13 Boundary Value Problems for Second Order Ordinary Differential Equations

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IN THIS CHAPTER we discuss boundary value problems and eigenvalue problems for linear second order ordinary differential equations.

Section 13.1 discusses point two-point boundary value problems for linear second order ordinary differential equations.

Section 13.2 deals with generalizations of the eigenvalue problems considered in Section 11.1

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Answers to selected exercises in this chapter

13.1 Two-Point Boundary Value Problems

In Section 5.3 we considered initial value problems for the linear second order equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x). \tag{13.1.1}$$

Suppose P_0, P_1, P_2, and F are continuous and P_0 has no zeros on an open interval (a, b). From Theorem 5.3.1, if x_0 is in (a, b) and k_1 and k_2 are arbitrary real numbers then (13.1.1) has a unique solution on (a, b) such that $y(x_0) = k_1$ and $y'(x_0) = k_2$. Now we consider a different problem for (13.1.1).

PROBLEM Suppose P_0, P_1, P_2, and F are continous and P_0 has no zeros on a closed interval $[a, b]$. Let α, β, ρ, and δ be real numbers such that

$$\alpha^2 + \beta^2 \neq 0 \quad \text{and} \quad \rho^2 + \delta^2 \neq 0, \tag{13.1.2}$$

and let k_1 and k_2 be arbitrary real numbers. Find a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{13.1.3}$$

on the closed interval $[a, b]$ such that

$$\alpha y(a)+\beta y'(a)=k_1 \tag{13.1.4}$$

and

$$\rho y(b)+\delta y'(b)=k_2. \tag{13.1.5}$$

The assumptions stated in this problem apply throughout this section and won't be repeated. Note that we imposed conditions on P_0, P_1, P_2, and F on the *closed* interval $[a,b]$, and we are interested in solutions of (13.1.3) on the closed interval. This is different from the situation considered in Chapter 5, where we imposed conditions on P_0, P_1, P_2, and F on the *open* interval (a,b) and we were interested in solutions on the open interval. There is really no problem here; we can always extend P_0, P_1, P_2, and F to an open interval (c,d) (for example, by defining them to be constant on $(c,d]$ and $[b,d)$) so that they are continuous and P_0 has no zeros on $[c,d]$. Then we can apply the theorems from Chapter 5 to the equation

$$y''+\frac{P_1(x)}{P_0(x)}y'+\frac{P_2(x)}{P_0(x)}y=\frac{F(x)}{P_0(x)}$$

on (c,d) to draw conclusions about solutions of (13.1.3) on $[a,b]$.

We call a and b *boundary points.* The conditions (13.1.4) and (13.1.5) are *boundary conditions*, and the problem is a *two-point boundary value problem* or, for simplicity, a *boundary value problem*. (We used similar terminology in Chapter 12 with a different meaning; both meanings are in common usage.) We require (13.1.2) to insure that we're imposing a sensible condition at each boundary point. For example, if $\alpha^2+\beta^2=0$ then $\alpha=\beta=0$, so $\alpha y(a)+\beta y'(a)=0$ for all choices of $y(a)$ and $y'(a)$. Therefore (13.1.4) is an impossible condition if $k_1\neq 0$, or no condition at all if $k_1=0$.

We abbreviate (13.1.1) as $Ly=F$, where

$$Ly=P_0(x)y''+P_1(x)y'+P_0(x)y,$$

and we denote

$$B_1(y)=\alpha y(a)+\beta y'(a)\ \text{ and }\quad B_2(y)=\rho y(b)+\delta y'(b).$$

We combine (13.1.3), (13.1.4), and (13.1.5) as

$$Ly=F,\qquad B_1(y)=k_1,\qquad B_2(y)=k_2. \tag{13.1.6}$$

This boundary value problem is *homogeneous* if $F=0$ and $k_1=k_2=0$; otherwise it's *nonhomogeneous*.

We leave it to you (Exercise 1) to verify that B_1 and B_2 are linear operators; that is, if c_1 and c_2 are constants then

$$B_i(c_1y_1 + c_2y_2) = c_1B_i(y_1) + c_2B_i(y_2), \quad i = 1, 2. \tag{13.1.7}$$

The next three examples show that the question of existence and uniqueness for solutions of boundary value problems is more complicated than for initial value problems.

Example 13.1.1

Consider the boundary value problem

$$y'' + y = 1, \quad y(0) = 0, \quad y(\pi/2) = 0.$$

The general solution of $y'' + y = 1$ is

$$y = 1 + c_1 \sin x + c_2 \cos x,$$

so $y(0) = 0$ if and only if $c_2 = -1$ and $y(\pi/2) = 0$ if and only if $c_1 = -1$. Therefore

$$y = 1 - \sin x - \cos x$$

is the unique solution of the boundary value problem.

Example 13.1.2

Consider the boundary value problem

$$y'' + y = 1, \quad y(0) = 0, \quad y(\pi) = 0.$$

Again, the general solution of $y'' + y = 1$ is

$$y = 1 + c_1 \sin x + c_2 \cos x,$$

so $y(0) = 0$ if and only if $c_2 = -1$, but $y(\pi) = 0$ if and only if $c_2 = 1$. Therefore the boundary value problem has no solution.

Example 13.1.3

Consider the boundary value problem

$$y'' + y = \sin 2x, \quad y(0) = 0, \quad y(\pi) = 0.$$

You can use the method of undetermined coefficients (Section 5.5) to find that the general solution of $y'' + y = \sin 2x$ is

$$y = -\frac{\sin 2x}{3} + c_1 \sin x + c_2 \cos x.$$

The boundary conditions $y(0) = 0$ and $y(\pi) = 0$ both require that $c_2 = 0$, but they don't restrict c_1. Therefore the boundary value problem has infinitely many solutions

$$y = -\frac{\sin 2x}{3} + c_1 \sin x,$$

where c_1 is arbitrary.

Theorem 13.1.1

If z_1 and z_2 are solutions of $Ly = 0$ such that either $B_1(z_1) = B_1(z_2) = 0$ or $B_2(z_1) = B_2(z_2) = 0$, then $\{z_1, z_2\}$ is linearly dependent. Equivalently, if $\{z_1, z_2\}$ is linearly independent, then

$$B_1^2(z_1) + B_1^2(z_2) \neq 0 \text{ and } \quad B_2^2(z_1) + B_2^2(z_2) \neq 0.$$

Proof Recall that $B_1(z) = \alpha z(a) + \beta z'(a)$ and $\alpha^2 + \beta^2 \neq 0$. Therefore, if $B_1(z_1) = B_1(z_2) = 0$ then (α, β) is a nontrivial solution of the system

$$\begin{aligned} \alpha z_1(a) + \beta z_1'(a) &= 0 \\ \alpha z_2(a) + \beta z'(a) &= 0. \end{aligned}$$

This implies that

$$z_1(a) z_2'(a) - z_1'(a) z_2(a) = 0,$$

so $\{z_1, z_2\}$ is linearly dependent, by Theorem 5.1.6. We leave it to you to show that $\{z_1, z_2\}$ is linearly dependent if $B_2(z_1) = B_2(z_2) = 0$.

Theorem 13.1.2

The following statements are equivalent; that is, they are either all true or all false.

a. There's a fundamental set $\{z_1, z_2\}$ of solutions of $Ly = 0$ such that

$$B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) \neq 0. \tag{13.1.8}$$

b. If $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$ then

$$B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) \neq 0. \tag{13.1.9}$$

c. For each continuous F and pair of constants (k_1, k_2), the boundary value problem

$$Ly = F, \quad B_1(y) = k_1, \quad B_2(y) = k_2$$

has a unique solution.

d. The homogeneous boundary value problem

$$Ly = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \tag{13.1.10}$$

has only the trivial solution $y = 0$.

e. The homogeneous equation $Ly = 0$ has linearly independent solutions z_1 and z_2 such that $B_1(z_1) = 0$ and $B_2(z_2) = 0$.

Proof We'll show that

$$\backslash\text{part}a \implies \backslash\text{part}b \implies \backslash\text{part}c \implies \backslash\text{part}d \implies \backslash\text{part}e \implies \backslash\text{part}a.$$

(a) $\implies$ **(b)**: Since $\{z_1, z_2\}$ is a fundamental set of solutions for $Ly = 0$, there are constants a_1, a_2, b_1, and b_2 such that

$$\begin{aligned} y_1 &= a_1z_1 + a_2z_2 \\ y_2 &= b_1z_1 + b_2z_2. \end{aligned} \tag{13.1.11}$$

Moreover,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0. \tag{13.1.12}$$

because if this determinant were zero, its rows would be linearly dependent and therefore $\{y_1, y_2\}$ would be linearly dependent, contrary to our assumption that $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$. From (13.1.7) and (13.1.11),

$$\begin{bmatrix} B_1(y_1) & B_2(y_1) \\ B_1(y_2) & B_2(y_2) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} B_1(z_1) & B_2(z_1) \\ B_1(z_2) & B_2(z_2) \end{bmatrix}.$$

Since the determinant of a product of matrices is the product of the determinants of the matrices, (13.1.8) and (13.1.12) imply (13.1.9).

(b) $\implies$ **(c)**: Since $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$, the general solution of $Ly = F$ is

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are arbitrary constants and y_p is a particular solution of $Ly = F$. To satisfy the boundary conditions, we must choose c_1 and c_2 so that

$$\begin{aligned} k_1 &= B_1(y_p) + c_1 B_1(y_1) + c_2 B_1(y_2) \\ k_2 &= B_2(y_p) + c_1 B_2(y_1) + c_2 B_2(y_2), \end{aligned}$$

(recall (13.1.7)), which is equivalent to

$$\begin{aligned} c_1 B_1(y_1) + c_2 B_1(y_2) &= k_1 - B_1(y_p) \\ c_1 B_2(y_1) + c_2 B_2(2_2) &= k_2 - B_2(y_p). \end{aligned}$$

From (13.1.9), this system always has a unique solution (c_1, c_2).

(c) $\implies$ **(d)**: Obviously, $y = 0$ is a solution of (13.1.10). From **(c)** with $F = 0$ and $k_1 = k_2 = 0$, it's the only solution.

(d) $\implies$ **(e)**: Let $\{y_1, y_2\}$ be a fundamental system for $Ly = 0$ and let

$$z_1 = B_1(y_2) y_1 - B_1(y_1) y_2 \quad \text{and} \quad z_2 = B_2(y_2) y_1 - B_2(y_1) y_2.$$

Then $B_1(z_1) = 0$ and $B_2(z_2) = 0$. To see that z_1 and z_2 are linearly independent, note that

$$\begin{aligned} a_1z_1 + a_2z_2 &= a_1[B_1(y_2)y_1 - B_1(y_1)y_2] + a_2[B_2(y_2)y_1 - B_2(y_1)y_2] \\ &= [B_1(y_2)a_1 + B_2(y_2)a_2]y_1 - [B_1(y_1)a_1 + B_2(y_1)a_2]y_2. \end{aligned}$$

Therefore, since y_1 and y_2 are linearly independent, $a_1z_1 + a_2z_2 = 0$ if and only if

$$\left[\begin{array}{cc} B_1(y_1) & B_2(y_1) \\ B_1(y_2) & B_2(y_2) \end{array}\right]\left[\begin{array}{c} a_1 \\ a_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right].$$

If this system has a nontrivial solution then so does the system

$$\left[\begin{array}{cc} B_1(y_1) & B_1(y_2) \\ B_2(y_1) & B_2(y_2) \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right].$$

This and (13.1.7) imply that $y = c_1z_1 + c_2z_2$ is a nontrivial solution of (13.1.10), which contradicts **(d)**.

(e) $\implies$ **(a)**. Theorem 13.1.1 implies that if $B_1(z_1) = 0$ and $B_2(z_2) = 0$ then $B_1(z_2) \neq 0$ and $B_2(z_1) \neq 0$. This implies (13.1.8), which completes the proof.

Example 13.1.4

Solve the boundary value problem

$$x^2y'' - 2xy' + 2y - 2x^3 = 0, \quad y(1) = 4, \quad y'(2) = 3, \tag{13.1.13}$$

given that $\{x, x^2\}$ is a fundamental set of solutions of the complementary equation

Solution Using variation of parameters (Section 5.7), you can show that $y_p = x^3$ is a solution of the complementary equation

$$x^2y'' - 2xy' + 2y = 2x^3 = 0.$$

Therefore the solution of (13.1.13) can be written as

$$y = x^3 + c_1x + c_2x^2.$$

Then

$$y' = 3x^2 + c_1 + 2c_2x,$$

and imposing the boundary conditions yields the system

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 + 4c_2 &= -9, \end{aligned}$$

so $c_1 = 7$ and $c_2 = -4$. Therefore

$$y = x^3 + 7x - 4x^2$$

is the unique solution of (13.1.13)

Example 13.1.5

Solve the boundary value problem

$$y'' - 7y' + 12y = 4e^{2x}, \quad y(0) = 3, \quad y(1) = 5e^2.$$

Solution From Example 5.4.1, $y_p = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{13.1.14}$$

Since $\{e^{3x}, e^{4x}\}$ is a fundamental set for the complementary equation, we could write the solution of (13.1.13) as

$$y = 2e^{2x} + c_1 e^{3x} + c_2 e^{4x}$$

and determine c_1 and c_2 by imposing the boundary conditions. However, this would lead to some tedious algebra, and the form of the solution would be very unappealing. (Try it!) In this case it's convenient to use the fundamental system $\{z_1, z_2\}$ mentioned in Theorem 13.1.2**(e)**; that is, we choose $\{z_1, z_2\}$ so that $B_1(z_1) = z_1(0) = 0$ and $B_2(z_2) = z_2(1) = 0$. It is easy to see that

$$z_1 = e^{3x} - e^{4x} \quad \text{and} \quad z_2 = e^{3(x-1)} - e^{4(x-1)}$$

satisfy these requirements. Now we write the solution of (13.1.14) as

$$y = 2e^{2x} + c_1 \left(e^{3x} - e^{4x}\right) + c_2 \left(e^{3(x-1)} - e^{4(x-1)}\right).$$

Imposing the boundary conditions $y(0)=3$ and $y(1)=5e^2$ yields

$$3=2+c_2e^{-4}(e-1)\ \text{ and }\quad 5e^2=2e^2+c_1e^3(1-e).$$

Therefore

$$c_1=\frac{3}{e(1-e)},\quad c_2=\frac{e^4}{e-1},$$

and

$$y=2e^{2x}+\frac{3}{e(1-e)}(e^{3x}-e^{4x})+\frac{e^4}{e-1}(e^{3(x-1)}-e^{4(x-1)}).$$

Sometimes it's useful to have a formula for the solution of a general boundary problem. Our next theorem addresses this question.

THEOREM 13.1.3

Suppose the homogeneous boundary value problem

$$Ly=0,\quad B_1(y)=0,\quad B_2(y)=0 \tag{13.1.15}$$

has only the trivial solution. Let y_1 and y_2 be linearly independent solutions of $Ly=0$ such that $B_1(y_1)=0$ and $B_2(y_2)=0$, and let

$$W=y_1y_2'-y_1'y_2.$$

Then the unique solution of

$$Ly=F,\quad B_1(y)=0,\quad B_2(y)=0 \tag{13.1.16}$$

is

$$y(x)=y_1(x)\int_x^b\frac{F(t)y_2(t)}{P_0(t)W(t)}\,dt+y_2(x)\int_a^x\frac{F(t)y_1(t)}{P_0(t)W(t)}\,dt. \tag{13.1.17}$$

PROOF In Section 5.7 we saw that if

$$y = u_1 y_1 + u_2 y_2 \tag{13.1.18}$$

where

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= F, \end{aligned}$$

then $Ly = F$. Solving for u_1' and u_2' yields

$$u_1' = -\frac{F y_2}{P_0 W} \quad \text{and} \quad u_2' = \frac{F y_1}{P_0 W},$$

which hold if

$$u_1(x) = \int_x^b \frac{F(t) y_2(t)}{P_0(t) W(t)}\, dt \quad \text{and} \quad u_2(x) = \int_a^x \frac{F(t) y_1(t)}{P_0(t) W(t)}\, dt.$$

This and (13.1.18) show that (13.1.17) is a solution of $Ly = F$. Differentiating (13.1.17) yields

$$y'(x) = y_1'(x) \int_x^b \frac{F(t) y_2(t)}{P_0(t) W(t)}\, dt + y_2'(x) \int_a^x \frac{F(t) y_1(t)}{P_0(t) W(t)}\, dt. \tag{13.1.19}$$

(Verify.) From (13.1.17) and (13.1.19),

$$B_1(y) = B_1(y_1) \int_a^b \frac{F(t) y_2(t)}{P_0(t) W(t)}\, dt = 0$$

because $B_1(y_1) = 0$, and

$$B_2(y) = B_2(y_2) \int_a^b \frac{F(t) y_1(t)}{P_0(t) W(t)}\, dt = 0$$

because $B_2(y_2) = 0$. Hence, y satisfies (13.1.16). This completes the proof.

We can rewrite (13.1.17) as

$$y = \int_a^b G(x,t) F(t)\, dt, \tag{13.1.20}$$

where

$$G(x,t)=\begin{cases} \backslash\text{dst}\,\frac{y_1(t)y_2(x)}{P_0(t)W(t)} & , \quad a\le t\le x, \\ \backslash\text{dst}\,\frac{y_1(x)y_2(t)}{P_0(t)W(t)} & , \quad x\le t\le b. \end{cases}.$$

This is the *Green's function* for the boundary value problem (13.1.16). The Green's function is related to the boundary value problem (13.1.16) in much the same way that the inverse of a square matrix $\mathbf{A}$ is related to the linear algebraic system $\mathbf{y}=\mathbf{A}\mathbf{x}$; just as we substitute the given vector $\mathbf{y}$ into the formula $\mathbf{x}=\mathbf{A}^{-1}\mathbf{y}$ to solve $\mathbf{y}=\mathbf{A}x$, we substitute the given function F into the formula (13.1.20) to obtain the solution of (13.1.16). The analogy goes further: just as $\mathbf{A}^{-1}$ exists if and only if $\mathbf{A}\mathbf{x}=\mathbf{0}$ has only the trivial solution, the boundary value problem (13.1.16) has a Green's function if and only the homogeneous boundary value problem (13.1.15) has only the trivial solution.

We leave it to you (Exercise 32) to show that the assumptions of Theorem 13.1.3 imply that the unique solution of the boundary value problem

$$Ly=F, \quad B_1(y)=k_1, \quad B_2(y)=k_2$$

is

$$y(x)=\int_a^b G(x,t)F(t)\,dt+\frac{k_2}{B_2(y_1)}y_1+\frac{k_1}{B_1(y_2)}y_2.$$

Example 13.1.6

Solve the boundary value problem

$$y''+y=F(x). \quad y(0)+y'(0)=0, \quad y(\pi)-y'(\pi)=0, \tag{13.1.21}$$

and find the Green's function for this problem.

Solution Here

$$B_1(y)=y(0)+y'(0) \ \text{ and } \quad B_2(y)=y(\pi)-y'(\pi).$$

Let $\{z_1,z_2\}=\{\cos x,\sin x\}$, which is a fundamental set of solutions of $y''+y=0$. Then

$$\begin{aligned} B_1(z_1)&=(\cos x-\sin x)\big|_{x=0}&=&\ \ 1\\ B_2(z_1)&=(\cos x+\sin x)\big|_{x=\pi}&=&-1\end{aligned}$$

and

$$\begin{aligned} B_1(z_2) &= (\sin x + \cos x)\big|_{x=0} = 1 \\ B_2(z_2) &= (\sin x - \cos x)\big|_{x=\pi} = 1. \end{aligned}$$

Therefore

$$B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) = 2,$$

so Theorem 13.1.2 implies that (13.1.21) has a unique solution. Let

$$y_1 = B_1(z_2)z_1 - B_1(z_1)z_2 = \cos x - \sin x$$

and

$$y_2 = B_2(z_2)z_1 - B_2(z_1)z_2 = \cos x + \sin x.$$

Then $B_1(y_1) = 0$, $B_2(y_2) = 0$, and the Wronskian of $\{y_1, y_2\}$ is

$$W(x) = \left|\begin{array}{cc} \cos x - \sin x & \cos x + \sin x \\ -\sin x - \cos x & -\sin x + \cos x \end{array}\right| = 2.$$

Since $P_0 = 1$, (13.1.17) yields the solution

$$\begin{aligned} y(x) = {}& \backslash\text{dst}\frac{\cos x - \sin x}{2}\int_x^\pi F(t)(\cos t + \sin t)\,dt \\ & + \backslash\text{dst}\frac{\cos x + \sin x}{2}\int_0^x F(t)(\cos t - \sin t)\,dt. \end{aligned}$$

The Green's function is

$$G(x,t) = \begin{cases} \backslash\text{dst}\frac{(\cos t - \sin t)(\cos x + \sin x)}{2}, & 0 \le t \le x, \\ \backslash\text{dst}\frac{(\cos x - \sin x)(\cos t + \sin t)}{2}, & x \le t \le \pi. \end{cases}$$

We'll now consider the situation not covered by Theorem 13.1.3.

THEOREM 13.1.4

Suppose the homogeneous boundary value problem

$$Ly = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \tag{13.1.22}$$

has a nontrivial solution y_1, and let y_2 be any solution of $Ly = 0$ that isn't a constant multiple of y_1. Let $W = y_1 y_2' - y_1' y_2$. If

$$\int_a^b \frac{F(t) y_1(t)}{P_0(t) W(t)}\, dt = 0, \tag{13.1.23}$$

then the homogeneous boundary value problem

$$Ly = F, \quad B_1(y) = 0, \quad B_2(y) = 0 \tag{13.1.24}$$

has infinitely many solutions, all of the form $y = y_p + c_1 y_1$, where

$$y_p = y_1(x) \int_x^b \frac{F(t) y_2(t)}{P_0(t) W(t)}\, dt + y_2(x) \int_a^x \frac{F(t) y_1(t)}{P_0(t) W(t)}\, dt$$

and c_1 is a constant. If

$$\int_a^b \frac{F(t) y_1(t)}{P_0(t) W(t)}\, dt \neq 0,$$

then (13.1.24) has no solution.

Proof From the proof of Theorem 13.1.3, y_p is a particular solution of $Ly = F$, and

$$y_p'(x) = y_1'(x) \int_x^b \frac{F(t) y_2(t)}{P_0(t) W(t)}\, dt + y_2'(x) \int_a^x \frac{F(t) y_1(t)}{P_0(t) W(t)}\, dt.$$

Therefore the general solution of (13.1.22) is of the form

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are constants. Then

$$\begin{aligned} B_1(y) &= B_1(y_p + c_1y_1 + c_2y_2) = B_1(y_p) + c_1B_1(y_1) + c_2B_1y_2 \\ &= B_1(y_1)\int_a^b \frac{F(t)y_2(t)}{P_0(t)W(t)}\,dt + c_1B_1(y_1) + c_2B_1(y_2) \\ &= c_2B_1(y_2) \end{aligned}$$

Since $B_1(y_1) = 0$, Theorem 13.1.1 implies that $B_1(y_2) \neq 0$; hence, $B_1(y) = 0$ if and only if $c_2 = 0$. Therefore $y = y_p + c_1y_1$ and

$$\begin{aligned} B_2(y) = B_2(y_p + c_1y_1) &= B_2(y_2)\int_a^b \frac{F(t)y_1(t)}{P_0(t)W(t)}\,dt + c_1B_2(y_1) \\ &= B_2(y_2)\int_a^b \frac{F(t)y_1(t)}{P_0(t)W(t)}\,dt, \end{aligned}$$

since $B_2(y_1) = 0$. From Theorem 13.1.1, $B_2(y_2) \neq 0$ (since $B_2(y_1 = 0)$. Therefore $Ly = 0$ if and only if (13.1.23) holds. This completes the proof.

Example 13.1.7

Applying Theorem 13.1.4 to the boundary value problem

$$y'' + y = F(x), \quad y(0) = 0, \quad y(\pi) = 0 \tag{13.1.25}$$

explains the Examples 13.1.2 and 13.1.3. The complementary equation $y'' + y = 0$ has the linear independent solutions $y_1 = \sin x$ and $y_2 = \cos x$, and y_1 satisfies both boundary conditions. Since $P_0 = 1$ and

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1,$$

(13.1.23) reduces to

$$\int_0^\pi F(x)\sin x\,dx = 0.$$

From Example 13.1.2, $F(x) = 1$ and

$$\int_0^\pi F(x)\sin x\,dx = \int_0^\pi \sin x\,dx = 2,$$

so Theorem 13.1.3 implies that (13.1.25) has no solution. In Example 13.1.3,

$$F(x) = \sin 2x = 2\sin x\cos x$$

and

$$\int_0^\pi F(x)\sin x\,dx = 2\int_0^\pi \sin^2 x\cos x\,dx = \frac{2}{3}\sin^3 x\Big|_0^\pi = 0,$$

so Theorem 13.1.3 implies that (13.1.25) has infinitely many solutions, differing by constant multiples of $y_1(x) = \sin x$.

13.1 Exercises

1. Verify that B_1 and B_2 are linear operators; that is, if c_1 and c_2 are constants then

$$B_i(c_1y_1 + c_2y_2) = c_1B_i(y_1) + c_2B_i(y_2), \quad i = 1, 2.$$

In Exercises 2–7 solve the boundary value problem.

2. $y'' - y = x, \quad y(0) = -2, \quad y(1) = 1$

Show answer

3. $y'' = 2 - 3x, \quad y(0) = 0, \quad y(1) - y'(1) = 0$

Show answer

4. $y'' - y = x, \quad y(0) + y'(0) = 3, \quad y(1) - y'(1) = 2$

Show answer

5. $y'' + 4y = 1, \quad y(0) = 3, \quad y(\pi/2) + y'(\pi/2) = -7$

Show answer

6. $y'' - 2y' + y = 2e^x, \quad y(0) - 2y'(0) = 3, \quad y(1) + y'(1) = 6e$

Show answer

7. $y'' - 7y' + 12y = 4e^{2x}, \quad y(0) + y'(0) = 8, \quad y(1) = -7e^2$ (see Example 13.1.5)

Show answer

8. State a condition on F such that the boundary value problem

$$y'' = F(x), \quad y(0) = 0, \quad y(1) - y'(1) = 0$$

has a solution, and find all solutions.

Show answer

9. a. State a condition on a and b such that the boundary value problem

$$y'' + y = F(x), \quad y(a) = 0, \quad y(b) = 0 \tag{A}$$

has a unique solution for every continuous F, and find the solution by the method used to prove Theorem 13.1.3

b. In the case where a and b don't satisfy the condition you gave for **(a)**, state necessary and sufficient on F such that (A) has a solution, and find all solutions by the method used to prove Theorem 13.1.4.

Show answer

10. Follow the instructions in Exercise 9 for the boundary value problem

$$y'' + y = F(x), \quad y(a) = 0, \quad y'(b) = 0.$$

Show answer

11. Follow the instructions in Exercise 9 for the boundary value problem

$$y'' + y = F(x), \quad y'(a) = 0, \quad y'(b) = 0.$$

Show answer

In Exercises 12–15 find a formula for the solution of the boundary problem by the method used to prove Theorem 13.1.3. Assume that $a < b$.

12. $y'' - y = F(x), \quad y(a) = 0, \quad y(b) = 0$

Show answer

13. $y'' - y = F(x), \quad y(a) = 0, \quad y'(b) = 0$

Show answer

14. $y'' - y = F(x), \quad y'(a) = 0, \quad y'(b) = 0$

Show answer

15. $y'' - y = F(x), \quad y(a) - y'(a) = 0, \quad y(b) + y'(b) = 0$

Show answer

In Exercises 16–19 find all values of ω such that boundary problem has a unique solution, and find the solution by the method used to prove Theorem 13.1.3. For other values of ω, find conditions on F such that the problem has a solution, and find all solutions by the method used to prove Theorem 13.1.4.

16. $y'' + \omega^2 y = F(x), \quad y(0) = 0, \quad y(\pi) = 0$

Show answer

17. $y'' + \omega^2 y = F(x), \quad y(0) = 0, \quad y'(\pi) = 0$

Show answer

18. $y'' + \omega^2 y = F(x), \quad y'(0) = 0, \quad y(\pi) = 0$

Show answer

19. $y'' + \omega^2 y = F(x), \quad y'(0) = 0, \quad y'(\pi) = 0$

Show answer

20. Let $\{z_1, z_2\}$ be a fundamental set of solutions of $Ly = 0$. Given that the homogeneous boundary value problem

$$Ly = 0, \quad B_1(y) = 0, \quad B_2(y) = 0$$

has a nontrivial solution, express it explicity in terms of z_1 and z_2.

Show answer

21. If the boundary value problem has a solution for every continuous F, then find the Green's function for the problem and use it to write an explicit formula for the solution. Otherwise, if the boundary value problem does not have a solution for every continuous F, find a necessary and sufficient condition on F for the problem to have a solution, and find all solutions. Assume that $a < b$.

a. $y'' = F(x), \quad y(a) = 0, \quad y(b) = 0$

b. $y'' = F(x), \quad y(a) = 0, \quad y'(b) = 0$

c. $y'' = F(x), \quad y'(a) = 0, \quad y(b) = 0$

d. $y'' = F(x), \quad y'(a) = 0, \quad y'(b) = 0$

Show answer

22. Find the Green's function for the boundary value problem

$$y'' = F(x), \quad y(0) - 2y'(0) = 0, \quad y(1) + 2y'(1) = 0. \tag{A}$$

Then use the Green's function to solve (A) with **(a)** $F(x) = 1$, **(b)** $F(x) = x$, and **(c)** $F(x) = x^2$.

Show answer

23. Find the Green's function for the boundary value problem

$$x^2y'' + xy' + (x^2 - 1/4)y = F(x), \quad y(\pi/2) = 0, \quad y(\pi) = 0, \tag{A}$$

given that

$$y_1(x) = \frac{\cos x}{\sqrt{x}} \quad \text{and} \quad y_2(x) = \frac{\sin x}{\sqrt{x}}$$

are solutions of the complementary equation. Then use the Green's function to solve (A) with **(a)** $F(x) = x^{3/2}$ and **(b)** $F(x) = x^{5/2}$.

Show answer

24. Find the Green's function for the boundary value problem

$$x^2y'' - 2xy' + 2y = F(x), \quad y(1) = 0, \quad y(2) = 0, \tag{A}$$

given that $\{x, x^2\}$ is a fundamental set of solutions of the complementary equation. Then use the Green's function to solve (A) with **(a)** $F(x) = 2x^3$ and **(b)** $F(x) = 6x^4$.

Show answer

25. Find the Green's function for the boundary value problem

$$x^2y'' + xy' - y = F(x), \quad y(1) - 2y'(1) = 0, \quad y'(2) = 0, \tag{A}$$

given that $\{x, 1/x\}$ is a fundamental set of solutions of the complementary equation. Then use the Green's function to solve (A) with **(a)** $F(x) = 1$, **(b)** $F(x) = x^2$, and **(c)** $F(x) = x^3$.

Show answer

In Exercises 26–30 find necessary and sufficient conditions on α, β, ρ, and δ for the boundary value problem to have a unique solution for every continuous F, and find the Green's function.

26. $y'' = F(x), \qquad \alpha y(0) + \beta y'(0) = 0, \qquad \rho y(1) + \delta y'(1) = 0$

Show answer

27. $y'' + y = F(x), \qquad \alpha y(0) + \beta y'(0) = 0, \qquad \rho y(\pi) + \delta y'(\pi) = 0$

Show answer

28. $y'' + y = F(x), \qquad \alpha y(0) + \beta y'(0) = 0, \qquad \rho y(\pi/2) + \delta y'(\pi/2) = 0$

Show answer

29. $y'' - 2y' + 2y = F(x), \qquad \alpha y(0) + \beta y'(0) = 0, \qquad \rho y(\pi) + \delta y'(\pi) = 0$

Show answer

30. $y'' - 2y' + 2y = F(x), \qquad \alpha y(0) + \beta y'(0) = 0, \qquad \rho y(\pi/2) + \delta y'(\pi/2) = 0$

Show answer

31. Find necessary and sufficient conditions on α, β, ρ, and δ for the boundary value problem

$$y'' - y = F(x), \quad \alpha y(a) + \beta y'(a) = 0, \quad \rho y(b) + \delta y'(b) = 0 \tag{A}$$

to have a unique solution for every continuous F, and find the Green's function for (A). Assume that $a < b$.

Show answer

32. Show that the assumptions of Theorem 13.1.3 imply that the unique solution of

$$Ly = F, \quad B_1(y) = k_1, \quad B_2(y) = f_2$$

is

$$y = \int_a^b G(x,t)F(t)\,dt + \frac{k_2}{B_2}(y_1)y_1 + \frac{k_1}{B_1(y_2)}y_2.$$

13.2 Sturm-Liouville Problems

In this section we consider eigenvalue problems of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0, \tag{13.2.1}$$

where

$$B_1(y) = \alpha y(a) + \beta y'(a) \quad \text{and} \quad B_2(y) = \rho y(b) + \delta y'(b).$$

As in Section 13.1, α, β, ρ, and δ are real numbers, with

$$\alpha^2 + \beta^2 > 0 \;\text{ and }\quad \rho^2 + \delta^2 > 0,$$

P_0, P_1, P_2, and R are continuous, and P_0 and R are positive on $[a, b]$.

We say that λ is an *eigenvalue* of (13.2.1) if (13.2.1) has a nontrivial solution y. In this case, y is an *eigenfunction associated with* λ, or a λ*-eigenfunction*. *Solving* the eigenvalue problem means finding all eigenvalues and associated eigenfunctions of (13.2.1).

Example 13.2.1

Solve the eigenvalue problem

$$y' + 3y' + 2y + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0. \tag{13.2.2}$$

Solution The characteristic equation of (13.2.2) is

$$r^2 + 3r + 2 + \lambda = 0,$$

with zeros

$$r_1 = \frac{-3 + \sqrt{1 - 4\lambda}}{2} \quad \text{and} \quad r_2 = \frac{-3 - \sqrt{1 - 4\lambda}}{2}.$$

If $\lambda < 1/4$ then r_1 and r_2 are real and distinct, so the general solution of the differential equation in (13.2.2) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

The boundary conditions require that

$$\begin{aligned} c_1 \quad + c_2 \quad &= 0 \\ c_1 e^{r_1} + c_2 e^{r_2} &= 0. \end{aligned}$$

Since the determinant of this system is $e^{r_2} - e^{r_1} \neq 0$, the system has only the trivial solution. Therefore λ isn't an eigenvalue of (13.2.2).

If $\lambda = 1/4$ then $r_1 = r_2 = -3/2$, so the general solution of (13.2.2) is

$$y = e^{-3x/2}(c_1 + c_2 x).$$

The boundary condition $y(0) = 0$ requires that $c_1 = 0$, so $y = c_2 x e^{-3x/2}$ and the boundary condition $y(0)$ requires that $c_2 = 0$. Therefore $\lambda = 1/4$ isn't an eigenvalue of (13.2.2).

If $\lambda > 1/4$ then

$$r_1 = -\frac{3}{2} + i\omega \quad \text{and} \quad r_2 = -\frac{3}{2} - i\omega,$$

with

$$\omega = \frac{\sqrt{4\lambda - 1}}{2} \quad \text{or, equivalently,} \quad \lambda = \frac{1 + 4\omega^2}{4}. \tag{13.2.3}$$

In this case the general solution of the differential equation in (13.2.2) is

$$y = e^{-3x/2}(c_1 \cos \omega x + c_2 \sin \omega x).$$

The boundary condition $y(0) = 0$ requires that $c_1 = 0$, so $y = c_2 e^{-3x/2} \sin \omega x$, which holds with $c_2 \neq 0$ if and only if $\omega = n\pi$, where n is an integer. We may assume that n is a positive integer. (Why?). From (13.2.3), the eigenvalues are $\lambda_n = (1 + 4n^2\pi^2)/4$, with associated eigenfunctions

$$y_n = e^{-3x/2} \sin n\pi x, \quad n = 1, 2, 3, \dots .$$

Example 13.2.2

Solve the eigenvalue problem

$$x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0. \tag{13.2.4}$$

Solution If $\lambda = 0$, the differential equation in (13.2.4) reduces to $x(xy')' = 0$, so $xy' = c_1$,

$$y' = \frac{c_1}{x}, \quad \text{and} \quad y = c_1 \ln x + c_2.$$

The boundary condition $y(1) = 0$ requires that $c_2 = 0$, so $y = c_1 \ln x$. The boundary condition $y(2) = 0$ requires that $c_1 \ln 2 = 0$, so $c_1 = 0$. Therefore zero isn't an eigenvalue of (13.2.4).

If $\lambda < 0$, we write $\lambda = -k^2$ with $k > 0$, so (13.2.4) becomes

$$x^2y'' + xy' - k^2y = 0,$$

an Euler equation (Section 7.4) with indicial equation

$$r^2 - k^2 = (r-k)(r+k) = 0.$$

Therefore

$$y = c_1x^k + c_2x^{-k}.$$

The boundary conditions require that

$$\begin{aligned} c_1 + \quad\; c_2 &= 0 \\ 2^kc_1 + 2^{-k}c_2 &= 0. \end{aligned}$$

Since the determinant of this system is $2^{-k} - 2^k \neq 0$, $c_1 = c_2 = 0$. Therefore (13.2.4) has no negative eigenvalues.

If $\lambda > 0$ we write $\lambda = k^2$ with $k > 0$. Then (13.2.4) becomes

$$x^2y'' + xy' + k^2y = 0,$$

an Euler equation with indicial equation

$$r^2 + k^2 = (r-ik)(r+ik) = 0,$$

so

$$y = c_1 \cos(k \ln x) + c_2 \sin(k \ln x).$$

The boundary condition $y(1) = 0$ requires that $c_1 = 0$. Therefore $y = c_2 \sin(k \ln x)$. This holds with $c_2 \neq 0$ if and only if $k = n\pi/\ln 2$, where n is a positive integer. Hence, the eigenvalues of (13.2.4) are $\lambda_n = (n\pi/\ln 2)^2$, with associated eigenfunctions

$$y_n = \sin\left(\frac{n\pi}{\ln 2}\ln x\right), \quad n = 1, 2, 3, \ldots.$$

For theoretical purposes, it's useful to rewrite the differential equation in (13.2.1) in a different form, provided by the next theorem.

Theorem 13.2.1

If $P_0,\ P_1,\ P_2,$ and R are continuous and P_0 and R are positive on a closed interval $[a, b]$, then the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0 \tag{13.2.5}$$

can be rewritten as

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0, \tag{13.2.6}$$

where p, p', q and r are continuous and p and r are positive on $[a, b]$.

Proof We begin by rewriting (13.2.5) as

$$y'' + u(x)y' + v(x)y + \lambda R_1(x)y = 0, \tag{13.2.7}$$

with $u = P_1/P_0$, $v = P_2/P_0$, and $R_1 = R/P_0$. (Note that R_1 is positive on $[a, b]$.) Now let $p(x) = e^{U(x)}$, where U is any antiderivative of u. Then p is positive on $[a, b]$ and, since $U' = u$,

$$p'(x) = p(x)u(x) \tag{13.2.8}$$

is continuous on $[a, b]$. Multiplying (13.2.7) by $p(x)$ yields

$$p(x)y'' + p(x)u(x)y' + p(x)v(x)y + \lambda p(x)R_1(x)y = 0. \tag{13.2.9}$$

Since p is positive on $[a, b]$, this equation has the same solutions as (13.2.5). From (13.2.8),

$$(p(x)y')' = p(x)y'' + p'(x)y' = p(x)y'' + p(x)u(x)y',$$

so (13.2.9) can be rewritten as in (13.2.6), with $q(x)=p(x)v(x)$ and $r(x)=p(x)R_1(x)$. This completes the proof.

It is to be understood throughout the rest of this section that p, q, and r have the properties stated in Theorem 13.2.1. Moreover, whenever we write Ly in a general statement, we mean

$$Ly=(p(x)y')'+q(x)y.$$

The differential equation (13.2.6) is called a Sturm (http://www-history.mcs.st-and.ac.uk/Mathematicians/Sturm.html)– Liouville equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Liouville.html), and the eigenvalue problem

$$(p(x)y')'+q(x)y+\lambda r(x)y=0,\quad B_1(y)=0,\quad B_2(y)=0, \tag{13.2.10}$$

which is equivalent to (13.2.1), is called a Sturm-Liouville problem (http://www-history.mcs.st-and.ac.uk/Mathematicians/Sturm.html).

Example 13.2.3

Rewrite the eigenvalue problem

$$y''+3y'+(2+\lambda)y=0,\quad y(0)=0,\quad y(1)=0 \tag{13.2.11}$$

of Example 13.2.1 as a Sturm-Liouville problem.

Solution Comparing (13.2.11) to (13.2.7) shows that $u(x)=3$, so we take $U(x)=3x$ and $p(x)=e^{3x}$. Multiplying the differential equation in (13.2.11) by e^{3x} yields

$$e^{3x}(y''+3y')+2e^{3x}y+\lambda e^{3x}y=0.$$

Since

$$e^{3x}(y''+3y')=(e^{3x}y')',$$

(13.2.11) is equivalent to the Sturm–Liouville problem

$$(e^{3x}y')'+2e^{3x}y+\lambda e^{3x}y=0,\quad y(0)=0,\quad y(1)=0. \tag{13.2.12}$$

Example 13.2.4

Rewrite the eigenvalue problem

$$x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0 \tag{13.2.13}$$

of Example 13.2.2 as a Sturm-Liouville problem.

Solution Dividing the differential equation in (13.2.13) by x^2 yields

$$y'' + \frac{1}{x}y' + \frac{\lambda}{x^2}y = 0.$$

Comparing this to (13.2.7) shows that $u(x) = 1/x$, so we take $U(x) = \ln x$ and $p(x) = e^{\ln x} = x$. Multiplying the differntial equation by x yields

$$xy'' + y' + \frac{\lambda}{x}y = 0.$$

Since

$$xy'' + y' = (xy')',$$

(13.2.13) is equivalent to the Sturm–Liouville problem

$$(xy')' + \frac{\lambda}{x}y = 0, \quad y(1) = 0, \quad y(2) = 0. \tag{13.2.14}$$

Problems 1–4 of Section 11.1 are Sturm–Liouville problems. (Problem 5 isn't , although some authors ∎ use a definition of Sturm-Liouville problem (http://www-history.mcs.st-and.ac.uk/Mathematicians/Sturm.html) that does include it.) We were able to find the eigenvalues of Problems 1-4 explicitly because in each problem the coefficients in the boundary conditions satisfy $\alpha\beta = 0$ and $\rho\delta = 0$; that is, each boundary condition involves either y or y', but not both. If this isn't true then the eigenvalues can't in general be expressed exactly by simple formulas; rather, approximate values must be obtained by numerical solution of equations derived by requiring the determinants of certain 2×2 systems of homogeneous equations to be zero. To apply the numerical methods effectively, graphical methods must be used to determine approximate locations of the zeros of these determinants. Then the zeros can be computed accurately by numerical methods.

Example 13.2.5

Solve the Sturm–Liouville problem

$$y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) + 3y'(1) = 0. \tag{13.2.15}$$

Solution If $\lambda = 0$, the differential equation in (13.2.15) reduces to $y'' = 0$, with general solution $y = c_1 + c_2 x$. The boundary conditions require that

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 4c_2 &= 0, \end{aligned}$$

so $c_1 = c_2 = 0$. Therefore zero isn't an eigenvalue of (13.2.15).

If $\lambda < 0$, we write $\lambda = -k^2$ where $k > 0$, and the differential equation in (13.2.15) becomes $y'' - k^2 y = 0$, with general solution

$$y = c_1 \cosh kx + c_2 \sinh kx, \tag{13.2.16}$$

so

$$y' = k(c_1 \sinh kx + c_2 \cosh kx).$$

The boundary conditions require that

$$\begin{aligned} c_1 + kc_2 &= 0 \\ (\cosh k + 3k \sinh k)c_1 + (\sinh k + 3k \cosh k)c_2 &= 0. \end{aligned} \tag{13.2.17}$$

The determinant of this system is

$$\begin{aligned} D_N(k) &= \begin{vmatrix} 1 & k \\ \cosh k + 3k \sinh k & \sinh k + 3k \cosh k \end{vmatrix} \\ &= (1 - 3k^2) \sinh k + 2k \cosh k. \end{aligned}$$

Therefore the system (13.2.17) has a nontrivial solution if and only if $D_N(k) = 0$ or, equivalently,

$$\tanh k = -\frac{2k}{1 - 3k^2}. \tag{13.2.18}$$

The graph of the right side (Figure 13.2.1) has a vertical asymptote at $k = 1/\sqrt{3}$. Since the two sides have different signs if $k < 1/\sqrt{3}$, this equation has no solution in $(0, 1/\sqrt{3})$. Figure 13.2.1 shows the graphs of the two sides of (13.2.18) on an interval to the right of the vertical asymptote, which is indicated by the dashed line. You can see that the two curves intersect near $k_0 = 1.2$, Given this estmate, you can use Newton's to compute k_0 more accurately. We computed $k_0 \approx 1.1219395$. Therefore $-k_0^2 \approx -1.2587483$ is an eigenvalue of (13.2.15). From (13.2.16) and the first equation in (13.2.17),

$$y_0 = k_0 \cosh k_0 x - \sinh k_0 x.$$

Figure 13.2.1. $u = \tanh k$ ***and*** $u = -2k/(1-3k^2)$

If $\lambda > 0$ we write $\lambda = k^2$ where $k > 0$, and differential equation in (13.2.15) becomes $y'' + k^2 y = 0$, with general solution

$$y = \cos kx + c_2 \sin kx, \tag{13.2.19}$$

so

$$y' = k(-c_1 \sin kx + c_2 \cos kx).$$

The boundary conditions require that

$$\begin{gathered} c_1 + kc_2 = 0 \\ (\cos k - 3k \sin k)c_1 + (\sin k + 3k \cos k)c_2 = 0. \end{gathered} \tag{13.2.20}$$

The determinant of this system is

$$\begin{aligned} D_P(k) &= \begin{vmatrix} 1 & k \\ \cos k - 3k \sin k & \sin k + 3k \cos k \end{vmatrix} \\ &= (1 + 3k^2) \sin k + 2k \cos k. \end{aligned}$$

The system (13.2.20) has a nontrivial solution if and only if $D_P(k) = 0$ or, equivalently,

$$\tan k = -\frac{2k}{1 + 3k^2}.$$

Figure 13.2.2 shows the graphs of the two sides of this equation. You can see from the figure that the graphs intersect at infinitely many points $k_n \approx n\pi$ ($n = 1, 2, 3, \ldots$), where the error in this approximation approaches zero as $n \to \infty$. Given this estimate, you can use Newton's method to compute k_n more accurately. We computed

$$\begin{aligned} k_1 &\approx 2.9256856, \\ k_2 &\approx 6.1765914, \\ k_3 &\approx 9.3538959, \\ k_4 &\approx 12.5132570. \end{aligned}$$

The estimates of the corresponding eigenvalues $\lambda_n = k_n^2$ are

$$\begin{aligned} \lambda_1 &\approx 8.5596361, \\ \lambda_2 &\approx 38.1502809, \\ \lambda_3 &\approx 87.4953676, \\ \lambda_4 &\approx 156.5815998. \end{aligned}$$

From (13.2.19) and the first equation in (13.2.20),

$$y_n = k_n \cos k_n x - \sin k_n x$$

is an eigenfunction associated with λ_n ∎

Figure 13.2.2. $u = \tan k$ *and* $u = -2k/(1+k)$

Since the differential equations in (13.2.12) and (13.2.14) are more complicated than those in (13.2.11) and (13.2.13) respectively, what is the point of Theorem 13.2.1? The point is this: to solve a *specific* problem, it may be better to deal with it directly, as we did in Examples 13.2.1 and 13.2.2; however, we'll see that transforming the *general* eigenvalue problem (13.2.1) to the Sturm–Liouville problem (13.2.10) leads to results applicable to *all* eigenvalue problems of the form (13.2.1).

Theorem 13.2.2

If

$$Ly = (p(x)y')' + q(x)y$$

and u and v are twice continuously functions on $[a,b]$ that satisfy the boundary conditions $B_1(y) = 0$ and $B_2(y) = 0$, then

$$\int_a^b [u(x)Lv(x) - v(x)Lu(x)]\,dx = 0. \tag{13.2.21}$$

Proof Integration by parts yields

$$\begin{aligned}\int_a^b [u(x)Lv(x) - v(x)Lu(x)]\,dx &= \int_a^b [u(x)(p(x)v'(x))' - v(x)(p(x)u'(x))']\,dx \\ &= p(x)[u(x)v'(x) - u'(x)v(x)]\Big|_a^b \\ &\quad - \int_a^b p(x)[u'(x)v'(x) - u'(x)v'(x)]\,dx.\end{aligned}$$

Since the last integral equals zero,

$$\int_a^b [u(x)Lv(x) - v(x)Lu(x)]\,dx = p(x)[u(x)v'(x) - u'(x)v'(x)]\Big|_a^b. \tag{13.2.22}$$

By assumption, $B_1(u) = B_1(v) = 0$ and $B_2(u) = B_2(v) = 0$. Therefore

$$\begin{aligned}\alpha u(a) + \beta u'(a) &= 0 \\ \alpha v(a) + \beta v'(a) &= 0\end{aligned} \qquad \text{and} \qquad \begin{aligned}\rho u(b) + \delta u'(b) &= 0 \\ \rho v(b) + \delta v'(b) &= 0.\end{aligned}$$

Since $\alpha^2 + \beta^2 > 0$ and $\rho^2 + \delta^2 > 0$, the determinants of these two systems must both be zero; that is,

$$u(a)v'(a) - u'(a)v(a) = u(b)v'(b) - u'(b)v(b) = 0.$$

This and (13.2.22) imply (13.2.21), which completes the proof.

The next theorem shows that a Sturm–Liouville problem has no complex eigenvalues.

Theorem 13.2.3

If $\lambda = p + qi$ with $q \neq 0$ then the boundary value problem

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0$$

has only the trivial solution.

Proof For this theorem to make sense, we must consider complex-valued solutions of

$$Ly + (p + iq)r(x,y)y = 0. \tag{13.2.23}$$

If $y = u + iv$ where u and v are real-valued and twice differentiable, we define $y' = u' + iv'$ and $y'' = u'' + iv''$. We say that y is a solution of (13.2.23) if the real and imaginary parts of the left side of (13.2.23) are both zero. Since $Ly = (p(x)'y)' + q(x)y$ and p, q, and r are real-valued,

$$\begin{aligned} Ly + \lambda r(x)y &= L(u+iv) + (p+iq)r(x)(u+iv) \\ &= Lu + r(x)(pu - qv) + i[Lv + r(x)(pu+qv)], \end{aligned}$$

so $Ly + \lambda r(x)y = 0$ if and only if

$$\begin{aligned} Lu + r(x)(pu - qv) = 0 \\ Lv + r(x)(qu + pv) = 0. \end{aligned}$$

Multiplying the first equation by v and the second by u yields

$$\begin{aligned} vLu + r(x)(puv - qv^2) = 0 \\ uLv + r(x)(qu^2 + puv) = 0. \end{aligned}$$

Subtracting the first equation from the second yields

$$uLv - vLu + qr(x)(u^2 + v^2) = 0,$$

so

$$\int_a^b [u(x)Lv(x) - v(x)Lu(x)]\,dx + \int_a^b r(x)[u^2(x) + v^2(x)]\,dx = 0. \tag{13.2.24}$$

Since

$$B_1(y) = B_1(u+iv) = B_1(u) + iB_1(v)$$

and

$$B_2(y) = B_2(u+iv) = B_2(u) + iB_2(v),$$

$B_1(y) = 0$ and $B_2(y) = 0$ implies that

$$B_1(u) = B_2(u) = B_1(v) = B_2(v) = 0.$$

Therefore Theorem 13.2.2 implies that first integral in (13.2.24) equals zero, so (13.2.24) reduces to

$$q \int_a^b r(x)[u^2(x) + v^2(x)]\, dx = 0.$$

Since r is positive on $[a, b]$ and $q \neq 0$ by assumption, this implies that $u \equiv 0$ and $v \equiv 0$ on $[a, b]$. Therefore $y \equiv 0$ on $[a, b]$, which completes the proof.

Theorem 13.2.4

If λ_1 and λ_2 are distinct eigenvalues of the Sturm–Liouville problem

$$Ly + \lambda r(x) y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \tag{13.2.25}$$

with associated eigenfunctions u and v respectively, then

$$\int_a^b r(x) u(x) v(x)\, dx = 0. \tag{13.2.26}$$

Proof Since u and v satisfy the boundary conditions in (13.2.25), Theorem 13.2.2 implies that

$$\int_a^b [u(x) Lv(x) - v(x) Lu(x)]\, dx = 0.$$

Since $Lu = -\lambda_1 ru$ and $Lv = -\lambda_2 rv$, this implies that

$$(\lambda_1 - \lambda_2) \int_a^b r(x) u(x) v(x)\, dx = 0.$$

Since $\lambda_1 \neq \lambda_2$, this implies (13.2.26), which completes the proof.

If u and v are any integrable functions on $[a, b]$ and

$$\int_a^b r(x) u(x) v(x)\, dx = 0,$$

we say that u and v *orthogonal on* $[a, b]$ *with respect to* $r = r(x)$.

Theorem 13.1.1 implies the next theorem.

Theorem 13.2.5

If $u \not\equiv 0$ and v both satisfy

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0,$$

then $v = cu$ for some constant c.

We've now proved parts of the next theorem. A complete proof is beyond the scope of this book.

Theorem 13.2.6

The set of all eigenvalues of the Sturm–Liouville problem

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0$$

can be ordered as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

and

$$\lim_{n \to \infty} \lambda_n = \infty.$$

For each n, if y_n is an arbitrary λ_n-eigenfunction, then every λ_n-eigenfunction is a constant multiple of y_n. If $m \neq n$, y_m and y_n are orthogonal $[a, b]$ with respect to $r = r(x)$; that is,

$$\int_a^b r(x)y_m(x)y_n(x)\,dx = 0. \tag{13.2.27}$$

You may want to verify (13.2.27) for the eigenfunctions obtained in Examples 13.2.1 and 13.2.2.

In conclusion, we mention the next theorem. The proof is beyond the scope of this book.

Theorem 13.2.7

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ be the eigenvalues of the Sturm–Liouville problem

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0,$$

with associated eigenvectors y_1, y_2, …, y_n, …. Suppose f is piecewise smooth (Definition ???) on $[a,b]$. For each n, let

$$c_n = \frac{\backslash\mathrm{dst}\int_a^b r(x)f(x)y_n(x)\,dx}{\backslash\mathrm{dst}\int_a^b r(x)y_n^2(x)\,dx}.$$

Then

$$\frac{f(x-)+f(x+)}{2} = \sum_{n=1}^{\infty} c_n y_n(x)$$

for all x in the open interval (a,b).

13.2 Exercises

In Exercises ???–??? rewrite the equation in Sturm–Liouville form (with $\lambda = 0$). Assume that b, c, α, and ν are contants.

1. $y'' + by' + cy = 0$

Show answer

2. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ (Bessel's equation)

Show answer

3. $(1 - x^2)y'' - xy' + \alpha^2 y = 0$ (Chebyshev's equation)

Show answer

4. $x^2y'' + bxy' + cy = 0$ (Euler's equation)

Show answer

5. $y'' - 2xy' + 2\alpha y = 0$ (Hermite's equation)

Show answer

6. $xy'' + (1 - x)y' + \alpha y = 0$ (Laguerre's equation)

Show answer

7. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ (Legendre's equation)

Show answer

8. In Example 13.2.4 we found that the eigenvalue problem

$$x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0 \tag{A}$$

is equivalent to the Sturm-Liouville problem

$$(xy')' + \frac{\lambda}{x}y = 0, \quad y(1) = 0, \quad y(2) = 0. \tag{B}$$

Multiply the differential equation in (B) by y and integrate to show that

$$\lambda \int_1^2 \frac{y^2(x)}{x}\,dx = \int_1^2 x(y'(x))^2\,dx.$$

Conclude from this that the eigenvalues of (A) are all positive.

9. Solve the eigenvalue problem

$$y'' + 2y' + y + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

Show answer

10. Solve the eigenvalue problem

$$y'' + 2y' + y + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0.$$

Show answer

In Exercises ???–??? : ***(a)*** *Determine whether* $\lambda = 0$ *is an eigenvalue. If it is, find an associated eigenfunction.*

(b) *Compute the negative eigenvalues with errors not greater than* 5×10^{-8}*. State the form of the associated eigenfunctions.*

(c) *Compute the first four positive eigenvalues with errors not greater than* $5 \times 10^{-8.}$ *State the form of the associated eigenfunctions.*

11. C $y'' + \lambda y = 0, \quad y(0) + 2y'(0) = 0, \quad y(2) = 0$

Show answer

12. C $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) - 2y'(1) = 0$

Show answer

13. C $y'' + \lambda y = 0, \quad y(0) - y'(0) = 0, \quad y'(\pi) = 0$

Show answer

14. C $y'' + \lambda y = 0, \quad y(0) + 2y'(0) = 0, \quad y(\pi) = 0$

Show answer

15. C $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(2) - y'(2) = 0$

Show answer

16. C $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(2) + 2y'(2) = 0$

Show answer

17. C $y'' + \lambda y = 0, \quad y(0) + 2y'(0) = 0, \quad y(3) - 2y'(3) = 0$

Show answer

18. C $y'' + \lambda y = 0, \quad 3y(0) + y'(0) = 0, \quad 3y(2) - 2y'(2) = 0$

Show answer

19. C $y'' + \lambda y = 0, \quad y(0) + 2y'(0) = 0, \quad y(3) - y'(3) = 0$

Show answer

20. C $y'' + \lambda y = 0, \quad 5y(0) + 2y'(0) = 0, \quad 5y(1) - 2y'(1) = 0$

Show answer

21. Find the first five eigenvalues of the boundary value problem

$$y'' + 2y' + y + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$$

with errors not greater than 5×10^{-8}. State the form of the associated eigenfunctions.

Show answer

In Exercises 22–24 take it as given that $\{xe^{kx}, xe^{-kx}\}$ *and* $\{x\cos kx, x\sin kx\}$ *are fundamental sets of solutions of*

$$x^2y'' - 2xy' + 2y - k^2x^2y = 0$$

and

$$x^2y'' - 2xy' + 2y + k^2x^2y = 0,$$

respectively.

22. Solve the eigenvalue problem for

$$x^2y'' - 2xy' + 2y + \lambda x^2 y = 0, \quad y(1) = 0, \quad y(2) = 0.$$

Show answer

23. C Find the first five eigenvalues of

$$x^2y'' - 2xy' + 2y + \lambda x^2 y = 0, \quad y'(1) = 0, \quad y(2) = 0$$

with errors no greater than 5×10^{-8}. State the form of the associated eienfunctions.

Show answer

24. C Find the first five eigenvalues of

$$x^2y'' - 2xy' + 2y + \lambda x^2 y = 0, \quad y(1) = 0, \quad y'(2) = 0$$

with errors no greater than 5×10^{-8}. State the form of the associated eienfunctions.

Show answer

25. Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) + \delta y'(L) = 0. \tag{A}$$

(a) Show that (A) can't have more than one negative eigenvalue, and find the values of δ for which it has one.

(b) Find all values of δ such that $\lambda = 0$ is an eigenvalue of (A).

(c) Show that $\lambda = k^2$ with $k > 0$ is an eigenvalue of (A) if and only if

$$\tan kL = -\delta k. \tag{B}$$

(d) For $n = 1, 2, \ldots$, let y_n be an eigenfunction associated with $\lambda_n = k_n^2$. From Theorem 13.2.4, y_m and y_n are orthogonal over $[0, L]$ if $m \neq n$. Verify this directly. *Hint: Integrate by parts twice and use* (B).

Show answer

26. Solve the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) + \alpha y'(0) = 0, \quad y(\pi) + \alpha y'(\pi) = 0,$$

where $\alpha \neq 0$.

Show answer

27. L Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) + \alpha y'(0) = 0, \quad y(1) + (\alpha - 1)y'(1) = 0, \tag{A}$$

where $0 < \alpha < 1$.

(a) Show that $\lambda = 0$ is an eigenvalue of (A), and find an associated eigenfunction.

(b) Show that (A) has a negative eigenvalue, and find the form of an associated eigenfunction.

(c) Give a graphical argument to show that (A) has infinitely many positive eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and state the form of the associated eigenfunctions.

Show answer

Exercises 28–30 deal with the Sturm–Liouville problem

$$y'' + \lambda y = 0, \quad \alpha y(0) + \beta y'(0), \quad \rho y(L) + \delta y'(L) = 0, \tag{SL}$$

where $\alpha^2 + \beta^2 > 0$ *and* $\rho^2 + \delta^2 > 0$.

28. Show that $\lambda = 0$ is an eigenvalue of (SL) if and only if

$$\alpha(\rho L + \delta) - \beta\rho = 0.$$

29. L The point of this exercise is that (SL) can't have more than two negative eigenvalues.

(a) Show that λ is a negative eigenvalue of (SL) if and only if $\lambda = -k^2$, where k is a positive solution of

$$(\alpha\rho - \beta\delta k^2)\sinh kL + k(\alpha\delta - \beta\rho)\cosh kL.$$

(b) Suppose $\alpha\delta - \beta\rho = 0$. Show that (SL) has a negative eigenvalue if and only if $\alpha\rho$ and $\beta\delta$ are both nonzero. Find the negative eigenvalue and an associated eigenfunction. *Hint: Show that in this case $\rho = p\alpha$ and $s = q\beta$, where $q \neq 0$.*

(c) Suppose $\beta\rho - \alpha\delta \neq 0$. We know from Section 11.1 that (SL) has no negative eigenvalues if $\alpha\rho = 0$ and $\beta\delta = 0$. Assume that either $\alpha\rho \neq 0$ or $\beta\delta \neq 0$. Then we can rewrite (A) as

$$\tanh kL = \frac{k(\beta\rho - \alpha\delta)}{\alpha\rho - \beta\delta k^2}.$$

By graphing both sides of this equation on the same axes (there are several possibilities for the right side), show that it has at most two positive solutions, so (SL) has at most two negative eigenvalues.

Show answer

30. L The point of this exercise is that (SL) has infinitely many positive eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and that $\lim_{n\to\infty} \lambda_n = \infty$.

(a) Show that λ is a positive eigenvalue of (SL) if and only if $\lambda = k^2$, where k is a positive solution of

$$(\alpha\rho + \beta\delta k^2)\sin kL + k(\alpha\delta - \beta\rho)\cos kL = 0. \tag{A}$$

(b) Suppose $\alpha\delta - \beta\rho = 0$. Show that the positive eigenvalues of (SL) are $\lambda_n = (n\pi/L)^2$, $n = 1, 2, 3, \ldots$. *Hint: Recall the hint in Exercise 29**(b)**.*

Now suppose $\alpha\delta - \beta\rho \neq 0$. From Section 11.1, if $\alpha\rho = 0$ and $\beta\delta = 0$, then (SL) has the eigenvalues

$$\lambda_n = [(2n-1)\pi/2L]^2, \quad n = 1, 2, 3, \ldots$$

(why?), so let's suppose in addition that at least one of the products $\alpha\rho$ and $\beta\delta$ is nonzero. Then we can rewrite (A) as

$$\tan kL = \frac{k(\beta\rho - \alpha\delta)}{\alpha\rho - \beta\delta k^2}. \tag{B}$$

By graphing both sides of this equation on the same axes (there are several possibilities for the right side), convince yourself of the following:

(c) If $\beta\delta = 0$, there's a positive integer N such that (B) has one solution k_n in each of the intervals

$$\left((2n-1)\pi/L, (2n+1)\pi/L)\right), \quad n = N, N+1, N+2, \ldots, \tag{C}$$

and either

$$\lim_{n\to\infty}\left(k_n - \frac{(2n-1)\pi}{2L}\right) = 0 \quad \text{or} \quad \lim_{n\to\infty}\left(k_n - \frac{(2n+1)\pi}{2L}\right) = 0.$$

(d) If $\beta\delta \neq 0$, there's a positive integer N such that (B) has one solution k_n in each of the intervals (C) and

$$\lim_{n\to\infty}\left(k_n - \frac{n\pi}{N}\right) = 0.$$

31. The following Sturm–Liouville problems are genera1izations of Problems 1–4 of Section 11.1.

Problem 1: $(p(x)y')' + \lambda r(x)y = 0,\quad y(a) = 0,\quad y(b) = 0$

Problem 2: $(p(x)y')' + \lambda r(x)y = 0,\quad y'(a) = 0,\quad y'(b) = 0$

Problem 3: $(p(x)y')' + \lambda r(x)y = 0,\quad y(a) = 0,\quad y'(b) = 0$

Problem 4: $(p(x)y')' + \lambda r(x)y = 0,\quad y'(a) = 0,\quad y(b) = 0$

Prove: Problems 1–4 have no negative eigenvalues. Moreover, $\lambda = 0$ is an eigenvalue of Problem 2 with associated eigenfunction $y_0 = 1$, but $\lambda = 0$ isn't an eigenvalue of Problems 1, 3, and 4. *Hint: See the proof of Theorem 11.1.1.*

32. Show that the eigenvalues of the Sturm–Liouville problem

$$(p(x)y')' + \lambda r(x)y = 0,\quad \alpha y(a) + \beta y'(a) = 0,\quad \rho y(b) + \delta y'(b)$$

are all positive if $\alpha\beta \le 0,\ \ \rho\delta \ge 0$, and $(\alpha\beta)^2 + (\rho\delta)^2 > 0$.

Chapter 13 Boundary Value Problems for Second Order Ordinary Differential Equations

13.1 Two-Point Boundary Value Problems

2 $y = -x +$ \dst $\frac{2}{e-1}\left(e^x - e^{(x-1)}\right)$

3 $y = x^2 -$ \dst $\frac{x^3}{3} + cx$ with c arbitrary

4 $y = -x + 2e^x + e^{-(x-1)}$

5 $y =$ \dst $\frac{1}{4} + \frac{11}{4}\cos 2x + \frac{9}{4}\sin 2x$

6 $y = (x^2 + 13 - 8x)e^x$

7 $y = 2e^{2x} +$ \dst $\frac{3(5e^{3x} - 4e^{4x})}{e(15-16e)} + \frac{2e^4(4e^{3(x-1)} - 3e^{4(x-1)})}{16e-15}$

8 \dst $\int_a^b tF(t)\,dt = 0$ $y =$ \dst $-x\int_x^1 F(t)\,dt - \int_0^x tF(t)\,dt + c_1 x$ with c_1 arbitrary

9 **(a)** $b - a \neq k\pi$ ($k =$ integer)

(b) \dst $\int_a^b F(t)\sin(t-a)\,dt = 0$

y

$=$ \dst $\frac{\sin(x-a)}{\sin(b-a)}\int_x^b F(t)\sin(t-b)\,dt + \frac{\sin(x-b)}{\sin(b-a)}\int_a^x F(t)\sin(t-a)\,dt$

$+c_1\sin(x-a)$ with c_1 arbitrary

y

$= -$\dst $\sin(x-a)\int_x^b F(t)\cos(t-a)\,dt - \cos(x-a)\int_a^x F(t)\sin(t-a)\,dt$

10 **(a)** $b - a \neq (k + 1/2)\pi$ ($k =$ integer)

(b) \dst $\int_a^b F(t)\sin(t-a)\,dt = 0$

y

$= -$\dst $\frac{\sin(x-a)}{\cos(b-a)}\int_x^b F(t)\cos(t-b)\,dt - \frac{\cos(x-b)}{\cos(b-a)}\int_x^b F(t)\sin(t-a)\,dt$

$+c_1\sin(x-a)$ with c_1 arbitrary

y

$= -$\dst $\sin(x-a)\int_x^b F(t)\cos(t-a)\,dt - \cos(x-a)\int_a^x F(t)\sin(t-a)\,dt$

11 **(a)** $b-a \neq k\pi$ ($k =$ integer)

(b) \dst$\int_a^b F(t)\cos(t-a)\,dt = 0$

$y = $ \dst$\frac{\cos(x-a)}{\sin(b-a)}\int_x^b F(t)\cos(t-b)\,dt + \frac{\cos(x-b)}{\sin(b-a)}\int_a^x F(t)\cos(t-a)\,dt$

$y = $ \dst$\cos(x-a)\int_x^b F(t)\sin(t-a)\,dt + \sin(x-a)\int_a^x F(t)\cos(t-a)\,dt$

$+c_1\cos(x-a)$ with c_1 arbitrary

12 $y = $ \dst$\frac{\sinh(x-a)}{\sinh(b-a)}\int_x^b F(t)\sinh(t-b)\,dt + \frac{\sinh(x-b)}{\sinh(b-a)}\int_a^x F(t)\sinh(t-a)\,dt$

13 $y = $ \dst$-\frac{\sinh(x-a)}{\cosh(b-a)}\int_x^b F(t)\cosh(t-b)\,dt - \frac{\cosh(x-b)}{\cosh(b-a)}\int_a^x F(t)\sinh(t-a)\,dt$

14 $y = -$\dst$\frac{\cosh(x-a)}{\sinh(b-a)}\int_x^b F(t)\cosh(t-b)\,dt - \frac{\cosh(x-b)}{\sinh(b-a)}\int_a^x F(t)\cosh(t-a)\,dt$

15 $y = -$\dst$\frac{1}{2}\left(e^x\int_x^b e^{-t}F(t)\,dt + e^{-x}\int_a^x e^tF(t)\,dt\right)$

16 If ω isn't a positive integer, then

$y = $ \dst$\frac{1}{\omega\sin\omega\pi}\left(\sin\omega x\int_x^\pi F(t)\sin\omega(t-\pi)\,dt + \sin\omega(x-\pi)\int_0^x F(t)\sin\omega t\,dt\right)$.

If $\omega = n$ (positive integer), then \dst$\int_0^\pi F(t)\sin nt\,dt = 0$ is necessary for existence of a solution. In this case,

$y = -$\dst$\frac{1}{n}\left(\sin nx\int_x^\pi F(t)\cos nt\,dt + \cos nx\int_0^x F(t)\sin nt\,dt\right) + c_1\sin nx$ with c_1 arbitrary.

17 If $\omega \ne n+1/2$ ($n =$ integer), then

$y = -\frac{}{\omega}$ If $\omega = n+1/2$ ($n =$ integer), then $\dst\int_0^\pi F(t)\sin(n+1/2)t\,dt = 0$ is necessary

for existence of a solution. In this case, $y = -\dst\frac{\sin(n+1/2)x}{n+1/2}\int_x^\pi F(t)\cos(n+1/2)t\,dt$

with c_1 arbitrary,

$-\dst\frac{\cos(n+1/2)x}{n+/2}\int_0^x F(t)\sin(n+1/2)t\,dt + c_1\sin(n+1/2)x$

18 If $\omega \ne n+1/2$ ($n =$ integer), then

$y = \frac{\mathrm{co}}{\omega\,\mathrm{c}}$ If $\omega = n+1/2$ ($n =$ integer), then $\dst\int_0^\pi F(t)\cos(n+1/2)t\,dt = 0$ is necessary

for existence of a solution. In this case, $y = \dst\frac{\cos(n+1/2)x}{n+1/2}\int_x^\pi F(t)\sin(n+1/2)t\,dt$

with c_1 arbitrary.

$+\dst\frac{\sin(n+1/2)x}{n+/2}\int_0^x F(t)\cos(n+1/2)t\,dt + c_1\cos(n+1/2)x$

19 If ω isn't a positive integer, then

y If $\omega = n$ (positive integer), then $\dst\int_0^\pi F(t)\cos nt\,dt = 0$ is necessary for existence
$= \dst\frac{1}{\omega\sin\omega\pi}\left(\cos\omega x\int_x^\pi F(t)\cos\omega(t-\pi)\,dt + \cos\omega(x-\pi)\int_0^x F(t)\cos\omega t\,dt\right)$.

of
a
solution.
In this
case,

y with c_1 arbitrary.
$= -\dst\frac{1}{n}\left(\cos nx\int_x^\pi F(t)\sin nt\,dt + \sin nx\int_0^x F(t)\cos nt\,dt\right) + c_1\cos nx$

20 $y_1 = B_1(z_2)z_1 - B_1(z_1)z_2$

21 **(a)** $G(x,t) = \begin{cases} \dst\frac{(t-a)(x-b)}{b-a} & a \le t \le x, \\ \dst\frac{(x-a)(t-b)}{b-a)} & x \le t \le b \end{cases}$

y **(b)** $G(x,t) = \begin{cases} a-t & a \le t \le x \\ a-x & x \le t \le b \end{cases}$ $y = \dst(a-x)\int_x^b F(t)\,dt + \int_a^x (a-t)F(t)\,dt$
$= \dst\frac{1}{b-a}\left((x-a)\int_x^b (t-b)F(t)\,dt + (x-b)\int_a^x (t-a)F(t)\,dt\right)$

(c) **(d)** $\dst\int_a^b F(t)\,dt = 0$ is a necessary condition for existence of a solution. Then
$G(x,t)$
$= \begin{cases} x-b & a \le t \le x \\ t-b & x \le t \le b \end{cases}$
y
$= \dst\int_x^b (t-b)F(t)\,dt + (x-b)\int_a^x F(t)\,dt$

y
$= \dst\int_x^b tF(t)\,dt + x\int_a^x F(t)\,dt + c_1$
with c_1
arbitrary.

22 $G(x,t)=\begin{cases}\displaystyle -\frac{(2+t)(3-x)}{5}, & 0\le t\le x,\\ -\displaystyle\frac{(2+x)(3-t)}{5}, & x\le t\le 1\end{cases}$ **(a)** $y=\displaystyle\frac{x^2-x-2}{2}$ **(b)** $y=\displaystyle\frac{5x^2-7x-14}{30}$

(c) $y=\displaystyle\frac{5x^4-9x-18}{60}$

23 $G(x,t)=\begin{cases}\displaystyle\frac{\cos t\sin x}{t^{3/2}\sqrt{x}}, & \displaystyle\frac{\pi}{2}\le t\le x,\\ \displaystyle\frac{\cos x\sin t}{t^{3/2}\sqrt{x}}, & x\le t\le \pi\end{cases}$

(a) $y=\displaystyle\frac{1+\cos x-\sin x}{\sqrt{x}}$
(b) $y=\displaystyle\frac{x+\pi\cos x-\pi/2\sin x}{\sqrt{x}}$

24 $G(x,t)=\begin{cases}\displaystyle\frac{(t-1)x(x-2)}{t^3}, & 1\le t\le x,\\ \displaystyle\frac{x(x-1)(t-2)}{t^3}, & x\le t\le 2\end{cases}$

(a) $y=x(x-1)(x-2)$
(b) $y=x(x-1)(x-2)(x+3)$

25 $G(x,t)=\begin{cases}-\displaystyle\frac{1}{22}\left(3+\frac{1}{t^2}\right)\left(x+\frac{4}{x}\right), & 1\le x\le t,\\ -\displaystyle\frac{1}{22}\left(3x+\frac{1}{x}\right)\left(1+\frac{4}{t^2}\right), & x\le t\le 2\end{cases}$

(a) $y=\displaystyle\frac{x^2-11x+4}{11x}$ **(b)** $y=\displaystyle\frac{11x^3-45x^2-4}{33x}$ **(c)** $y=\displaystyle\frac{11x^4-139x^2-28}{88x}$

26 $\alpha(\rho+\delta)-\beta\rho\ne 0$ $\quad G(x,t)=\begin{cases}\displaystyle\frac{(\beta-\alpha t)(\rho+\delta-\rho x)}{\alpha(\rho+\delta)-\beta\rho}, & 0\le t\le x,\\ \displaystyle\frac{(\beta-\alpha x)(\rho+\delta-\rho t)}{\alpha(\rho+\delta)-\beta\rho}, & x\le t\le 1\end{cases}$

27 $\alpha\delta-\beta\rho\neq 0 \qquad G(x,t)=\begin{cases}\backslash\mathrm{dst}\frac{(\beta\cos t-\alpha\sin t)(\delta\cos x-\rho\sin x)}{\alpha\delta-\beta\rho}, & 0\le t\le x,\\ \backslash\mathrm{dst}\frac{(\beta\cos x-\alpha\sin x)(\delta\cos t-\rho\sin t)}{\alpha\delta-\beta\rho}, & x\le t\le\pi\end{cases}$

28 $\alpha\rho+\beta\delta\neq 0 \quad G(x,t)=\begin{cases}\backslash\mathrm{dst}\frac{(\beta\cos t-\alpha\sin t)(\rho\cos x+\delta\sin x)}{\alpha\rho+\beta\delta} & x\le t\le\pi\\ \backslash\mathrm{dst}\frac{(\beta\cos x-\alpha\sin x)(\rho\cos t+\delta\sin t)}{\alpha\rho+\beta\delta} & 0\le t\le x\end{cases}$

29 $\alpha\delta-\beta\rho\neq 0 \quad G(x,t)=\begin{cases}\backslash\mathrm{dst}\frac{e^{x-t)}(\beta\cos t-(\alpha+\beta)\sin t)(\delta\cos x-(\rho+\delta)\sin x)}{\alpha\delta-\beta\rho} & 0\le t\le x,\\ \backslash\mathrm{dst}\frac{e^{x-t}(\beta\cos x-(\alpha+\beta)\sin x)(\delta\cos t-(\rho+\delta)\sin t)}{\alpha\delta-\beta\rho}, & x\le t\le\pi\end{cases}$

30 $\beta\delta+(\alpha+\beta)(\rho+\delta)\neq 0 \quad G(x,t)=\begin{cases}\backslash\mathrm{dst}\frac{e^{x-t}(\beta\cos t-(\alpha+\beta)\sin t)((\rho+\delta)\cos x+\delta\sin x)}{\beta\delta+(\alpha+\beta)(\rho+\delta}, & 0\le t\le x,\\ \backslash\mathrm{dst}\frac{e^{x-t}(\beta\cos x-(\alpha+\beta)\sin x)((\rho+\delta)\cos t+\delta\sin t)}{\beta\delta+(\alpha+\beta)(\rho+\delta}, & x\le t\le\pi/2\end{cases}$

31 $(\rho+\delta)(\alpha-\beta)e^{(b-a)}-(\rho-\delta)(\alpha+\beta)e^{(a-b)}\neq 0$

$$G(x,t)\\ =\begin{cases}\backslash\mathrm{dst}\frac{((\alpha-\beta)e^{(t-a)}-(\alpha+\beta)e^{-(t-a)})((\rho-\delta)e^{(x-b)}-(\rho+\delta)e^{-(x-b)})}{2[(\rho+\delta)(\alpha-\beta)e^{(b-a)}-(\rho-\delta)(\alpha+\beta)e^{(a-b)}]}, & 0\le t\le x,\\ \backslash\mathrm{dst}\frac{((\alpha-\beta)e^{(x-a)}-(\alpha+\beta)e^{-(x-a)})((\rho-\delta)e^{(t-b)}-(\rho+\delta)e^{-(t-b)})}{2[(\rho+\delta)(\alpha-\beta)e^{(b-a)}-(\rho-\delta)(\alpha+\beta)e^{(a-b)}]} & x\le t\le\pi\end{cases}$$

13.2 Sturm-Liouville Problems

1 $(e^{bx}y')' + ce^{bx}y = 0$

2 $\dst(xy')' + \left(x - \frac{\nu^2}{x}\right)y = 0$

3 $\dst(\sqrt{1-x^2}y')' + \frac{\alpha^2}{\sqrt{1-x^2}}y = 0$

4 $(x^b y')' + cx^{b-2}y = 0$

5 $(e^{-x^2}y')' + 2\alpha e^{-x^2}y = 0$

6 $(xe^{-x}y')' + \alpha e^{-x}y = 0$

7 $((1-x^2)y')' + \alpha(\alpha+1)y = 0$

9 $\lambda_n = n^2\pi^2,\ y_n = e^{-x}\sin n\pi x$ ($n =$ positive integer)

10 $\lambda_0 = -1$, $y_0 = 1$ $\lambda_n = n^2\pi^2$, $y_n = e^{-x}(n\pi\cos n\pi x + \sin n\pi x)$ ($n =$ positive

integer)

11 **(a)** $\lambda = 0$ is an eigenvalue $y_0 = 2 - x$ **(b)** none **(c)** 5.0476821, 14.9198790,

29.7249673,
49.4644528
y
$= 2\sqrt{\lambda}\cos\sqrt{\lambda}$
x
$-\sin\sqrt{\lambda}x$

12 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** -0.5955245 $y = \cosh\sqrt{-\lambda}\,x$ **(c)** 8.8511386,

38.4741053,
87.8245457,
156.9126094
y
$= \cos\sqrt{\lambda}x$

13 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** none **(c)** 0.1470328, 1.4852833, 4.5761411,

9.6059439
y
$= \sqrt{\lambda}\cos\sqrt{\lambda}x$
$+ \sin\sqrt{\lambda}x$

14 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** -0.1945921

y 19.9317507 $y = 2\sqrt{\lambda}\,\cos\sqrt{\lambda}\,x - \sin\sqrt{\lambda}\,x$
$= 2\sqrt{-\lambda}$
$\cosh\sqrt{-\lambda}$
x
$- \sinh\sqrt{-\lambda}$
x **(c)**
1.9323619,
5.9318981,
11.9317920,

15 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** -1.0664054 $y = \cosh\sqrt{-\lambda}\,x$ **(c)** 1.5113188,

8.8785880,
21.2104662,
38.4805610
y
$= \cos\sqrt{\lambda}$
x

16 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** -1.0239346

$y = \sqrt{-\lambda}\,\cosh\sqrt{-\lambda}\,x - \sinh\sqrt{-\lambda}\,x$ **(c)** 2.0565705, 9.3927144, 21.7169130, 38.9842177 $y = \sqrt{\lambda}\,\cos\sqrt{\lambda}\,x - \sin\sqrt{\lambda}\,x$

17 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** -0.4357577,

$y = 2\sqrt{-\lambda}\,\cosh\sqrt{-\lambda}\,x - \sinh\sqrt{-\lambda}\,x$ **(c)** 0.3171423, 3.7055350, 9.1970150, 16.8760401 $y = 2\sqrt{\lambda}\,\cos\sqrt{\lambda}\,x - \sin\sqrt{\lambda}\,x$

18 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** -2.1790546, -9.0006633

$y = \sqrt{-\lambda}\,\cosh\sqrt{-\lambda}\,x - 3\sinh\sqrt{-\lambda}\,x$ **(c)** 5.8453181, 17.9260967, 35.1038567, 57.2659330 $y = \sqrt{\lambda}\,\cos\sqrt{\lambda}\,x - 3\sin\sqrt{\lambda}\,x$

19 **(a)** $\lambda = 0$ is an eigenvalue $y_0 = 2 - x$ **(b)** -1.0273046

y 38.4784094 $y = 2\sqrt{\lambda}\,\cos\sqrt{\lambda}\,x - \sin\sqrt{\lambda}\,x$

$= 2\sqrt{-\lambda}$

$\cosh\sqrt{-\lambda}$

x

$-\sinh\sqrt{-\lambda}$

x **(c)**

$8.8694608,$

$16.5459202,$

$26.4155505,$

20 **(a)** $\lambda = 0$ isn't an eigenvalue **(b)** $-7.9394171,\ -3.1542806$

y 236.7229622 $y = 2\sqrt{\lambda}\,\cos\sqrt{\lambda}\,x - 5\sin\sqrt{\lambda}\,x$

$= 2\sqrt{-\lambda}$

$\cosh\sqrt{-\lambda}$

x

$-5\sinh\sqrt{-\lambda}$

x **(c)**

$29.3617465,$

$78.777456,$

$147.8866417,$

21 $\lambda = 0,\ y = xe^{-x}$ $20.1907286,\ 118.8998692,\ 296.5544121,\ 553.1646458$

y

$= e^{-x}\sin\sqrt{\lambda}$

x

22 $\lambda_n = n^2\pi^2,\ y_n = x\sin n\pi(x-2)$ ($n =$ positive integer)

23 $\lambda = 0,\ y = x(2-x)$ $20.1907286,\ 118.8998692,\ 296.5544121$

$553.1646458,$

y

$= x\sin\sqrt{\lambda}$

$(x-2)$

24 3.3730893, 23.1923372, 62.6797232, 121.8999231, 200.8578309

y
$= x \sin \sqrt{\lambda}$
$(x-1)$

25 **(a)** $-L < \delta < 0$ **(b)** $\delta = -L$

26 $\lambda_0 = -1/\alpha^2$ $y_0 = e^{-x/\alpha}$ $\lambda_n = n^2$, $y_n = n\alpha \cos nx - \sin nx$, $n = 1, 2, \ldots$

27 **(a)** $y = x - \alpha$ **(b)** $y = \alpha k \cosh kx - \sin kx$ **(c)** $y = \alpha k \cos kx - \sin kx$

29 **(b)** $\lambda = -\alpha^2/\beta^2$ $y = e^{-\alpha x/\beta}$