.2.1. *Approximations to e obtained by three numerical methods.*

n	Euler	Improved Euler	Runge-Kutta	Exact
12	2.613035290	2.707188994	2.718069764	2.7182818
24	2.663731258	2.715327371	2.718266612	2.7182818
48	2.690496599	2.717519565	2.718280809	2.7182818

The Improved Euler Method

The improved Euler method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html) for solving the initial value problem (3.2.1) is based on approximating the integral curve of (3.2.1) at $(x_i, y(x_i))$ by the line through $(x_i, y(x_i))$ with slope

$$m_i = \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2};$$

that is, m_i is the average of the slopes of the tangents to the integral curve at the endpoints of $[x_i, x_{i+1}]$. The equation of the approximating line is therefore

$$y = y(x_i) + \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2}(x - x_i). \tag{3.2.2}$$

Setting $x = x_{i+1} = x_i + h$ in (3.2.2) yields

$$y_{i+1} = y(x_i) + \frac{h}{2}\left(f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))\right) \tag{3.2.3}$$

as an approximation to $y(x_{i+1})$. As in our derivation of Euler's method, we replace $y(x_i)$ (unknown if $i > 0$) by its approximate value y_i; then (3.2.3) becomes

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y(x_{i+1}))\right).$$

However, this still won't work, because we don't know $y(x_{i+1})$, which appears on the right. We overcome this by replacing $y(x_{i+1})$ by $y_i + hf(x_i, y_i)$, the value that the Euler method would assign to y_{i+1}. Thus, the improved Euler method starts with the known value $y(x_0) = y_0$ and computes y_1, y_2, …, y_n successively with the formula

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right). \tag{3.2.4}$$

The computation indicated here can be conveniently organized as follows: given y_i, compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + h, y_i + hk_{1i}\right), \\ y_{i+1} &= y_i + \frac{h}{2}(k_{1i} + k_{2i}). \end{aligned}$$

The improved Euler method requires two evaluations of $f(x, y)$ per step, while Euler's method requires only one. However, we'll see at the end of this section that if f satisfies appropriate assumptions, the local truncation error with the improved Euler method is $O(h^3)$, rather than $O(h^2)$ as with Euler's method. Therefore the global truncation error with the improved Euler method is $O(h^2)$; however, we won't prove this.

We note that the magnitude of the local truncation error in the improved Euler method and other methods discussed in this section is determined by the third derivative y''' of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y'''|$ is large, or smaller where $|y'''|$ is small.

The next example, which deals with the initial value problem considered in Example 3.1.1, illustrates the computational procedure indicated in the improved Euler method.

Example 3.2.1

Use the improved Euler method with $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3e^{-2x}, \quad y(0) = 1 \tag{3.2.5}$$

at $x = 0.1, 0.2, 0.3$.

Solution As in Example 3.1.1, we rewrite (3.2.5) as

$$y' = -2y + x^3e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.2.1), with

$$f(x, y) = -2y + x^3e^{-2x},\ x_0 = 0, \ \text{ and } y_0 = 1.$$

The improved Euler method yields

$$\begin{aligned}
k_{10} &= f(x_0, y_0) = f(0,1) = -2, \\
k_{20} &= f(x_1, y_0 + hk_{10}) = f(.1, 1 + (.1)(-2)) \\
&= f(.1, .8) = -2(.8) + (.1)^3 e^{-.2} = -1.599181269, \\
y_1 &= y_0 + \frac{h}{2}(k_{10} + k_{20}), \\
&= 1 + (.05)(-2 - 1.599181269) = .820040937,
\end{aligned}$$

$$\begin{aligned}
k_{11} &= f(x_1, y_1) = f(.1, .820040937) = -2(.820040937) + (.1)^3 e^{-.2} = -1.639263142, \\
k_{21} &= f(x_2, y_1 + hk_{11}) = f(.2, .820040937 + .1(-1.639263142)), \\
&= f(.2, .656114622) = -2(.656114622) + (.2)^3 e^{-.4} = -1.306866684, \\
y_2 &= y_1 + \frac{h}{2}(k_{11} + k_{21}), \\
&= .820040937 + (.05)(-1.639263142 - 1.306866684) = .672734445,
\end{aligned}$$

$$\begin{aligned}
k_{12} &= f(x_2, y_2) = f(.2, .672734445) = -2(.672734445) + (.2)^3 e^{-.4} = -1.340106330, \\
k_{22} &= f(x_3, y_2 + hk_{12}) = f(.3, .672734445 + .1(-1.340106330)), \\
&= f(.3, .538723812) = -2(.538723812) + (.3)^3 e^{-.6} = -1.062629710, \\
y_3 &= y_2 + \frac{h}{2}(k_{12} + k_{22}) \\
&= .672734445 + (.05)(-1.340106330 - 1.062629710) = .552597643.
\end{aligned}$$

Example 3.2.2

Table 3.2.2 shows results of using the improved Euler method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. For comparison, it also shows the corresponding approximate values obtained with Euler's method in 3.1.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the improved Euler method with $h = 0.1$ are better than those obtained by Euler's method with $h = 0.05$.

Table 3.2.2. *Numerical solution of* $y' + 2y = x^3 e^{-2x},\ y(0) = 1$, *by Euler's method and the improved Euler method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h =$
0.0	1.000000000	1.000000000	1.000000000	1.0000000
0.1	0.800000000	0.810005655	0.820040937	0.8190505
0.2	0.640081873	0.656266437	0.672734445	0.6710864
0.3	0.512601754	0.532290981	0.552597643	0.5505438
0.4	0.411563195	0.432887056	0.455160637	0.4528906
0.5	0.332126261	0.353785015	0.376681251	0.3743357
0.6	0.270299502	0.291404256	0.313970920	0.3116522
0.7	0.222745397	0.242707257	0.264287611	0.2620676
0.8	0.186654593	0.205105754	0.225267702	0.2231942
0.9	0.159660776	0.176396883	0.194879501	0.1929817
1.0	0.139778910	0.154715925	0.171388070	0.1696806
	Euler	Improved Euler	Exact	

Example 3.2.3

Table 3.2.3 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1.$$

We applied Euler's method to this problem in Example 3.1.3.

Table 3.2.3. *Numerical solution of* $y' = -2y^2 + xy + x^2,\ y(0) = 1$, *by Euler's method and the improved Euler method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = ($
0.0	1.000000000	1.000000000	1.000000000	1.0000000
0.1	0.800000000	0.821375000	0.840500000	0.838288
0.2	0.681000000	0.707795377	0.733430846	0.730556
0.3	0.605867800	0.633776590	0.661600806	0.658552
0.4	0.559628676	0.587454526	0.615961841	0.612884
0.5	0.535376972	0.562906169	0.591634742	0.588558
0.6	0.529820120	0.557143535	0.586006935	0.582927
0.7	0.541467455	0.568716935	0.597712120	0.594618
0.8	0.569732776	0.596951988	0.626008824	0.622898
0.9	0.614392311	0.641457729	0.670351225	0.667237
1.0	0.675192037	0.701764495	0.730069610	0.726985
	Euler	Improved Euler	"Exact"	

Example 3.2.4

Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of

$$y' - 2xy = 1, \quad y(0) = 3 \tag{3.2.6}$$

at $x = 0,\ 0.2,\ 0.4,\ 0.6,\ \dots,\ 2.0$ by **(a)** the improved Euler method; **(b)** the improved Euler semilinear method. (We used Euler's method and the Euler semilinear method on this problem in 3.1.4.)

Solution (a) Rewriting (3.2.6) as

$$y' = 1 + 2xy, \quad y(0) = 3$$

and applying the improved Euler method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.2.4.

Solution (b) Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the improved Euler semilinear method to (3.2.6), with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.2.5 are clearly better than those obtained by the improved Euler method.

Table 3.2.4. Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the improved Euler method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	
0.0	3.000000000	3.000000000	3.000000000	
0.2	3.328000000	3.328182400	3.327973600	
0.4	3.964659200	3.966340117	3.966216690	
0.6	5.057712497	5.065700515	5.066848381	
0.8	6.900088156	6.928648973	6.934862367	
1.0	10.065725534	10.154872547	10.177430736	1
1.2	15.708954420	15.970033261	16.041904862	1
1.4	26.244894192	26.991620960	27.210001715	2
1.6	46.958915746	49.096125524	49.754131060	5
1.8	89.982312641	96.200506218	98.210577385	9
2.0	184.563776288	203.151922739	209.464744495	21

Table 3.2.5. Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the improved Euler semilinear method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	
0.0	3.000000000	3.000000000	3.000000000	3
0.2	3.326513400	3.327518315	3.327768620	3
0.4	3.963383070	3.965392084	3.965892644	3
0.6	5.063027290	5.066038774	5.066789487	5
0.8	6.931355329	6.935366847	6.936367564	6
1.0	10.178248417	10.183256733	10.184507253	10
1.2	16.059110511	16.065111599	16.066611672	16
1.4	27.280070674	27.287059732	27.288809058	27
1.6	49.989741531	49.997712997	49.999711226	50
1.8	98.971025420	98.979972988	98.982219722	98
2.0	211.941217796	211.951134436	211.953629228	211

A Family of Methods with $O(h^3)$ Local Truncation Error

We'll now derive a class of methods with $O(h^3)$ local truncation error for solving (3.2.1). For simplicity, we assume that f, f_x, f_y, f_{xx}, f_{yy}, and f_{xy} are continuous and bounded for all (x,y). This implies that if y is the solution of (3.2.1 then y'' and y''' are bounded (Exercise 31).

We begin by approximating the integral curve of (3.2.1) at $(x_i, y(x_i))$ by the line through $(x_i, y(x_i))$ with slope

$$m_i = \sigma y'(x_i) + \rho y'(x_i + \theta h),$$

where σ, ρ, and θ are constants that we'll soon specify; however, we insist at the outset that $0 < \theta \le 1$, so that

$$x_i < x_i + \theta h \le x_{i+1}.$$

The equation of the approximating line is

$$\begin{aligned} y &= y(x_i) + m_i(x - x_i) \\ &= y(x_i) + \left[\sigma y'(x_i) + \rho y'(x_i + \theta h)\right](x - x_i). \end{aligned} \tag{3.2.7}$$

Setting $x = x_{i+1} = x_i + h$ in (3.2.7) yields

$$\hat{y}_{i+1} = y(x_i) + h\left[\sigma y'(x_i) + \rho y'(x_i + \theta h)\right]$$

as an approximation to $y(x_{i+1})$.

To determine σ, ρ, and θ so that the error

$$\begin{aligned} E_i &= y(x_{i+1}) - \hat{y}_{i+1} \\ &= y(x_{i+1}) - y(x_i) - h\left[\sigma y'(x_i) + \rho y'(x_i + \theta h)\right] \end{aligned} \tag{3.2.8}$$

in this approximation is $O(h^3)$, we begin by recalling from Taylor's theorem that

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(\hat{x}_i),$$

where $\hat{x}_i$ is in (x_i, x_{i+1}). Since y''' is bounded this implies that

$$y(x_{i+1}) - y(x_i) - hy'(x_i) - \frac{h^2}{2}y''(x_i) = O(h^3).$$

Comparing this with (3.2.8) shows that $E_i = O(h^3)$ if

$$\sigma y'(x_i) + \rho y'(x_i + \theta h) = y'(x_i) + \frac{h}{2}y''(x_i) + O(h^2). \tag{3.2.9}$$

However, applying Taylor's theorem to y' shows that

$$y'(x_i + \theta h) = y'(x_i) + \theta h y''(x_i) + \frac{(\theta h)^2}{2}y'''(\overline{x}_i),$$

where $\overline{x}_i$ is in $(x_i, x_i + \theta h)$. Since y''' is bounded, this implies that

$$y'(x_i + \theta h) = y'(x_i) + \theta h y''(x_i) + O(h^2).$$

Substituting this into (3.2.9) and noting that the sum of two $O(h^2)$ terms is again $O(h^2)$ shows that $E_i = O(h^3)$ if

$$(\sigma + \rho)y'(x_i) + \rho\theta h y''(x_i) = y'(x_i) + \frac{h}{2}y''(x_i),$$

which is true if

$$\sigma + \rho = 1 \quad \text{and} \quad \rho\theta = \frac{1}{2}. \tag{3.2.10}$$

Since $y' = f(x, y)$, we can now conclude from (3.2.8) that

$$y(x_{i+1}) = y(x_i) + h\left[\sigma f(x_i, y_i) + \rho f(x_i + \theta h, y(x_i + \theta h))\right] + O(h^3) \tag{3.2.11}$$

if σ, ρ, and θ satisfy (3.2.10). However, this formula would not be useful even if we knew $y(x_i)$ exactly (as we would for $i = 0$), since we still wouldn't know $y(x_i + \theta h)$ exactly. To overcome this difficulty, we again use Taylor's theorem to write

$$y(x_i + \theta h) = y(x_i) + \theta h y'(x_i) + \frac{h^2}{2}y''(\tilde{x}_i),$$

where $\tilde{x}_i$ is in $(x_i, x_i + \theta h)$. Since $y'(x_i) = f(x_i, y(x_i))$ and y'' is bounded, this implies that

$$|y(x_i + \theta h) - y(x_i) - \theta h f(x_i, y(x_i))| \le K h^2 \tag{3.2.12}$$

for some constant K. Since f_y is bounded, the mean value theorem implies that

$$|f(x_i + \theta h, u) - f(x_i + \theta h, v)| \le M|u - v|$$

for some constant M. Letting

$$u = y(x_i + \theta h) \quad \text{and} \quad v = y(x_i) + \theta h f(x_i, y(x_i))$$

and recalling (3.2.12) shows that

$$f(x_i + \theta h, y(x_i + \theta h)) = f(x_i + \theta h, y(x_i) + \theta h f(x_i, y(x_i))) + O(h^2).$$

Substituting this into (3.2.11) yields

$$\begin{aligned} y(x_{i+1}) = y(x_i) + h\,[\sigma f(x_i, y(x_i)) + \\ \rho f(x_i + \theta h, y(x_i) + \theta h f(x_i, y(x_i)))] + O(h^3). \end{aligned}$$

This implies that the formula

$$y_{i+1} = y_i + h\,[\sigma f(x_i, y_i) + \rho f(x_i + \theta h, y_i + \theta h f(x_i, y_i))]$$

has $O(h^3)$ local truncation error if σ, ρ, and θ satisfy (3.2.10). Substituting $\sigma = 1 - \rho$ and $\theta = 1/2\rho$ here yields

$$y_{i+1} = y_i + h\left[(1-\rho) f(x_i, y_i) + \rho f\left(x_i + \frac{h}{2\rho}, y_i + \frac{h}{2\rho} f(x_i, y_i)\right)\right]. \tag{3.2.13}$$

The computation indicated here can be conveniently organized as follows: given y_i, compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2\rho}, y_i + \frac{h}{2\rho} k_{1i}\right), \\ y_{i+1} &= y_i + h[(1-\rho)k_{1i} + \rho k_{2i}]. \end{aligned}$$

Consistent with our requirement that $0 < \theta < 1$, we require that $\rho \geq 1/2$. Letting $\rho = 1/2$ in (3.2.13) yields the improved Euler method (3.2.4). Letting $\rho = 3/4$ yields Heun's method (http://www-history.mcs.st-and.ac.uk/Mathematicians/Heun.html),

$$y_{i+1} = y_i + h\left[\frac{1}{4} f(x_i, y_i) + \frac{3}{4} f\left(x_i + \frac{2}{3}h, y_i + \frac{2}{3} h f(x_i, y_i)\right)\right],$$

which can be organized as

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{2h}{3}, y_i + \frac{2h}{3} k_{1i}\right), \\ y_{i+1} &= y_i + \frac{h}{4}(k_{1i} + 3k_{2i}). \end{aligned}$$

Letting $\rho = 1$ yields the midpoint method,

$$y_{i+1} = y_i + hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}f(x_i, y_i)\right),$$

which can be organized as

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right), \\ y_{i+1} &= y_i + hk_{2i}. \end{aligned}$$

Examples involving the midpoint method and Heun's method are given in Exercises 23-30.

3.2 Exercises

Most of the following numerical exercises involve initial value problems considered in the exercises in Section 3.1. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in Section 3.1.

In Exercises 1–5 use the improved Euler method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1$, 2, 3.

1. C $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$

Show answer

2. C $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$

Show answer

3. C $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$

Show answer

4. C $y' = \backslash\mathrm{dst}\,\frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$

Show answer

5. C $y' + x^2y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

Show answer

6. C Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \ldots,\ 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.2.2.

Show answer

7. C Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0,\ 1.1,\ 1.2,\ 1.3,\ \ldots,\ 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9\ln x + x^3 + 2)$$

which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.2.2.

Show answer

8. C Use the improved Euler method with step sizes $h=0.05$, $h=0.025$, and $h=0.0125$ to find approximate values of the solution of the initial value problem

$$y'=\frac{y^2+xy-x^2}{x^2},\quad y(1)=2,$$

at $x=1.0,\,1.05,\,1.10,\,1.15,\,\dots,\,1.5$. Compare these approximate values with the values of the exact solution

$$y=\frac{x(1+x^2/3)}{1-x^2/3}$$

obtained in Example 2.4.3. Present your results in a table like Table 3.2.2.

Show answer

9. C In Example 3.2.2 it was shown that

$$y^5+y=x^2+x-4$$

is an implicit solution of the initial value problem

$$y'=\frac{2x+1}{5y^4+1},\quad y(2)=1. \tag{A}$$

Use the improved Euler method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of (A) at $x=2.0,\,2.1,\,2.2,\,2.3,\,\dots,\,3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x,y)=y^5+y-x^2-x+4$$

for each value of (x,y) appearing in the first table.

Show answer

10. C You can see from Example 2.5.1 that

$$x^4y^3 + x^2y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3y^3 + 2xy^5 + 2y}{3x^4y^2 + 5x^2y^4 + 2x}, \quad y(1) = 1. \tag{A}$$

Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0,\ 1.14,\ 1.2,\ 1.3,\ \dots,\ 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4y^3 + x^2y^5 + 2xy - 4$$

for each value of (x, y) appearing in the first table.

Show answer

11. C Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

Missing dimension or its units for \hspace

at $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \dots,\ 1.0$.

Show answer

12. C Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

Missing dimension or its units for \hspace

at $x = 1.0,\ 1.1,\ 1.2,\ 1.3,\ \dots,\ 2.0$.

Show answer

13. C Use the improved Euler method and the improved Euler semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = e^{-3x}(1 - 2x), \quad y(0) = 2,$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution $y = e^{-3x}(2 + x - x^2)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

Show answer

The linear initial value problems in Exercises 14–19 can't be solved exactly in terms of known elementary functions. In each exercise use the improved Euler and improved Euler semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. C $y' - 2y =$ \dst $\frac{1}{1+x^2}, \quad y(2) = 2; \qquad h = 0.1, 0.05, 0.025$ on $[2, 3]$

Show answer

15. C $y' + 2xy = x^2, \quad y(0) = 3; \quad h = 0.2, 0.1, 0.05$ on $[0, 2]$ (Exercise 2.1. 38)

Show answer

16. C \dst $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2, \qquad h = 0.2, 0.1, 0.05$ on $[1, 3]$ (Exercise 2.1.39)

Show answer

17. C \dst $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0; \quad h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$ (Exercise 2.1. 40),

Show answer

18. C \dst $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1; \; h = 0.2, 0.1, 0.05$ on $[0, 2]$ (Exercise 2.1. 41)

Show answer

19. C $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2; \; h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$ (Exercise 2.1. 42)

Show answer

In Exercises 20–22 use the improved Euler method and the improved Euler semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. C $y' + 3y = xy^2(y+1), \quad y(0) = 1; \qquad h = 0.1, 0.05, 0.025$ on $[0,1]$

Show answer

21. C $\dst y' - 4y = \frac{x}{y^2(y+1)}, \quad y(0) = 1; \qquad h = 0.1, 0.05, 0.025$ on $[0,1]$

Show answer

22. C $\dst y' + 2y = \frac{x^2}{1+y^2}, \quad y(2) = 1; \qquad h = 0.1, 0.05, 0.025$ on $[2,3]$

Show answer

23. C Do Exercise 7 with “improved Euler method” replaced by “midpoint method.”

Show answer

24. C Do Exercise 7 with “improved Euler method” replaced by “Heun’s method.”

Show answer

25. C Do Exercise 8 with “improved Euler method” replaced by “midpoint method.”

Show answer

26. C Do Exercise 8 with “improved Euler method” replaced by “Heun’s method.”

Show answer

27. C Do Exercise 11 with “improved Euler method” replaced by “midpoint method.”

Show answer

28. C Do Exercise 11 with “improved Euler method” replaced by “Heun’s method.”

Show answer

29. C Do Exercise 12 with “improved Euler method” replaced by “midpoint method.”

Show answer

30. C Do Exercise 12 with “improved Euler method” replaced by “Heun’s method.”

Show answer

31. Show that if f, f_x, f_y, f_{xx}, f_{yy}, and f_{xy} are continuous and bounded for all (x,y) and y is the solution of the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0,$$

then y'' and y''' are bounded.

32. Numerical Quadrature (see Exercise 3.1. 23).

a. Derive the quadrature formula

$$\int_a^b f(x)\,dx \approx .5h(f(a)+f(b)) + h\sum_{i=1}^{n-1} f(a+ih) \quad (\text{where } h=(b-a)/n) \tag{A}$$

by applying the improved Euler method to the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

b. The quadrature formula (A) is called the trapezoid rule. Draw a figure that justifies this terminology.

c. L For several choices of a, b, A, and B, apply (A) to $f(x) = A + Bx$, with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.

d. L For several choices of a, b, A, B, and C, apply (A) to $f(x) = A + Bx + Cx^2$, with $n = 10$, 20, 40, 80, 160, 320. Compare your results with the exact answers and explain what you find.

3.3 The Runge-Kutta Method

In general, if k is any positive integer and f satisfies appropriate assumptions, there are numerical methods with local truncation error $O(h^{k+1})$ for solving an initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0. \tag{3.3.1}$$

Moreover, it can be shown that a method with local truncation error $O(h^{k+1})$ has global truncation error $O(h^k)$. In Sections 3.1 and 3.2 we studied numerical methods where $k = 1$ and $k = 2$. We'll skip methods for which $k = 3$ and proceed to the Runge (http://www-history.mcs.st-and.ac.uk/Mathematicians/Runge.html)-Kutta (http://www-history.mcs.st-and.ac.uk/Mathematicians/Kutta.html) method, the most widely used method, for which $k = 4$. The magnitude of the local truncation error is determined by the fifth derivative $y^{(5)}$ of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y^{(5)}|$ is large, or smaller where $|y^{(5)}|$ is small. The Runge-Kutta method computes approximate values y_1, y_2, …, y_n of the solution of (3.3.1) at x_0, $x_0 + h$, …, $x_0 + nh$ as follows: Given y_i, compute

$$\begin{aligned}
k_{1i} &= f(x_i, y_i),\\
k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right),\\
k_{3i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{2i}\right),\\
k_{4i} &= f(x_i + h, y_i + hk_{3i}),
\end{aligned}$$

and

$$y_{i+1}=y_i+\frac{h}{6}(k_{1i}+2k_{2i}+2k_{3i}+k_{4i}).$$

The next example, which deals with the initial value problem considered in Examples 3.1.1 and 3.2.1, illustrates the computational procedure indicated in the Runge-Kutta method.

Example 3.3.1

Use the Runge-Kutta method with $h=0.1$ to find approximate values for the solution of the initial value problem

$$y'+2y=x^3e^{-2x},\quad y(0)=1, \tag{3.3.2}$$

at $x=0.1,0.2$.

Solution Again we rewrite (3.3.2) as

$$y'=-2y+x^3e^{-2x},\quad y(0)=1,$$

which is of the form (3.3.1), with

$$f(x,y)=-2y+x^3e^{-2x},\ x_0=0,\ \text{and}\ y_0=1.$$

The Runge-Kutta method yields

$$\begin{aligned}
k_{10} &= f(x_0,y_0)=f(0,1)=-2,\\
k_{20} &= f(x_0+h/2,y_0+hk_{10}/2)=f(.05,1+(.05)(-2))\\
&= f(.05,.9)=-2(.9)+(.05)^3e^{-.1}=-1.799886895,\\
k_{30} &= f(x_0+h/2,y_0+hk_{20}/2)=f(.05,1+(.05)(-1.799886895))\\
&= f(.05,.910005655)=-2(.910005655)+(.05)^3e^{-.1}=-1.819898206,\\
k_{40} &= f(x_0+h,y_0+hk_{30})=f(.1,1+(.1)(-1.819898206))\\
&= f(.1,.818010179)=-2(.818010179)+(.1)^3e^{-.2}=-1.635201628,\\
y_1 &= y_0+\frac{h}{6}(k_{10}+2k_{20}+2k_{30}+k_{40}),\\
&= 1+\frac{.1}{6}(-2+2(-1.799886895)+2(-1.819898206)-1.635201628)=.818753803,
\end{aligned}$$

$$\begin{aligned}
k_{11} &= f(x_1,y_1)=f(.1,.818753803)=-2(.818753803))+(.1)^3e^{-.2}=-1.636688875,\\
k_{21} &= f(x_1+h/2,y_1+hk_{11}/2)=f(.15,.818753803+(.05)(-1.636688875))\\
&= f(.15,.736919359)=-2(.736919359)+(.15)^3e^{-.3}=-1.471338457,\\
k_{31} &= f(x_1+h/2,y_1+hk_{21}/2)=f(.15,.818753803+(.05)(-1.471338457))\\
&= f(.15,.745186880)=-2(.745186880)+(.15)^3e^{-.3}=-1.487873498,\\
k_{41} &= f(x_1+h,y_1+hk_{31})=f(.2,.818753803+(.1)(-1.487873498))\\
&= f(.2,.669966453)=-2(.669966453)+(.2)^3e^{-.4}=-1.334570346,\\
y_2 &= y_1+\frac{h}{6}(k_{11}+2k_{21}+2k_{31}+k_{41}),\\
&= .818753803+\frac{.1}{6}(-1.636688875+2(-1.471338457)+2(-1.487873498)-1.334570346)\\
&= .670592417.
\end{aligned}$$

The Runge-Kutta method is sufficiently accurate for most applications.

Example 3.3.2

Table 3.3.1 shows results of using the Runge-Kutta method with step sizes $h=0.1$ and $h=0.05$ to find approximate values of the solution of the initial value problem

$$y'+2y=x^3e^{-2x},\quad y(0)=1$$

at $x=0$, 0.1, 0.2, 0.3, …, 1.0. For comparison, it also shows the corresponding approximate values obtained with the improved Euler method in Example 3.2.2, and the values of the exact solution

$$y=\frac{e^{-2x}}{4}(x^4+4).$$

The results obtained by the Runge-Kutta method are clearly better than those obtained by the improved Euler method in fact; the results obtained by the Runge-Kutta method with $h=0.1$ are better than those obtained by the improved Euler method with $h=0.05$.

Table 3.3.1. *Numerical solution of $y' + 2y = x^3e^{-2x},\ y(0) = 1$, by the Runge-Kuttta method and the improved Euler method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h =$
0.0	1.000000000	1.000000000	1.000000000	1.000000
0.1	0.820040937	0.819050572	0.818753803	0.818751
0.2	0.672734445	0.671086455	0.670592417	0.670588
0.3	0.552597643	0.550543878	0.549928221	0.549923
0.4	0.455160637	0.452890616	0.452210430	0.452205
0.5	0.376681251	0.374335747	0.373633492	0.373627
0.6	0.313970920	0.311652239	0.310958768	0.310953
0.7	0.264287611	0.262067624	0.261404568	0.261399
0.8	0.225267702	0.223194281	0.222575989	0.222571
0.9	0.194879501	0.192981757	0.192416882	0.192412
1.0	0.171388070	0.169680673	0.169173489	0.169169
	Improved Euler	Runge-Kutta	Exact	

Example 3.3.3

Table 3.3.2 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1.$$

We applied the improved Euler method to this problem in Example 3.

Table 3.3.2. *Numerical solution of* $y' = -2y^2 + xy + x^2,\ y(0) = 1$, *by the Runge-Kuttta method and the improved Euler method.*

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h =$
0.0	1.000000000	1.000000000	1.000000000	1.0000000
0.1	0.840500000	0.838288371	0.837587192	0.8375847
0.2	0.733430846	0.730556677	0.729644487	0.7296421
0.3	0.661600806	0.658552190	0.657582449	0.6575805
0.4	0.615961841	0.612884493	0.611903380	0.6119019
0.5	0.591634742	0.588558952	0.587576716	0.5875756
0.6	0.586006935	0.582927224	0.581943210	0.5819423
0.7	0.597712120	0.594618012	0.593630403	0.5936296
0.8	0.626008824	0.622898279	0.621908378	0.6219075
0.9	0.670351225	0.667237617	0.666251988	0.6662509
1.0	0.730069610	0.726985837	0.726017378	0.7260159
	Improved Euler	Runge-Kutta	"Exact"	

Example 3.3.4

Tables 3.3.3 and 3.3.4 show results obtained by applying the Runge-Kutta and Runge-Kutta semilinear methods to to the initial value problem

$$y' - 2xy = 1,\ y(0) = 3,$$

which we considered in Examples 3.1.4 and 3.2.4.

Table 3.3.3. *Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the Runge-Kutta method.*

x	$h = 0.2$	$h = 0.1$	$h = 0.05$
0.0	3.000000000	3.000000000	3.000000000
0.2	3.327846400	3.327851633	3.327851952
0.4	3.966044973	3.966058535	3.966059300
0.6	5.066996754	5.067037123	5.067039396
0.8	6.936534178	6.936690679	6.936700320
1.0	10.184232252	10.184877733	10.184920997
1.2	16.064344805	16.066915583	16.067098699
1.4	27.278771833	27.288605217	27.289338955
1.6	49.960553660	49.997313966	50.000165744
1.8	98.834337815	98.971146146	98.982136702
2.0	211.393800152	211.908445283	211.951167637

Table 3.3.4. *Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the Runge-Kutta semilinear method.*

x	$h = 0.2$	$h = 0.1$	$h = 0.05$
0.0	3.000000000	3.000000000	3.000000000
0.2	3.327853286	3.327852055	3.327851978
0.4	3.966061755	3.966059497	3.966059357
0.6	5.067042602	5.067039725	5.067039547
0.8	6.936704019	6.936701137	6.936700957
1.0	10.184926171	10.184924093	10.184923963
1.2	16.067111961	16.067111696	16.067111678
1.4	27.289389418	27.289392167	27.289392335
1.6	50.000370152	50.000377302	50.000377745
1.8	98.982955511	98.982968633	98.982969450
2.0	211.954439983	211.954460825	211.954462127

The Case Where x_0 Isn't The Left Endpoint

So far in this chapter we've considered numerical methods for solving an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{3.3.3}$$

on an interval $[x_0, b]$, for which x_0 is the left endpoint. We haven't discussed numerical methods for solving (3.3.3) on an interval $[a, x_0]$, for which x_0 is the right endpoint. To be specific, how can we obtain approximate values y_{-1}, y_{-2}, …, y_{-n} of the solution of (3.3.3) at $x_0 - h, \dots, x_0 - nh$, where $h = (x_0 - a)/n$? Here's the answer to this question:

Consider the initial value problem

$$z' = -f(-x, z), \quad z(-x_0) = y_0, \tag{3.3.4}$$

on the interval $[-x_0, -a]$, for which $-x_0$ is the left endpoint. Use a numerical method to obtain approximate values z_1, z_2, …, z_n of the solution of $(???)$ at $-x_0 + h$, $-x_0 + 2h$, …, $-x_0 + nh = -a$. Then $y_{-1} = z_1$, $y_{-2} = z_2$, …, $y_{-n} = z_n$ are approximate values of the solution of $(???)$ at $x_0 - h$, $x_0 - 2h$, …, $x_0 - nh = a$.

The justification for this answer is sketched in Exercise 23. Note how easy it is to make the change the given problem (3.3.3) to the modified problem (3.3.4): first replace f by $-f$ and then replace x, x_0, and y by $-x$, $-x_0$, and z, respectively.

Example 3.3.5

Use the Runge-Kutta method with step size $h = 0.1$ to find approximate values of the solution of

$$(y - 1)^2 y' = 2x + 3, \quad y(1) = 4 \tag{3.3.5}$$

at $x = 0$, 0.1, 0.2, …, 1.

Solution We first rewrite (3.3.5) in the form (3.3.3) as

$$y' = \frac{2x + 3}{(y - 1)^2}, \quad y(1) = 4. \tag{3.3.6}$$

Since the initial condition $y(1) = 4$ is imposed at the right endpoint of the interval $[0, 1]$, we apply the Runge-Kutta method to the initial value problem

$$z' = \frac{2x - 3}{(z - 1)^2}, \quad z(-1) = 4 \tag{3.3.7}$$

on the interval $[-1, 0]$. (You should verify that (3.3.7) is related to (3.3.6) as (3.3.4) is related to (3.3.3).) Table 3.3.5 shows the results. Reversing the order of the rows in Table 3.3.5 and changing the signs of the values of x yields the first two columns of Table 3.3.6. The last column of Table 3.3.6 shows the exact values of y, which are given by

$$y = 1 + (3x^2 + 9x + 15)^{1/3}.$$

(Since the differential equation in (3.3.6) is separable, this formula can be obtained by the method of Section 2.2.)

Table 3.3.5. *Numerical solution of* $z' = \dst\frac{2x-3}{(z-1)^2},\ z(-1)=4$, *on* $[-1,0]$.

x	z
-1.0	4.000000000
-0.9	3.944536474
-0.8	3.889298649
-0.7	3.834355648
-0.6	3.779786399
-0.5	3.725680888
-0.4	3.672141529
-0.3	3.619284615
-0.2	3.567241862
-0.1	3.516161955
0.0	3.466212070

Table 3.3.6. *Numerical solution of* $(y-1)^2y' = 2x+3,\ y(1)=4$, *on* $[0,1]$.

x	y	Exact
0.00	3.466212070	3.466212074
0.10	3.516161955	3.516161958
0.20	3.567241862	3.567241864
0.30	3.619284615	3.619284617
0.40	3.672141529	3.672141530
0.50	3.725680888	3.725680889
0.60	3.779786399	3.779786399
0.70	3.834355648	3.834355648
0.80	3.889298649	3.889298649
0.90	3.944536474	3.944536474
1.00	4.000000000	4.000000000

We leave it to you to develop a procedure for handling the numerical solution of (3.3.3) on an interval $[a,b]$ such that $a < x_0 < b$ (Exercises 26 and 27).

3.3 Exercises

Most of the following numerical exercises involve initial value problems considered in the exercises in Sections 3.2. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in those sections.

In Exercises 1–5 use the Runge-Kutta method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1$, 2.

1. C $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$

Show answer

2. C $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$

Show answer

3. C $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$

Show answer

4. C $y' = \dst\frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$

Show answer

5. C $y' + x^2y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

Show answer

6. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2,$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.3.1.

Show answer

7. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9\ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.3.1.

Show answer

8. C Use the Runge-Kutta method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0, 1.05, 1.10, 1.15 \dots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3},$$

which was obtained in Example 2.2.3. Present your results in a table like Table 3.3.1.

Show answer

9. C In Example 2.2.3 it was shown that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \dots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of (x, y) appearing in the first table.

Show answer

10. C You can see from Example 2.5.1 that

$$x^4y^3 + x^2y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3y^3 + 2xy^5 + 2y}{3x^4y^2 + 5x^2y^4 + 2x}, \quad y(1) = 1. \tag{A}$$

Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4y^3 + x^2y^5 + 2xy - 4$$

for each value of (x, y) appearing in the first table.

Show answer

11. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

Missing dimension or its units for \hspace

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$.

Show answer

12. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

Missing dimension or its units for \hspace

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$.

Show answer

13. C Use the Runge-Kutta method and the Runge-Kutta semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = e^{-3x}(1 - 4x + 3x^2 - 4x^3), \quad y(0) = -3$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution $y = -e^{-3x}(3 - x + 2x^2 - x^3 + x^4)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

Show answer

The linear initial value problems in Exercises 14–19 can't be solved exactly in terms of known elementary functions. In each exercise use the Runge-Kutta and the Runge-Kutta semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. C $y' - 2y = \dst \frac{1}{1+x^2}, \quad y(2) = 2; \quad h = 0.1, 0.05, 0.025$ on $[2, 3]$

Show answer

15. C $y' + 2xy = x^2, \quad y(0) = 3; \quad h = 0.2, 0.1, 0.05$ on $[0, 2]$ (Exercise 2.1. 38)

Show answer

16. C $\dst y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2; \quad h = 0.2, 0.1, 0.05$ on $[1, 3]$ (Exercise 2.1. 39)

Show answer

17. C $\dst y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0; \quad h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$ (Exercise 2.1. 40)

Show answer

18. C $\dst y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1; \quad h = 0.2, 0.1, 0.05$ on $[0, 2]$ (Exercise 2.1, 41)

Show answer

19. C $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2; \quad h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$ (Exercise 2.1. 42)

Show answer

In Exercises 20–22 use the Runge-Kutta method and the Runge-Kutta semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. C $y' + 3y = xy^2(y+1), \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025$ on $[0,1]$

Show answer

21. C $\dst y' - 4y = \frac{x}{y^2(y+1)}, \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025$ on $[0,1]$

Show answer

22. C $\dst y' + 2y = \frac{x^2}{1+y^2}, \quad y(2) = 1; \quad h = 0.1, 0.05, 0.025$ on $[2,3]$

Show answer

23. C Suppose $a < x_0$, so that $-x_0 < -a$. Use the chain rule to show that if z is a solution of

$$z' = -f(-x, z), \quad z(-x_0) = y_0,$$

on $[-x_0, -a]$, then $y = z(-x)$ is a solution of

$$y' = f(x, y), \quad y(x_0) = y_0,$$

on $[a, x_0]$.

24. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(2) = -1$$

at $x = 1.1,\ 1.2,\ 1.3,\ \dots\ 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(4 - 3x^2)}{4 + 3x^2},$$

which can be obtained by referring to Example 2.4.3.

Show answer

25. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$y' = -x^2 y - xy^2, \quad y(1) = 1$$

at $x = 0,\ 0.1,\ 0.2,\ \dots,\ 1$.

Show answer

26. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$y' + \frac{1}{x}y = \frac{7}{x^2} + 3, \quad y(1) = \frac{3}{2}$$

at $x = 0.5,\ 0.6, \dots,\ 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{7\ln x}{x} + \frac{3x}{2},$$

which can be obtained by the method discussed in Section 2.1.

Show answer

27. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$xy' + 2y = 8x^2, \quad y(2) = 5$$

at $x = 1.0,\ 1.1,\ 1.2,\ \dots,\ 3.0$. Compare these approximate values with the values of the exact solution

$$y = 2x^2 - \frac{12}{x^2},$$

which can be obtained by the method discussed in Section 2.1.

Show answer

28. Numerical Quadrature (see Exercise 3.1. 23).

a. Derive the quadrature formula

$$\int_a^b f(x)\,dx \approx \frac{h}{6}(f(a)+f(b)) + \frac{h}{3}\sum_{i=1}^{n-1} f(a+ih) + \frac{2h}{3}\sum_{i=1}^{n} f\left(a+(2i-1)h/2\right) \tag{A}$$

(where $h=(b-a)/n$) by applying the Runge-Kutta method to the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

This quadrature formula is called Simpson’s Rule (http://www-history.mcs.st-and.ac.uk/Mathematicians/Simpson.html).

b. L For several choices of a, b, A, B, C, and D apply (A) to $f(x) = A + Bx + Cx + Dx^3$, with $n = 10$, 20, 40, 80, 160, 320. Compare your results with the exact answers and explain what you find.

c. L For several choices of a, b, A, B, C, D, and E apply (A) to $f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.

Chapter 3 Numerical Methods

3.1 Euler's Method

1 $y_1 = 1.450000000,\ y_2 = 2.085625000,\ y_3 = 3.079099746$

2 $y_1 = 1.200000000,\ y_2 = 1.440415946,\ y_3 = 1.729880994$

3 $y_1 = 1.900000000,\ y_2 = 1.781375000,\ y_3 = 1.646612970$

4 $y_1 = 2.962500000,\ y_2 = 2.922635828,\ y_3 = 2.880205639$

5 $y_1 = 2.513274123,\ y_2 = 1.814517822,\ y_3 = 1.216364496$

6

x
1.0

7

x
2.0

8

x
1.50

9

x
3.0

10

x
2.0

11

x
1.0

12

x
2.0

13

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	0.538871178	0.593002325	0.620131525	0.647231889

Euler semilinear method
x
1.0

Applying variation of parameters to the given initial value problem yields

> $y = ue^{-3x}$, where (A) $u' = 7, \quad u(0) = 6$. Since $u'' = 0$, Euler's method yields the exact solution of (A). Therefore the Euler semilinear method produces the exact solution of the given problem

.

14

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	12.804226135	13.912944662	14.559623055	15.282004826

Euler semilinear method
x
3.0

15

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.867565004	0.885719263	0.895024772	0.904276722

Eule semilir meth
x
2.0

16

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.922094379	0.945604800	0.956752868	0.967523153

Eule semilir meth
x
3.0

17

Euler's method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.319892131	0.330797109	0.337020123	0.343780513

Eule semilir meth
x
1.5

18

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.754572560	0.743869878	0.738303914	0.732638628

Eule semilir meth
x
2.0

19

Euler's method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.175959970	2.210259554	2.227207500	2.244023982

Eule semilir meth
x
1.5

20

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.032105117	0.043997045	0.050159310	0.056415515

Eule semilir meth
x
1.0

21

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	28.987816656	38.426957516	45.367269688	54.729594761

Eule semilii meth
x
1.(

22

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.361427907	1.361320824	1.361332589	1.361383810

Eule semilii meth
x
3.(

3.2 The Improved Euler Method and Related Methods

1 $y_1 = 1.542812500,\ y_2 = 2.421622101,\ y_3 = 4.208020541$

2 $y_1 = 1.220207973,\ y_2 = 1.489578775\ y_3 = 1.819337186$

3 $y_1 = 1.890687500,\ y_2 = 1.763784003,\ y_3 = 1.622698378$

4 $y_1 = 2.961317914\ \ y_2 = 2.920132727\ \ y_3 = 2.876213748.$

5 $y_1 = 2.478055238,\ y_2 = 1.844042564,\ y_3 = 1.313882333$

6

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	56.134480009	55.003390448	54.734674836	54.647937102

7

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.353501839	1.353288493	1.353219485	1.353193719

8

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.141969585	10.396770409	10.472502111	10.500000000

9

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
3.0	1.455674816	1.455935127	1.456001289	-0.00818	-0.00207	-0.000518
	Approximate Solutions	Residuals				

10

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
2.0	0.492862999	0.492709931	0.492674855	0.00335	0.000777	0.000187
	Approximate Solutions	Residuals				

11

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.660268159	0.660028505	0.659974464	0.659957689

12

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.749751364	-0.750637632	-0.750845571	-0.750912371

13 Applying variation of parameters to the given initial value problem

> $y = ue^{-3x}$, where (A) $u' = 1 - 2x,\quad u(0) = 2$. Since $u''' = 0$, the improved Euler method yields the exact solution of (A). Therefore the improved Euler semilinear method produces the exact solution of the given problem.

Improv Eule meth
x
1.0

Impro Eule semili meth
x
1.0

14

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
3.0	15.107600968	15.234856000	15.269755072	15.282004826

Impro Eule semili meth
x
3.0

15

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact"
2.0	0.924335375	0.907866081	0.905058201	0.904276722

Impro Eule semili meth
x
2.0

16

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.967473721	0.967510790	0.967520062	0.967523153

Impro Eule semili meth
x
3.0

17

Improved Euler method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.349176060	0.345171664	0.344131282	0.343780513

Impro Eule semili meth
x
1.5

18

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.732679223	0.732721613	0.732667905	0.732638628

Impro Eule semili meth
x
2.0

19

Improved Euler method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	“Exact"
1.50	2.247880315	2.244975181	2.244260143	2.244023982

Impro Eule semili meth
x
1.5

20

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
1.0	0.059071894	0.056999028	0.056553023	0.056415515

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
1.0	0.056295914	0.056385765	0.056408124	0.056415515

21

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
1.0	50.534556346	53.483947013	54.391544440	54.729594761

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
1.0	54.709041434	54.724083572	54.728191366	54.729594761

22

Improved Euler method				
x	$h=0.1$	$h=0.05$	$h=0.025$	“Exact"
3.0	1.361395309	1.361379259	1.361382239	1.361383810

Improved Euler semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	“Exact"
3.0	1.375699933	1.364730937	1.362193997	1.361383810

23

x	$h=0.1$	$h=0.05$	$h=0.025$	Exact
2.0	1.349489056	1.352345900	1.352990822	1.353193719

24

x	$h=0.1$	$h=0.05$	$h=0.025$	Exact
2.0	1.350890736	1.352667599	1.353067951	1.353193719

25

x	$h=0.05$	$h=0.025$	$h=0.0125$	Exact
1.50	10.133021311	10.391655098	10.470731411	10.500000000

26

x	$h=0.05$	$h=0.025$	$h=0.0125$	Exact
1.50	10.136329642	10.393419681	10.470731411	10.500000000

27

x	$h=0.1$	$h=0.05$	$h=0.025$	“Exact"
1.0	0.660846835	0.660189749	0.660016904	0.659957689

28

x	$h=0.1$	$h=0.05$	$h=0.025$	“Exact"
1.0	0.660658411	0.660136630	0.660002840	0.659957689

29

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
2.0	-0.750626284	-0.750844513	-0.750895864	-0.751331499

30

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
2.0	-0.750335016	-0.750775571	-0.750879100	-0.751331499

3.3 The Runge-Kutta Method

1 $y_1 = 1.550598190,\ y_2 = 2.469649729$

2 $y_1 = 1.221551366,\ y_2 = 1.492920208$

3 $y_1 = 1.890339767,\ y_2 = 1.763094323$

4 $y_1 = 2.961316248\ \ y_2 = 2.920128958.$

5 $y_1 = 2.475605264,\ y_2 = 1.825992433$

6

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	54.654509699	54.648344019	54.647962328	54.647937102

7

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.353191745	1.353193606	1.353193712	1.353193719

8

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.498658198	10.499906266	10.499993820	10.500000000

9

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
3.0	1.456023907	1.456023403	1.456023379	0.0000124	0.000000611	0.0000000333
	Approximate Solutions	Residuals				

10

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
2.0	0.492663789	0.492663738	0.492663736	0.000000902	0.0000000508	0.00000000302
	Approximate Solutions	Residuals				

11

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.659957046	0.659957646	0.659957686	0.659957689

12

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.750911103	-0.750912294	-0.750912367	-0.750912371

13 Applying variation of parameters to the given initial value problem yields $y = ue^{-3x}$, where (A) $u' = 1 - 4x + 3x^2 - 4x^3$, $u(0) = -3$. Since $u^{(5)} = 0$, the Runge-Kutta method yields the exact solution of (A). Therefore the Euler semilinear method produces the exact solution of the given problem.

	Runge-Kutta method			
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	-3.000000000	-3.000000000	-3.000000000	-3.000000000
0.1	-2.162598011	-2.162526572	-2.162522707	-2.162522468
0.2	-1.577172164	-1.577070939	-1.577065457	-1.577065117
0.3	-1.163350794	-1.163242678	-1.163236817	-1.163236453
0.4	-0.868030294	-0.867927182	-0.867921588	-0.867921241
0.5	-0.655542739	-0.655450183	-0.655445157	-0.655444845
0.6	-0.501535352	-0.501455325	-0.501450977	-0.501450707
0.7	-0.389127673	-0.389060213	-0.389056546	-0.389056318
0.8	-0.306468018	-0.306412184	-0.306409148	-0.306408959
0.9	-0.245153433	-0.245107859	-0.245105379	-0.245105226
1.0	-0.199187198	-0.199150401	-0.199148398	-0.199148273

Rung Kutt semili meth
x
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

14

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
3.0	15.281660036	15.281981407	15.282003300	15.282004826

Rung Kutt semili meth
x
3.0

15

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact"
2.0	0.904678156	0.904295772	0.904277759	0.904276722

Rung Kutt semili meth
x
2.0

16

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact"
3.0	0.967523147	0.967523152	0.967523153	0.967523153

Rung Kutt semili meth
x
3.0

17

Runge-Kutta method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	“Exact"
1.50	0.343839158	0.343784814	0.343780796	0.343780513

Rung Kutt semili meth
x
1.0
1.0
1.1
1.1
1.2
1.2
1.3
1.3
1.4
1.4
1.5

18

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact"
2.0	0.732633229	0.732638318	0.732638609	0.732638628

Rung Kutt semili meth
x
2.(

19

Runge-Kutta method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.244025683	2.244024088	2.244023989	2.244023982

Rung Kutt semili meth
x
1.5

20

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.056426886	0.056416137	0.056415552	0.056415515

Rung Kutt semili meth
x
1.0

21

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	54.695901186	54.727111858	54.729426250	54.729594761

Rung Kutt semili meth
x
1.0

22

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact"
3.0	1.361384082	1.361383812	1.361383809	1.361383810

Rung Kutt semili meth
x
3.(

24

x	$h = .1$	$h = .05$	$h = .025$	Exact
2.00	-1.000000000	-1.000000000	-1.000000000	-1.000000000

25

x	$h = .1$	$h = .05$	$h = .025$	“Exact"
1.00	1.000000000	1.000000000	1.000000000	1.000000000

26

x	$h = .1$	$h = .05$	$h = .025$	Exact
1.50	4.142171279	4.142170553	4.142170508	4.142170505

27

x	$h = .1$	$h = .05$	$h = .025$	Exact
3.0	16.666666988	16.666666687	16.666666668	16.666666667