12 Fourier Solutions of Partial Differential Equations

12 Fourier Solutions of Partial Differential Equations

IN THIS CHAPTER we use the series discussed in Chapter 11 to solve partial differential equations that arise in problems of mathematical physics.

SECTION 12.1 deals with the partial differential equation

$$u_t = a^2 u_{xx},$$

which arises in problems of conduction of heat.

SECTION 12.2 deals with the partial differential equation

$$u_{tt} = a^2 u_{xx},$$

which arises in the problem of the vibrating string.

SECTION 12.3 deals with the partial differential equation

$$u_{xx} + u_{yy} = 0,$$

which arises in steady state problems of heat conduction and potential theory.

SECTION 12.4 deals with the partial differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

which is the equivalent to the equation studied in Section 1.3 when the independent variables are polar coordinates.

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Answers to selected exercises in this chapter

12.1 The Heat Equation

We begin the study of partial differential equations with the problem of heat flow in a uniform bar of length L, situated on the x axis with one end at the origin and the other at $x = L$ (Figure 12.1.1).

We assume that the bar is perfectly insulated except possibly at its endpoints, and that the temperature is constant on each cross section and therefore depends only on x and t. We also assume that the thermal properties of the bar are independent of x and t. In this case, it can be shown that the temperature $u = u(x,t)$ at time t at a point x units from the origin satisfies the partial differential equation

$$u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$

where a is a positive constant determined by the thermal properties. This is the heat equation.

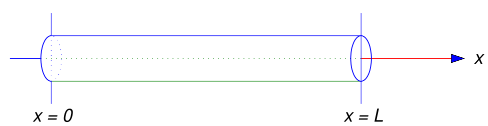

Figure 12.1.1. *A uniform bar of length L*

To determine u, we must specify the temperature at every point in the bar when $t = 0$, say

$$u(x,0) = f(x), \quad 0 \le x \le L.$$

We call this the initial condition. We must also specify boundary conditions that u must satisfy at the ends of the bar for all $t > 0$. We'll call this problem an initial-boundary value problem.

We begin with the boundary conditions $u(0,t) = u(L,t) = 0$, and write the initial-boundary value problem as

$$
\begin{aligned}
u_t &= a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\
u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\
u(x,0) &= f(x), \quad 0 \le x \le L.
\end{aligned} \tag{12.1.1}
$$

Our method of solving this problem is called separation of variables (not to be confused with method of separation of variables used in Section 2.2 for solving ordinary differential equations). We begin by looking for functions of the form

$$v(x,t) = X(x)T(t)$$

that are not identically zero and satisfy

$$v_t = a^2 v_{xx}, \quad v(0,t) = 0, \quad v(L,t) = 0$$

for all (x,t). Since

$$v_t = XT' \quad \text{and} \quad v_{xx} = X''T,$$

$v_t = a^2 v_{xx}$ if and only if

$$XT' = a^2 X''T,$$

which we rewrite as

$$\frac{T'}{a^2T} = \frac{X''}{X}.$$

Since the expression on the left is independent of x while the one on the right is independent of t, this equation can hold for all (x,t) only if the two sides equal the same constant, which we call a separation constant, and write it as $-\lambda$; thus,

$$\frac{X''}{X} = \frac{T'}{a^2T} = -\lambda.$$

This is equivalent to

$$X'' + \lambda X = 0$$

and

$$T' = -a^2\lambda T. \tag{12.1.2}$$

Since $v(0,t) = X(0)T(t) = 0$ and $v(L,t) = X(L)T(t) = 0$ and we don't want T to be identically zero, $X(0) = 0$ and $X(L) = 0$. Therefore λ must be an eigenvalue of the boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0, \tag{12.1.3}$$

and X must be a λ-eigenfunction. From Theorem 11.1.2, the eigenvalues of (12.1.3) are $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$X_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots.$$

Substituting $\lambda = n^2\pi^2/L^2$ into (12.1.2) yields

$$T' = -(n^2\pi^2a^2/L^2)T,$$

which has the solution

$$T_n = e^{-n^2\pi^2a^2t/L^2}.$$

Now let

$$v_n(x,t) = X_n(x)T_n(t) = e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

Since

$$v_n(x,0) = \sin\frac{n\pi x}{L},$$

v_n satisfies (12.1.1) with $f(x) = \sin n\pi x/L$. More generally, if $\alpha_1, \dots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m}\alpha_n e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L},$$

then u_m satisfies (12.1.1) with

$$f(x) = \sum_{n=1}^{m} \alpha_n \sin \frac{n\pi x}{L}.$$

This motivates the next definition.

Definition 12.1.1

The formal solution of the initial-boundary value problem

$$\begin{gathered} u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{gathered} \tag{12.1.4}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2a^2t/L^2} \sin \frac{n\pi x}{L}, \tag{12.1.5}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

We use the term “formal solution” in this definition because it’s not in general true that the infinite series in (12.1.5) actually satisfies all the requirements of the initial-boundary value problem (12.1.4) when it does, we say that it’s an actual solution of (12.1.4).

Because of the negative exponentials in (12.1.5), u converges for all (x, t) with $t > 0$ (Exercise 54). Since each term in (12.1.5) satisfies the heat equation and the boundary conditions in (12.1.4), u also has these properties if u_t and u_{xx} can be obtained by differentiating the series in (12.1.5) term by term once with respect to t and twice with respect to x, for $t > 0$. However, it’s not always legitimate to differentiate an infinite series term by term. The next theorem gives a useful sufficient condition for legitimacy of term by term differentiation of an infinite series. We omit the proof.

Theorem 12.1.2

A convergent infinite series

$$W(z)=\sum_{n=1}^{\infty} w_n(z)$$

can be differentiated term by term on a closed interval $[z_1, z_2]$ to obtain

$$W'(z)=\sum_{n=1}^{\infty} w_n'(z)$$

(where the derivatives at $z=z_1$ and $z=z_2$ are one-sided) provided that w_n' is continuous on $[z_1, z_2]$ and

$$|w_n'(z)| \le M_n, \quad z_1 \le z \le z_2, \quad n=1,2,3,\dots,$$

where M_1, M_2, …, M_n, …, are constants such that the series $\sum_{n=1}^{\infty} M_n$ converges.

Theorem 12.1.2, applied twice with $z=x$ and once with $z=t$, shows that u_{xx} and u_t can be obtained by differentiating u term by term if $t>0$ (Exercise 54). Therefore u satisfies the heat equation and the boundary conditions in (12.1.4) for $t>0$. Therefore, since $u(x,0)=S(x)$ for $0 \le x \le L$, u is an actual solution of (12.1.4) if and only if $S(x)=f(x)$ for $0 \le x \le L$. From Theorem 11.3.2, this is true if f is continuous and piecewise smooth on $[0,L]$, and $f(0)=f(L)=0$.

In this chapter we'll define formal solutions of several kinds of problems. When we ask you to solve such problems, we always mean that you should find a formal solution.

Example 12.1.1

Solve (12.1.4) with $f(x)=x(x^2-3Lx+2L^2)$.

Solution From Example 11.3.6, the Fourier sine series of f on $[0,L]$ is

$$S(x)=\dst\frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{L}.$$

Therefore

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If both ends of bar are insulated so that no heat can pass through them, then the boundary conditions are

$$u_x(0,t)=0,\quad u_x(L,t)=0,\quad t>0.$$

We leave it to you (Exercise 1) to use the method of separation of variables and Theorem 11.1.3 to motivate the next definition.

Definition 12.1.3

The formal solution of the initial-boundary value problem

$$\begin{array}{c} u_t=a^2u_{xx},\quad 0<x<L,\quad t>0,\\ u_x(0,t)=0,\quad u_x(L,t)=0,\quad t>0,\\ u(x,0)=f(x),\quad 0\le x\le L \end{array} \tag{12.1.6}$$

is

$$u(x,t)=\alpha_0+\sum_{n=1}^{\infty}\alpha_n e^{-n^2\pi^2a^2t/L^2}\cos\frac{n\pi x}{L},$$

where

$$C(x)=\alpha_0+\sum_{n=1}^{\infty}\alpha_n\cos\frac{n\pi x}{L}$$

is the Fourier cosine series of f on $[0,L]$; that is,

$$\alpha_0=\frac{1}{L}\int_0^L f(x)\,dx\quad\text{and}\quad \alpha_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\quad n=1,2,3,\dots.$$

Example 12.1.2

Solve (12.1.6) with $f(x)=x$.

Solution From Example 11.3.1, the Fourier cosine series of f on $[0, L]$ is

$$C(x) = \dst\frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}.$$

Therefore

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We leave it to you (Exercise 2) to use the method of separation of variables and Theorem 11.1.4 to motivate the next definition.

Definition 12.1.4

The formal solution of the initial-boundary value problem

$$\begin{gathered} u_t = a^2u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{gathered} \tag{12.1.7}$$

is

$$u(x,t) = \sum_{n=1}^{\infty}\alpha_n e^{-(2n-1)^2\pi^2a^2t/4L^2}\sin\frac{(2n-1)\pi x}{2L},$$

where

$$S_M(x) = \sum_{n=1}^{\infty}\alpha_n\sin\frac{(2n-1)\pi x}{2L}$$

is the mixed Fourier sine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx.$$

Example 12.1.3

> Solve (12.1.7) with $f(x)=x$.

Solution From Example 11.3.4, the mixed Fourier sine series of f on $[0,L]$ is

$$S_M(x)=-\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2L}.$$

Therefore

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Figure 12.1.2 shows a graph of $u=u(x,t)$ plotted with respect to x for various values of t. The line $y=x$ corresponds to $t=0$. The other curves correspond to positive values of t. As t increases, the graphs approach the line $u=0$.

Figure 12.1.2.

We leave it to you (Exercise 3) to use the method of separation of variables and Theorem 11.1.5 to motivate the next definition.

Definition 12.1.5

The formal solution of the initial-boundary value problem

$$\begin{gathered} u_t=a^2u_{xx},\quad 0<x<L,\quad t>0,\\ u_x(0,t)=0,\quad u(L,t)=0,\quad t>0,\\ u(x,0)=f(x),\quad 0\le x\le L \end{gathered}\tag{12.1.8}$$

is

$$u(x,t)=\sum_{n=1}^{\infty}\alpha_n e^{-(2n-1)^2\pi^2a^2t/4L^2}\cos\frac{(2n-1)\pi x}{2L},$$

where

$$C_M(x)=\sum_{n=1}^{\infty}\alpha_n\cos\frac{(2n-1)\pi x}{2L}$$

is the mixed Fourier cosine series of f on $[0,L]$; that is,

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.$$

Example 12.1.4

Solve (12.1.8) with $f(x) = x - L$.

Solution From Example 11.3.3, the mixed Fourier cosine series of f on $[0, L]$ is

$$C_M(x) = -\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2L}.$$

Therefore

$$u(x,t) = -\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2a^2t/4L^2}\cos\frac{(2n-1)\pi x}{2L}.$$

Nonhomogeneous Problems

A problem of the form

$$\begin{gathered} u_t = a^2u_{xx} + h(x), \quad 0 < x < L, \quad t > 0, \\ u(0,t) = u_0, \quad u(L,t) = u_L, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{gathered} \tag{12.1.9}$$

can be transformed to a problem that can be solved by separation of variables. We write

$$u(x,t) = v(x,t) + q(x), \tag{12.1.10}$$

where q is to be determined. Then

$$u_t = v_t \quad \text{and} \quad u_{xx} = v_{xx} + q''$$

so u satisfies (12.1.9) if v satisfies

$$\begin{gathered} v_t = a^2 v_{xx} + a^2 q''(x) + h(x), \quad 0 < x < L, \quad t > 0, \\ v(0,t) = u_0 - q(0), \quad v(L,t) = u_L - q(L), \quad t > 0, \\ v(x,0) = f(x) - q(x), \quad 0 \le x \le L. \end{gathered}$$

This reduces to

$$\begin{gathered} v_t = a^2 v_{xx}, \quad 0 < x < L, \quad t > 0, \\ v(0,t) = 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) = f(x) - q(x), \quad 0 \le x \le L \end{gathered} \tag{12.1.11}$$

if

$$a^2 q'' + h(x) = 0, \quad q(0) = u_0, \quad q(L) = u_L.$$

We can obtain q by integrating $q'' = -h/a^2$ twice and choosing the constants of integration so that $q(0) = u_0$ and $q(L) = u_L$. Then we can solve (12.1.11) for v by separation of variables, and (12.1.10) is the solution of (12.1.9).

Example 12.1.5

Solve

$$\begin{gathered} u_t = u_{xx} - 2, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = -1, \quad u(1,t) = 1, \quad t > 0, \\ u(x,0) = x^3 - 2x^2 + 3x - 1, \quad 0 \le x \le 1. \end{gathered}$$

Solution We leave it to you to show that

$$q(x) = x^2 + x - 1$$

satisfies

$$q'' - 2 = 0, \quad q(0) = -1, \quad q(1) = 1.$$

Therefore

$$u(x,t) = v(x,t) + x^2 + x - 1,$$

where

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0,$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0,$$

and

$$v(x,0) = x^3 - 2x^2 + 3x - 1 - x^2 - x + 1 = x(x^2 - 3x + 2).$$

From Example 12.1.1 with $a = 1$ and $L = 1$,

$$v(x,t) = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2\pi^2 t} \sin n\pi x.$$

Therefore

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A similar procedure works if the boundary conditions in (12.1.11) are replaced by mixed boundary conditions

$$u_x(0,t) = u_0, \quad u(L,t) = u_L, \quad t > 0$$

or

$$u(0,t) = u_0, \quad u_x(L,t) = u_L, \quad t > 0;$$

however, this isn’t true in general for the boundary conditions

$$u_x(0,t) = u_0, \quad u_x(L,t) = u_L, \quad t > 0.$$

(See Exercise 47.)

Using Technology

Numerical experiments can enhance your understanding of the solutions of initial-boundary value problems. To be specific, consider the formal solution

$$u(x,t)=\sum_{n=1}^{\infty}\alpha_n e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L},$$

of (12.1.4), where

$$S(x)=\sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0,L]$. Consider the m-th partial sum

$$u_m(x,t)=\sum_{n=1}^{m}\alpha_n e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L}. \tag{12.1.12}$$

For several fixed values of t (including $t=0$), graph $u_m(x,t)$ versus t. In some cases it may be useful to graph the curves corresponding to the various values of t on the same axes in other cases you may want to graph the various curves sucessively (for increasing values of t), and create a primitive motion picture on your monitor. Repeat this experiment for several values of m, to compare how the results depend upon m for small and large values of t. However, keep in mind that the meanings of “small” and “large” in this case depend upon the constants a^2 and L^2. A good way to handle this is to rewrite (12.1.12) as

$$u_m(x,t)=\sum_{n=1}^{m}\alpha_n e^{-n^2\tau}\sin\frac{n\pi x}{L},$$

where

$$\tau=\frac{\pi^2a^2t}{L^2}, \tag{12.1.13}$$

and graph u_m versus x for selected values of τ.

These comments also apply to the situations considered in Definitions 12.1.3-12.1.5, except that (12.1.13) should be replaced by

$$\tau=\frac{\pi^2a^2t}{4L^2},$$

in Definitions 12.1.4 and 12.1.5.

In some of the exercises we say “perform numerical experiments.” This means that you should perform the computations just described with the formal solution obtained in the exercise.

12.1 Exercises

1. Explain Definition 12.1.3.
2. Explain Definition 12.1.4.
3. Explain Definition 12.1.5.
4. C Perform numerical experiments with the formal solution obtained in Example 12.1.1.
5. C Perform numerical experiments with the formal solution obtained in Example 12.1.2.
6. C Perform numerical experiments with the formal solution obtained in Example 12.1.3.
7. C Perform numerical experiments with the formal solution obtained in Example 12.1.4.

In Exercises 8-42 solve the initial-boundary value problem. Where indicated by C *, perform numerical experiments. To simplify the computation of coefficients in some of these problems, check first to see if* $u(x,0)$ *is a polynomial that satisfies the boundary conditions. If it does, apply Theorem 11.3.5; also, see Exercises* $11.3.$ ***35(b)****,* $11.3.$ ***42(b)****, and* $11.3.$ ***50(b)****.*

8. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(1-x), \quad 0 \le x \le 1$

Show answer

9. $u_t = 9u_{xx}, \quad 0 < x < 4, \quad t > 0,$
$u(0,t) = 0, \quad u(4,t) = 0, \quad t > 0,$
$u(x,0) = 1, \quad 0 \le x \le 4$

Show answer

10. $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x\sin x, \quad 0 \le x \le \pi$

Show answer

11. C $u_t = 9u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$u(x,0) = x^2(2-x), \quad 0 \le x \le 2$

Show answer

12. $u_t = 4u_{xx}, \quad 0 < x < 3, \quad t > 0,$
$u(0,t) = 0, \quad u(3,t) = 0, \quad t > 0,$
$u(x,0) = x(9-x^2), \quad 0 \le x \le 3$

Show answer

13. $u_t = 4u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$$u(x,0) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2. \end{cases}$$

Show answer

14. $u_t = 7u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(3x^4 - 10x^2 + 7), \quad 0 \le x \le 1$

Show answer

15. $u_t = 5u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(x^3 - 2x^2 + 1), \quad 0 \le x \le 1$

Show answer

16. $u_t = 2u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(3x^4 - 5x^3 + 2), \quad 0 \le x \le 1$

Show answer

17. C $u_t = 9u_{xx}, \quad 0 < x < 4, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(4,t) = 0, \quad t > 0,$
$u(x,0) = x^2, \quad 0 \le x \le 4$

Show answer

18. $u_t = 4u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = x(x-4), \quad 0 \le x \le 2$

Show answer

19. C $u_t = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(1-x), \quad 0 \le x \le 1$

Show answer

20. $u_t = 3u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = 2x^2(3-x), \quad 0 \le x \le 2$

Show answer

21. $u_t = 5u_{xx}, \quad 0 < x < \sqrt{2}, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(\sqrt{2},t) = 0, \quad t > 0,$
$u(x,0) = 3x^2(x^2 - 4), \quad 0 \le x \le \sqrt{2}$

Show answer

22. C $u_t = 3u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^3(3x-4), \quad 0 \le x \le 1$

Show answer

23. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^2(3x^2 - 8x + 6), \quad 0 \le x \le 1$

Show answer

24. $u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x^2(x-\pi)^2, \quad 0 \le x \le \pi$

Show answer

25. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = \sin \pi x, \quad 0 \le x \le 1$

Show answer

26. C $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x(\pi - x), \quad 0 \le x \le \pi$

Show answer

27. $u_t = 5u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = x(4-x), \quad 0 \le x \le 2$

Show answer

28. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^2(3-2x), \quad 0 \le x \le 1$

Show answer

29. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = (x-1)^3 + 1, \quad 0 \le x \le 1$

Show answer

30. C $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(x^2-3), \quad 0 \le x \le 1$

Show answer

31. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^3(3x-4), \quad 0 \le x \le 1$

Show answer

32. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(x^3 - 2x^2 + 2), \quad 0 \le x \le 1$

Show answer

33. $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x^2(\pi - x), \quad 0 \le x \le \pi$

Show answer

34. $u_t = 16u_{xx}, \quad 0 < x < 2\pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(2\pi,t) = 0, \quad t > 0,$
$u(x,0) = 4, \quad 0 \le x \le 2\pi$

Show answer

35. $u_t = 9u_{xx}, \quad 0 < x < 4, \quad t > 0,$
$u_x(0,t) = 0, \quad u(4,t) = 0, \quad t > 0,$
$u(x,0) = x^2, \quad 0 \le x \le 4$

Show answer

36. C $u_t = 3u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 1 - x, \quad 0 \le x \le 1$

Show answer

37. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 1 - x^3, \quad 0 \le x \le 1$

Show answer

38. $u_t = 7u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = \pi^2 - x^2, \quad 0 \le x \le \pi$

Show answer

39. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 4x^3 + 3x^2 - 7, \quad 0 \le x \le 1$

Show answer

40. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 2x^3 + 3x^2 - 5, \quad 0 \le x \le 1$

Show answer

41. C $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^4 - 4x^3 + 6x^2 - 3, \quad 0 \le x \le 1$

Show answer

42. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^4 - 2x^3 + 1, \quad 0 \le x \le 1$

Show answer

In Exercises 43-46 solve the initial-boundary value problem. Perform numerical experiments for specific values of L and a.

43. C $u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0,$
$u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{L}{2}, \\ 0, & \frac{L}{2} < x < L \end{cases}$

Show answer

44. C $u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$
$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$
$u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{L}{2}, \\ 0, & \frac{L}{2} < x < L \end{cases}$

Show answer

45. C $u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$
$u_x(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$
$u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{L}{2}, \\ 0, & \frac{L}{2} < x < L \end{cases}$

Show answer

46. C $u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$
$u(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0,$
$u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{L}{2}, \\ 0, & \frac{L}{2} < x < L \end{cases}$

Show answer

47. Let h be continuous on $[0,L]$ and let u_0, u_L, and a be constants, with $a > 0$. Show that it's always possible to find a function q that satisfies **(a)**, **(b)**, or **(c)**, but that this isn't so for **(d)**.

a. $a^2 q'' + h = 0, \quad q(0) = u_0, \quad q(L) = u_L$

b. $a^2 q'' + h = 0, \quad q'(0) = u_0, \quad q(L) = u_L$

c. $a^2 q'' + h = 0, \quad q(0) = u_0, \quad q'(L) = u_L$

d. $a^2 q'' + h = 0, \quad q'(0) = u_0, \quad q'(L) = u_L$

In Exercises 48-53 solve the nonhomogeneous initial-boundary value problem

48. $u_t = 9u_{xx} - 54x, \quad 0 < x < 4, \quad t > 0,$
$u(0,t) = 1, \quad u(4,t) = 61, \quad t > 0,$
$u(x,0) = 2 - x + x^3, \quad 0 \le x \le 4$

Show answer

49. $u_t = u_{xx} - 2, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 1, \quad u(1,t) = 3, \quad t > 0,$
$u(x,0) = 2x^2 + 1, \quad 0 \le x \le 1$

Show answer

50. $u_t = 3u_{xx} - 18x, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = -1, \quad u(1,t) = -1, \quad t > 0,$
$u(x,0) = x^3 - 2x, \quad 0 \le x \le 1$

Show answer

51. $u_t = 9u_{xx} - 18, \quad 0 < x < 4, \quad t > 0,$
$u_x(0,t) = -1, \quad u(4,t) = 10, \quad t > 0,$
$u(x,0) = 2x^2 - x - 2, \quad 0 \le x \le 4$

Show answer

52. $u_t = u_{xx} + \pi^2 \sin \pi x, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = -\pi, \quad t > 0,$
$u(x,0) = 2 \sin \pi x, \quad 0 \le x \le 1$

Show answer

53. $u_t = u_{xx} - 6x, \quad 0 < x < L, \quad t > 0,$
$u(0,t) = 3, \quad u_x(1,t) = 2, \quad t > 0,$
$u(x,0) = x^3 - x^2 + x + 3, \quad 0 \le x \le 1$

Show answer

54. In this exercise take it as given that the infinite series $\sum_{n=1}^{\infty} n^p e^{-qn^2}$ converges for all p if $q > 0$, and, where appropriate, use the comparison test for absolute convergence of an infinite series.

Let

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L}$$

where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

and f is piecewise smooth on $[0, L]$.

a. Show that u is defined for (x, t) such that $t > 0$.

b. For fixed $t > 0$, use Theorem 12.1.2 with $z = x$ to show that

$$u_x(x,t) = \frac{\pi}{L} \sum_{n=1}^{\infty} n\alpha_n e^{-n^2\pi^2 a^2 t/L^2} \cos \frac{n\pi x}{L}, \quad -\infty < x < \infty.$$

c. Starting from the result of **(a)**, use Theorem 12.1.2 with $z = x$ to show that, for a fixed $t > 0$,

$$u_{xx}(x,t) = -\frac{\pi^2}{L^2} \sum_{n=1}^{\infty} n^2\alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L}, \quad -\infty < x < \infty.$$

d. For fixed but arbitrary x, use Theorem 12.1.2 with $z = t$ to show that

$$u_t(x,t) = -\frac{\pi^2 a^2}{L^2} \sum_{n=1}^{\infty} n^2\alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L},$$

if $t > t_0 > 0$, where t_0 is an arbitrary positive number. Then argue that since t_0 is arbitrary, the conclusion holds for all $t > 0$.

e. Conclude from **(c)** and **(d)** that

$$u_t = a^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0.$$

By repeatedly applying the arguments in **(a)** and **(c)**, it can be shown that u can be differentiated term by term any number of times with respect to x and/or t if $t > 0$.

12.2 The Wave Equation

In this section we consider initial-boundary value problems of the form

$$\begin{aligned} u_{tt} = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L, \end{aligned} \tag{12.2.1}$$

where a is a constant and f and g are given functions of x.

The partial differential equation $u_{tt} = a^2 u_{xx}$ is called the wave equation. It is necessary to specify both f and g because the wave equation is a second order equation in t for each fixed x.

This equation and its generalizations

$$u_{tt} = a^2(u_{xx} + u_{yy}) \quad \text{and} \quad u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz})$$

to two and three space dimensions have important applications to the the propagation of electromagnetic, sonic, and water waves.

The Vibrating String

We motivate the study of the wave equation by considering its application to the vibrations of a string – such as a violin string – tightly stretched in equilibrium along the x-axis in the xu-plane and tied to the points $(0,0)$ and $(L,0)$ (Figure 12.2.1).

Figure 12.2.1. *A stretched string*

If the string is plucked in the vertical direction and released at time $t = 0$, it will oscillate in the xu-plane. Let $u(x,t)$ denote the displacement of the point on the string above (or below) the abscissa x at time t.

We'll show that it's reasonable to assume that u satisfies the wave equation under the following assumptions:

1. The mass density (mass per unit length) ρ of the string is constant throughout the string.
2. The tension T induced by tightly stretching the string along the x-axis is so great that all other forces, such as gravity and air resistance, can be neglected.
3. The tension at any point on the string acts along the tangent to the string at that point, and the magnitude of its horizontal component is always equal to T, the tension in the string in equilibrium.
4. The slope of the string at every point remains sufficiently small so that we can make the approximation

$$\sqrt{1+u_x^2} \approx 1. \tag{12.2.2}$$

Figure 12.2.2 shows a segment of the displaced string at a time $t > 0$. (Don't think that the figure is necessarily inconsistent with Assumption 4; we exaggerated the slope for clarity.)

Figure 12.2.2. *A segment of the displaced string*

The vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ are the forces due to tension, acting along the tangents to the segment at its endpoints. From Newton's second law of motion, $\mathbf{T}_1 - \mathbf{T}_2$ is equal to the mass times the acceleration of the center of mass of the segment. The horizontal and vertical components of $\mathbf{T}_1 - \mathbf{T}_2$ are

$$|\mathbf{T}_2|\cos\theta_2 - |\mathbf{T}_1|\cos\theta_1 \quad \text{and} \quad |\mathbf{T}_2|\sin\theta_2 - |\mathbf{T}_1|\sin\theta_1,$$

respectively. Since

$$|\mathbf{T}_2|\cos\theta_2 = |\mathbf{T}_1|\cos\theta_1 = T \tag{12.2.3}$$

by assumption, the net horizontal force is zero, so there's no horizontal acceleration. Since the initial horizontal velocity is zero, there's no horizontal motion.

Applying Newton's second law of motion in the vertical direction yields

$$|\mathbf{T}_2|\sin\theta_2 - |\mathbf{T}_1|\sin\theta_1 = \rho\Delta s\, u_{tt}(\overline{x}, t), \tag{12.2.4}$$

where Δs is the length of the segment and $\overline{x}$ is the abscissa of the center of mass; hence,

$$x < \overline{x} < x + \Delta x.$$

From calculus, we know that

$$\Delta s = \int_x^{x+\Delta x} \sqrt{1+u_x^2(\sigma,t)}\, d\sigma;$$

however, because of (12.2.2), we make the approximation

$$\Delta s \approx \int_x^{x+\Delta x} 1\, d\sigma = \Delta x,$$

so (12.2.4) becomes

$$|\mathbf{T}_2|\sin\theta_2 - |\mathbf{T}_1|\sin\theta_1 = \rho\Delta x\, u_{tt}(\overline{x},t).$$

Therefore

$$\frac{|\mathbf{T}_2|\sin\theta_2 - |\mathbf{T}_1|\sin\theta_1}{\Delta x} = \rho u_{tt}(\overline{x},t).$$

Recalling (12.2.3), we divide by T to obtain

$$\frac{\tan\theta_2 - \tan\theta_1}{\Delta x} = \frac{\rho}{T} u_{tt}(\overline{x},t). \tag{12.2.5}$$

Since $\tan\theta_1 = u_x(x,t)$ and $\tan\theta_2 = u_x(x+\Delta x,t)$, (12.2.5) is equivalent to

$$\frac{u_x(x+\Delta x) - u_x(x,t)}{\Delta x} = \frac{\rho}{T} u_{tt}(\overline{x},t).$$

Letting $\Delta x \to 0$ yields

$$u_{xx}(x,t) = \frac{\rho}{T} u_{tt}(x,t),$$

which we rewrite as $u_{tt} = a^2 u_{xx}$, with $a^2 = T/\rho$.

The Formal Solution

As in Section 12.1, we use separation of variables to obtain a suitable definition for the formal solution of (12.2.1). We begin by looking for functions of the form $v(x,t) = X(x)T(t)$ that are not identically zero and satisfy

$$v_{tt} = a^2 v_{xx}, \quad v(0,t) = 0, \quad v(L,t) = 0$$

for all (x,t). Since

$$v_{tt} = XT'' \quad \text{and} \quad v_{xx} = X''T,$$

$v_{tt} = a^2 v_{xx}$ if and only if

$$XT'' = a^2 X''T,$$

which we rewrite as

$$\frac{T''}{a^2T} = \frac{X''}{X}.$$

For this to hold for all (x,t), the two sides must equal the same constant; thus,

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda,$$

which is equivalent to

$$X'' + \lambda X = 0$$

and

$$T'' + a^2\lambda T = 0. \tag{12.2.6}$$

Since $v(0,t) = X(0)T(t) = 0$ and $v(L,t) = X(L)T(t) = 0$ and we don't want T to be identically zero, $X(0) = 0$ and $X(L) = 0$. Therefore λ must be an eigenvalue of

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0, \tag{12.2.7}$$

and X must be a λ-eigenfunction. From Theorem 11.1.2, the eigenvalues of (12.2.7) are $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$X_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots .$$

Substituting $\lambda = n^2\pi^2/L^2$ into (12.2.6) yields

$$T'' + (n^2\pi^2 a^2/L^2)T = 0,$$

which has the general solution

$$T_n = \alpha_n \cos\frac{n\pi at}{L} + \frac{\beta_n L}{n\pi a}\sin\frac{n\pi at}{L},$$

where α_n and β_n are constants. Now let

$$v_n(x,t) = X_n(x)T_n(t) = \left(\alpha_n \cos\frac{n\pi at}{L} + \frac{\beta_n L}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L}.$$

Then

$$\frac{\partial v_n}{\partial t}(x,t) = \left(-\frac{n\pi a}{L}\alpha_n \sin\frac{n\pi at}{L} + \beta_n \cos\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L},$$

so

$$v_n(x,0) = \alpha_n \sin\frac{n\pi x}{L} \quad \text{and} \quad \frac{\partial v_n}{\partial t}(x,0) = \beta_n \sin\frac{n\pi x}{L}.$$

Therefore v_n satisfies (12.2.1) with $f(x) = \alpha_n \sin n\pi x/L$ and $g(x) = \beta_n \cos n\pi x/L$. More generally, if α_1, α_2, …, α_m and β_1, β_2,…, β_m are constants and

$$u_m(x,t) = \sum_{n=1}^{m}\left(\alpha_n \cos\frac{n\pi at}{L} + \frac{\beta_n L}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L},$$

then u_m satisfies (12.2.1) with

$$f(x) = \sum_{n=1}^{m}\alpha_n \sin\frac{n\pi x}{L} \quad \text{and} \quad g(x) = \sum_{n=1}^{m}\beta_n \sin\frac{n\pi x}{L}.$$

This motivates the next definition.

Definition 12.2.1

If f and g are piecewise smooth of $[0,L]$, then the formal solution of (12.2.1) is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\alpha_n \cos\frac{n\pi at}{L} + \frac{\beta_n L}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L}, \tag{12.2.8}$$

where

$$S_f(x) = \sum_{n=1}^{\infty}\alpha_n \sin\frac{n\pi x}{L} \quad \text{and} \quad S_g(x) = \sum_{n=1}^{\infty}\beta_n \sin\frac{n\pi x}{L}$$

are the Fourier sine series of f and g on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{and} \quad \beta_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Since there are no convergence-producing factors in (12.2.8) like the negative exponentials in t that appear in formal solutions of initial-boundary value problems for the heat equation, it isn't obvious that (12.2.8) even converges for any values of x and t, let alone that it can be differentiated term by term to show that $u_{tt} = a^2 u_{xx}$. However, the next theorem guarantees that the series converges not only for $0 \le x \le L$ and $t \ge 0$, but for $-\infty < x < \infty$ and $-\infty < t < \infty$.

Theorem 12.2.2

If f and g are pieceswise smooth on $[0, L]$, then u in $(???)$ converges for all (x, t), and can be written as

$$u(x,t) = \frac{1}{2}[S_f(x+at) + S_f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} S_g(\tau)\,d\tau. \tag{12.2.9}$$

Proof Setting $A = n\pi x/L$ and $B = n\pi at/L$ in the identities

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

and

$$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$$

yields

$$\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}=\frac{1}{2}\left[\sin\frac{n\pi(x+at)}{L}+\sin\frac{n\pi(x-at)}{L}\right] \tag{12.2.10}$$

and

$$\begin{aligned}\backslash\text{dst}\sin\tfrac{n\pi at}{L}\sin\tfrac{n\pi x}{L} &= -\backslash\text{dst}\tfrac{1}{2}\left[\cos\tfrac{n\pi(x+at)}{L}-\cos\tfrac{n\pi(x-at)}{L}\right]\\ &= \backslash\text{dst}\tfrac{n\pi}{2L}\int_{x-at}^{x+at}\sin\tfrac{n\pi\tau}{L}\,d\tau.\end{aligned} \tag{12.2.11}$$

From (12.2.10),

$$\begin{aligned}\backslash\text{dst}\textstyle\sum_{n=1}^{\infty}\alpha_n\cos\tfrac{n\pi at}{L}\sin\tfrac{n\pi x}{L} &= \backslash\text{dst}\tfrac{1}{2}\textstyle\sum_{n=1}^{\infty}\alpha_n\left(\sin\tfrac{n\pi(x+at)}{L}+\sin\tfrac{n\pi(x-at)}{L}\right)\\ &= \backslash\text{dst}\tfrac{1}{2}[S_f(x+at)+S_f(x-at)].\end{aligned} \tag{12.2.12}$$

Since it can be shown that a Fourier sine series can be integrated term by term between any two limits, (12.2.11) implies that

$$\begin{aligned}\backslash\text{dst}\textstyle\sum_{n=1}^{\infty}\tfrac{\beta_nL}{n\pi a}\sin\tfrac{n\pi at}{L}\sin\tfrac{n\pi x}{L} &= \backslash\text{dst}\tfrac{1}{2a}\textstyle\sum_{n=1}^{\infty}\beta_n\int_{x-at}^{x+at}\sin\tfrac{n\pi\tau}{L}\,d\tau\\ &= \backslash\text{dst}\tfrac{1}{2a}\int_{x-at}^{x+at}\left(\textstyle\sum_{n=1}^{\infty}\beta_n\sin\tfrac{n\pi\tau}{L}\right)d\tau\\ &= \backslash\text{dst}\tfrac{1}{2a}\int_{x-at}^{x+at}S_g(\tau)\,d\tau.\end{aligned}$$

This and (12.2.12) imply (12.2.9), which completes the proof.

As we'll see below, if S_g is differentiable and S_f is twice differentiable on $(-\infty,\infty)$, then (12.2.9) satisfies $u_{tt}=a^2u_{xx}$ for all (x,t). We need the next theorem to formulate conditions on f and g such that S_f and S_g to have these properties.

Theorem 12.2.3

Suppose h is differentiable on $[0,L]$; that is, $h'(x)$ exists for $0<x<L$, and the one-sided derivatives

$$h'_+(0)=\lim_{x\to 0+}\frac{h(x)-h(0)}{x}\quad\text{and}\quad h'_-(L)=\lim_{x\to L-}\frac{h(x)-h(L)}{x-L}$$

both exist.

(a) Let p be the odd periodic extension of h to $(-\infty,\infty)$; that is,

$$p(x)=\left\{\begin{array}{cc} h(x), & 0\le x\le L,\\ -h(-x), & -L<x<0,\end{array}\right. \quad \text{and } p(x+2L)=p(x), \quad -\infty<x<\infty.$$

Then p is differentiable on $(-\infty,\infty)$ if and only if

$$h(0)=h(L)=0. \tag{12.2.13}$$

(b) Let q be the even periodic extension of h to $(-\infty,\infty)$; that is,

$$q(x)=\left\{\begin{array}{cc} h(x), & 0\le x\le L,\\ h(-x), & -L<x<0,\end{array}\right. \quad \text{and } q(x+2L)=q(x), \quad -\infty<x<\infty.$$

Then q is differentiable on $(-\infty,\infty)$ if and only if

$$h_+'(0)=h_-'(L)=0. \tag{12.2.14}$$

Proof Throughout this proof, k denotes an integer. Since f is differentiable on the open interval $(0,L)$, both p and q are differentiable on every open interval $((k-1)L,kL)$. Thus, we need only to determine whether p and q are differentiable at $x=kL$ for every k.

(a) From Figure 12.2.3, p is discontinuous at $x=2kL$ if $h(0)\ne 0$ and discontinuous at $x=(2k-1)L$ if $h(L)\ne 0$. Therefore p is not differentiable on $(-\infty,\infty)$ unless $h(0)=h(L)=0$. From Figure 12.2.4, if $h(0)=h(L)=0$, then

$$p'(2kL)=h_+'(0) \quad \text{and} \quad p'((2k-1)L)=h_-'(L)$$

for every k; therefore, p is differentiable on $(-\infty,\infty)$.

Figure 12.2.3. *The odd extension of a function that does not satisfy (12.2.13)*

Figure 12.2.4. *The odd extension of a function that satisfies (12.2.13)*

(b) From Figure 12.2.5,

$$q_-'(2kL)=-h_+'(0) \quad \text{and} \quad q_+'(2kL)=h_+'(0),$$

so q is differentiable at $x=2kL$ if and only if $h'_+(0)=0$. Also,

$$q'_-((2k-1)L)=h'_-(L) \quad \text{and} \quad q'_+((2k-1)L)=-h'_-(L),$$

so q is differentiable at $x=(2k-1)L$ if and only if $h'_-(L)=0$. Therefore q is differentiable on $(-\infty,\infty)$ if and only if $h'_+(0)=h'_-(L)=0$, as in Figure 12.2.6. This completes the proof.

Figure 12.2.5. *The even extension of a function that doesn't satisfy (12.2.14)*

Figure 12.2.6. *The even extension of a function that satisfies (12.2.14)*

Theorem 12.2.4

The formal solution of (???) is an actual solution if g is differentiable on $[0,L]$ and

$$g(0)=g(L)=0, \tag{12.2.15}$$

while f is twice differentiable on $[0,L]$ and

$$f(0)=f(L)=0 \tag{12.2.16}$$

and

$$f''_+(0)=f''_-(L)=0. \tag{12.2.17}$$

Proof We first show that S_g is differentiable and S_f is twice differentiable on $(-\infty,\infty)$. We'll then differentiate (12.2.9) twice with respect to x and t and verify that (12.2.9) is an actual solution of (12.2.1).

Since f and g are continuous on $(0,L)$, Theorem 11.3.2 implies that $S_f(x)=f(x)$ and $S_g(x)=g(x)$ on $[0,L]$. Therefore S_f and S_g are the odd periodic extensions of f and g. Since f and g are differentiable on $[0,L]$, (12.2.15), (12.2.16), and Theorem 12.2.3**(a)** imply that S_f and S_g are differentiable on $(-\infty,\infty)$.

Since $S'_f(x)=f'(x)$ on $[0,L]$ (one-sided derivatives at the endpoints), and S'_f is even (the derivative of an odd function is even), S'_f is the even periodic extension of f'. By assumption, f' is differentiable on $[0,L]$. Because of (12.2.17), Theorem 12.2.3**(b)** with $h=f'$ and $q=S'_f$ implies that S''_f exists on $(-\infty,\infty)$.

Now we can differentiate (12.2.9) twice with respect to x and t:

$$u_x(x,t) = \frac{1}{2}[S'_f(x+at) + S'_f(x-at)] + \frac{1}{2a}[S_g(x+at) - S_g(x-at)],$$

$$u_{xx}(x,t) = \frac{1}{2}[S''_f(x+at) + S''_f(x-at)] + \frac{1}{2a}[S'_g(x+at) - S'_g(x-at)], \tag{12.2.18}$$

$$u_t(x,t) = \frac{a}{2}[S'_f(x+at) - S'_f(x-at)] + \frac{1}{2}[S_g(x+at) + S_g(x-at)], \tag{12.2.19}$$

and

$$u_{tt}(x,t) = \frac{a^2}{2}[S''_f(x+at) - S''_f(x-at)] + \frac{a}{2}[S'_g(x+at) - S'_g(x-at)]. \tag{12.2.20}$$

Comparing (12.2.18) and (12.2.20) shows that $u_{tt}(x,t) = a^2 u_{xx}(x,t)$ for all (x,t).

From (12.2.8), $u(0,t) = u(L,t) = 0$ for all t. From (12.2.9), $u(x,0) = S_f(x)$ for all x, and therefore, in particular,

$$u(x,0) = f(x), \quad 0 \le x < L.$$

From (12.2.19), $u_t(x,0) = S_g(x)$ for all x, and therefore, in particular,

$$u_t(x,0) = g(x), \quad 0 \le x < L.$$

Therefore u is an actual solution of (12.2.1). This completes the proof.

Eqn (12.2.9) is called d'Alembert's solution (http://www-history.mcs.st-and.ac.uk/Mathematicians/D'Alembert.html) of (12.2.1). Although d'Alembert's solution was useful for proving Theorem 12.2.4 and is very useful in a slightly different context (Exercises 63-68), (12.2.8) is preferable for computational purposes.

Example 12.2.1

Solve (12.2.1) with

$$f(x) = x(x^3 - 2Lx^2 + L^2) \quad \text{and} \quad g(x) = x(L-x).$$

Solution We leave it to you to verify that f and g satisfy the assumptions of Theorem 12.2.4.

From Exercise 11.3. 39,

$$S_f(x) = \frac{96L^4}{\pi^5} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5} \sin \frac{(2n-1)\pi x}{L}.$$

From Exercise 11.3. 36,

$$S_g(x) = \frac{8L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{L}.$$

From (12.2.8),

$$\begin{aligned} u(x,t) \quad &= \quad \backslash\mathrm{dst}\, \tfrac{96L^4}{\pi^5} \textstyle\sum_{n=1}^{\infty} \tfrac{1}{(2n-1)^5} \cos \tfrac{(2n-1)\pi at}{L} \sin \tfrac{(2n-1)\pi x}{L} \\ & \quad\quad + \backslash\mathrm{dst}\, \tfrac{8L^3}{a\pi^4} \textstyle\sum_{n=1}^{\infty} \tfrac{1}{(2n-1)^4} \sin \tfrac{(2n-1)\pi at}{L} \sin \tfrac{(2n-1)\pi x}{L}. \end{aligned}$$

Theorem 12.1.2 implies that u_{xx} and u_{tt} can be obtained by term by term differentiation, for all (x,t), so $u_{tt} = a^2 u_{xx}$ for all (x,t) (Exercise 62). Moreover, Theorem 11.3.2 implies that $S_f(x) = f(x)$ and $S_g(x) = g(x)$ if $0 \le x \le L$. Therefore $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ if $0 \le x \le L$. Hence, u is an actual solution of the initial-boundary value problem.

Remark

In solving a specific initial-boundary value problem (12.2.1), it's convenient to solve the problem with $g \equiv 0$, then with $f \equiv 0$, and add the solutions to obtain the solution of the given problem. Because of this, either $f \equiv 0$ or $g \equiv 0$ in all the specific initial-boundary value problems in the exercises.

The Plucked String

If f and g don't satisfy the assumptions of Theorem 12.2.4, then (12.2.8) isn't an actual solution of (12.2.1) in fact, it can be shown that (12.2.1) doesn't have an actual solution in this case. Nevertheless, u is defined for all (x,t), and we can see from (12.2.18) and (12.2.20) that $u_{tt}(x,t) = a^2 u_{xx}(x,t)$ for all (x,t) such that $S_f''(x \pm at)$ and $S_g'(x \pm at)$ exist. Moreover, u may still provide a useful approximation to the vibration of the string; a laboratory experiment can confirm or deny this.

We'll now consider the initial-boundary value problem (12.2.1) with

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{L}{2}, \\ L - x, & \frac{L}{2} \le x \le L \end{cases} \tag{12.2.21}$$

and $g \equiv 0$. Since f isn't differentiable at $x = L/2$, it does'nt satisfy the assumptions of Theorem 12.2.4, so the formal solution of (12.2.1) can't be an actual solution. Nevertheless, it's instructive to investigate the properties of the formal solution.

The graph of f is shown in Figure 12.2.7. Intuitively, we are plucking the string by half its length at the middle. You're right if you think this is an extraordinarily large displacement; however, we could remove this objection by multiplying the function in Figure 12.2.7 by a small constant. Since this would just multiply the formal solution by the same constant, we'll leave f as we've defined it. Similar comments apply to the exercises.

Figure 12.2.7. *Graph of (12.2.21)*

From Exercise 11.3. 15, the Fourier sine series of f on $[0, L]$ is

$$S_f(x) = \dst\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{L},$$

which converges to f for all x in $[0, L]$, by Theorem 11.3.2. Therefore

$$u(x,t) = \dst\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi at}{L}\sin\frac{(2n-1)\pi x}{L}. \tag{12.2.22}$$

This series converges absolutely for all (x, t) by the comparison test, since the series

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

converges. Moreover, (12.2.22) satisfies the boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \quad 0 \le x \le L.$$

However, we can't justify differentiating (12.2.22) term by term even once, and formally differentiating it twice term by term produces a series that diverges for all (x, t). (Verify.). Therefore we use d'Alembert's form

$$u(x,t) = \frac{1}{2}[S_f(x+at) + S_f(x-at)] \tag{12.2.23}$$

for u to study its derivatives. Figure 12.2.8 shows the graph of S_f, which is the odd periodic extension of f. You can see from the graph that S_f is differentiable at x (and $S_f'(x) = \pm 1$) if and only if x isn't an odd multiple of $L/2$.

Figure 12.2.8. *The odd periodic extension of (12.2.21)*

Figure 12.2.9. *Graphs of $y = S_f(x-at)$ (dashed) and $y = S_f(x-at)$ (solid), with f as in (12.2.21)*

In Figure 12.2.9 the dashed and solid curves are the graphs of $y = S_f(x-at)$ and $y = S_f(x+at)$ respectively, for a fixed value of t. As t increases the dashed curve moves to the right and the solid curve moves to the left. For this reason, we say that the functions $u_1(x,t) = S_f(x+at)$ and $u_2(x,t) = S_f(x-at)$ are traveling waves. Note that u_1 satisfies the wave equation at (x,t) if $x+at$ isn't an odd multiple of $L/2$ and u_2 satisfies the wave equation at (x,t) if $x-at$ isn't an odd multiple of $L/2$. Therefore (12.2.23) (or, equivalently, (12.2.22)) satisfies $u_{tt}(x,t) = a^2 u_{xx}(x,t) = 0$ for all (x,t) such that neither $x-at$ nor $x+at$ is an odd multiple of $L/2$.

We conclude by finding an explicit formula for $u(x,t)$ under the assumption that

$$0 \le x \le L \quad \text{and} \quad 0 \le t \le L/2a. \tag{12.2.24}$$

To see how this formula can be used to compute $u(x,t)$ for $0 \le x \le L$ and arbitrary t, we refer you to Exercise 16.

From Figure 12.2.10,

$$S_f(x-at) = \begin{cases} x-at, & 0 \le x \le \frac{L}{2}+at, \\ L-x+at, & \frac{L}{2}+at \le x \le L \end{cases}$$

and

$$S_f(x+at) = \begin{cases} x+at, & 0 \le x \le \frac{L}{2}-at, \\ L-x-at, & \frac{L}{2}-at \le x \le L \end{cases}$$

if (x,t) satisfies (12.2.24).

Therefore, from (12.2.23),

$$u(x,t)=\begin{cases} x, & 0\le x\le \frac{L}{2}-at,\\ \frac{L}{2}-at, & \frac{L}{2}-at\le x\le \frac{L}{2}+at,\\ L-x, & \frac{L}{2}-at\le x\le L\end{cases}$$

if (x,t) satisfies (12.2.24). Figure 12.2.11 is the graph of this function on $[0,L]$ for a fixed t in $(0,L/2a)$.

Figure 12.2.10. *The part of the graph from Figure 12.2.9 on* $[0,L]$

Figure 12.2.11. *The graph of (12.2.23) on* $[0,L]$ *for a fixed* t *in* $(0,L/2a)$

Using Technology

Although the formal solution

$$u(x,t)=\sum_{n=1}^{\infty}\left(\alpha_n\cos\frac{n\pi at}{L}+\frac{\beta_n L}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L}$$

of (12.2.1) is defined for all (x,t), we're mainly interested in its behavior for $0\le x\le L$ and $t\ge 0$. In fact, it's sufficient to consider only values of t in the interval $0\le t<2L/a$, since

$$u(x,t+2kL/a)=u(x,t)$$

for all (x,t) if k is an integer. (Verify.)

You can create an animation of the motion of the string by performing the following numerical experiment.

Let m and k be positive integers. Let

$$t_j=\frac{2Lj}{ka},\quad j=0,1,\dots k;$$

thus, t_0, t_1, … t_k are equally spaced points in $[0,2L/a]$. For each $j=0$, 1 ,2, … k, graph the partial sum

$$u_m(x,t_j)=\sum_{n=1}^{m}\left(\alpha_n\cos\frac{n\pi at_j}{L}+\frac{\beta_n L}{n\pi a}\sin\frac{n\pi at_j}{L}\right)\sin\frac{n\pi x}{L}$$

on $[0, L]$ as a function of x. Write your program so that each graph remains displayed on the monitor for a short time, and is then deleted and replaced by the next. Repeat this procedure for various values of m and k.

We suggest that you perform experiments of this kind in the exercises marked C , without other specific instructions. (These exercises were chosen arbitrarily; the experiment is worthwhile in all the exercises dealing with specific initial-boundary value problems.) In some of the exercises the formal solutions have other forms, defined in Exercises 17, 34, and 49; however, the idea of the experiment is the same.

12.2 Exercises

*In Exercises 1-15 solve the initial-boundary value problem. In some of these exercises, Theorem 11.3.5**(b)** or Exercise* $11.3.$ *35 will simplify the computation of the coefficients in the Fourier sine series.*

1. $u_{tt}=9u_{xx},\quad 0<x<1,\quad t>0,$
$u(0,t)=0,\quad u(1,t)=0,\quad t>0,$
$u(x,0)=0,\quad u_t(x,0)=\begin{cases} x, & 0\le x\le \frac{1}{2},\\ 1-x, & \frac{1}{2}\le x\le 1\end{cases},\quad 0\le x\le 1$

Show answer

2. $u_{tt}=9u_{xx},\quad 0<x<1,\quad t>0,$
$u(0,t)=0,\quad u(1,t)=0,\quad t>0,$
$u(x,0)=x(1-x),\quad u_t(x,0)=0,\quad 0\le x\le 1$

Show answer

3. $u_{tt}=7u_{xx},\quad 0<x<1,\quad t>0,$
$u(0,t)=0,\quad u(1,t)=0,\quad t>0,$
$u(x,0)=x^2(1-x),\quad u_t(x,0)=0,\quad 0\le x\le 1$

Show answer

4. C $u_{tt}=9u_{xx},\quad 0<x<1,\quad t>0,$
$u(0,t)=0,\quad u(1,t)=0,\quad t>0,$
$u(x,0)=0,\quad u_t(x,0)=x(1-x),\quad 0\le x\le 1$

Show answer

5. $u_{tt}=7u_{xx},\quad 0<x<1,\quad t>0,$
$u(0,t)=0,\quad u(1,t)=0,\quad t>0,$
$u(x,0)=0\quad u_t(x,0)=x^2(1-x),\quad 0\le x\le 1$

Show answer

6. $u_{tt}=64u_{xx},\quad 0<x<3,\quad t>0,$
$u(0,t)=0,\quad u(3,t)=0,\quad t>0,$
$u(x,0)=x(x^2-9),\quad u_t(x,0)=0,\quad 0\le x\le 3$

Show answer

7. $u_{tt}=4u_{xx},\quad 0<x<1,\quad t>0,$
$u(0,t)=0,\quad u(1,t)=0,\quad t>0,$
$u(x,0)=x(x^3-2x^2+1),\quad u_t(x,0)=0,\quad 0\le x\le 1$

Show answer

8. C $u_{tt}=64u_{xx},\quad 0<x<3,\quad t>0,$
$u(0,t)=0,\quad u(3,t)=0,\quad t>0,$
$u(x,0)=0,\quad u_t(x,0)=x(x^2-9),\quad 0\le x\le 3$

Show answer

9. $u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x(x^3 - 2x^2 + 1), \quad 0 \le x \le 1$

Show answer

10. $u_{tt} = 5u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x\sin x, \quad u_t(x,0) = 0, \quad 0 \le x \le \pi$

Show answer

11. $u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(3x^4 - 5x^3 + 2), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

12. C $u_{tt} = 5u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x\sin x, \quad 0 \le x \le \pi$

Show answer

13. $u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x(3x^4 - 5x^3 + 2), \quad 0 \le x \le 1$

Show answer

14. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(3x^4 - 10x^2 + 7), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

15. C $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0 \quad u_t(x,0) = x(3x^4 - 10x^2 + 7), \quad 0 \le x \le 1$

Show answer

16. We saw that the displacement of the plucked string is, on the one hand,

$$u(x,t) = \dst\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi at}{L}\sin\frac{(2n-1)\pi x}{L},\ 0\le x\le L,\ t\ge 0, \tag{A}$$

and, on the other hand,

$$u(x,\tau) = \begin{cases} x, & 0\le x\le \frac{L}{2}-a\tau, \\ \frac{L}{2}-a\tau, & \frac{L}{2}-a\tau\le x\le\frac{L}{2}+a\tau, \\ L-x, & \frac{L}{2}-a\tau\le x\le L. \end{cases} \tag{B}$$

if $0\le\tau\le L/2a$. The first objective of this exercise is to show that (B) can be used to compute $u(x,t)$ for $0\le x\le L$ and all $t>0$.

a. Show that if $t>0$, there's a nonnegative integer m such that either

$$\textbf{(i)}\quad t=\frac{mL}{a}+\tau \quad\text{or}\quad \textbf{(ii)}\quad t=\frac{(m+1)L}{a}-\tau,$$

where $0\le\tau\le L/2a$.

b. Use (A) to show that $u(x,t)=(-1)^m u(x,\tau)$ if **(i)** holds, while $u(x,t)=(-1)^{m+1}u(x,\tau)$ if **(ii)** holds.

c. L Perform the following experiment for specific values of L and a and various values of m and k: Let

$$t_j=\frac{Lj}{2ka},\quad j=0,1,\dots k;$$

thus, t_0, t_1, …, t_k are equally spaced points in $[0,L/2a]$. For each $j=0$, 1 , 2,…, k, graph the mth partial sum of (A) and $u(x,t_j)$ computed from (B) on the same axis. Create an animation, as described in the remarks on using technology at the end of the section.

17. If a string vibrates with the end at $x = 0$ free to move in a frictionless vertical track and the end at $x = L$ fixed, then the initial-boundary value problem for its displacement takes the form

$$\begin{gathered} u_{tt} = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u_x(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L. \end{gathered} \tag{A}$$

Justify defining the formal solution of (A) to be

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{(2n-1)\pi at}{2L} + \frac{2L\beta_n}{(2n-1)\pi a} \sin \frac{(2n-1)\pi at}{2L} \right) \cos \frac{(2n-1)\pi x}{2L},$$

where

$$C_{Mf}(x) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{(2n-1)\pi x}{2L} \quad \text{and} \quad C_{Mg}(x) = \sum_{n=1}^{\infty} \beta_n \cos \frac{(2n-1)\pi x}{2L}$$

are the mixed Fourier cosine series of f and g on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L}\, dx \quad \text{and} \quad \beta_n = \frac{2}{L} \int_0^L g(x) \cos \frac{(2n-1)\pi x}{2L}\, dx.$$

*In Exercises 18-31, use Exercise 17 to solve the initial-boundary value problem. In some of these exercises Theorem 11.3.5**(c)** or Exercise* $11.3.$ *42**(b)** will simplify the computation of the coefficients in the mixed Fourier cosine series.*

18. $u_{tt} = 9u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$u(x,0) = 4 - x^2, \quad u_t(x,0) = 0, \quad 0 \le x \le 2$

Show answer

19. $u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^2(1 - x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

20. $u_{tt} = 9u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = 4 - x^2, \quad 0 \le x \le 2$

Show answer

21. $u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x^2(1 - x), \quad 0 \le x \le 1$

Show answer

22. C $u_{tt} = 5u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 2x^3 + 3x^2 - 5, \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

23. $u_{tt} = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = \pi^3 - x^3, \quad u_t(x,0) = 0, \quad 0 \le x \le \pi$

Show answer

24. $u_{tt} = 5u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = 2x^3 + 3x^2 - 5, \quad 0 \le x \le 1$

Show answer

25. C $u_{tt} = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = \pi^3 - x^3, \quad 0 \le x \le \pi$

Show answer

26. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^4 - 2x^3 + 1, \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

27. $u_{tt} = 7u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 4x^3 + 3x^2 - 7, \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

28. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x^4 - 2x^3 + 1, \quad 0 \le x \le 1$

Show answer

29. C $u_{tt} = 7u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = 4x^3 + 3x^2 - 7, \quad 0 \le x \le 1$

Show answer

30. $u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^4 - 4x^3 + 6x^2 - 3, \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

31. $u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x^4 - 4x^3 + 6x^2 - 3, \quad 0 \le x \le 1$

Show answer

32. Adapt the proof of Theorem 12.2.2 to find d'Alembert's solution of the initial-boundary value problem in Exercise 17.

Show answer

33. Use the result of Exercise 32 to show that the formal solution of the initial-boundary value problem in Exercise 17 is an actual solution if g is differentiable and f is twice differentiable on $[0, L]$ and

$$g'_{+}(0) = g(L) = f'_{+}(0) = f(L) = f''_{-}(L) = 0.$$

Hint: See Exercise $11.3.$ *57, and apply Theorem 12.2.3 with* L *replaced by* $2L$*.*

34. Justify defining the formal solution of the initial-boundary value problem

$$\begin{gathered} u_{tt} = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L \end{gathered}$$

to be

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{(2n-1)\pi a t}{2L} + \frac{2L\beta_n}{(2n-1)\pi a} \sin \frac{(2n-1)\pi a t}{2L} \right) \sin \frac{(2n-1)\pi x}{2L},$$

where

$$S_{Mf}(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L} \quad \text{and} \quad S_{Mg}(x) = \sum_{n=1}^{\infty} \beta_n \sin \frac{(2n-1)\pi x}{2L}$$

are the mixed Fourier sine series of f and g on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx \quad \text{and} \quad \beta_n = \frac{2}{L} \int_0^L g(x) \sin \frac{(2n-1)\pi x}{2L}\, dx.$$

*In Exercises 35-46 use Exercise 34 to solve the initial-boundary value problem. In some of these exercises Theorem 11.3.5**(d)** or Exercise* $11.3.$ *50**(b)** will simplify the computation of the coefficients in the mixed Fourier sine series.*

35. $u_{tt} = 64u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x(2\pi - x), \quad u_t(x,0) = 0, \quad 0 \le x \le \pi$

Show answer

36. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^2(3 - 2x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

37. $u_{tt} = 64u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x(2\pi - x), \quad 0 \le x \le \pi$

Show answer

38. C $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x^2(3 - 2x), \quad 0 \le x \le 1$

Show answer

39. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = (x-1)^3 + 1, \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

40. $u_{tt} = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x(x^2 - 3\pi^2), \quad u_t(x,0) = 0, \quad 0 \le x \le \pi$

Show answer

41. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = (x-1)^3 + 1, \quad 0 \le x \le 1$

Show answer

42. $u_{tt} = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x(x^2 - 3\pi^2), \quad 0 \le x \le \pi$

Show answer

43. $u_{tt} = 5u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^3(3x-4), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

44. C $u_{tt} = 16u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(x^3 - 2x^2 + 2), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

Show answer

45. $u_{tt} = 5u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x^3(3x-4), \quad 0 \le x \le 1$

Show answer

46. C $u_{tt} = 16u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x(x^3 - 2x^2 + 2), \quad 0 \le x \le 1$

Show answer

47. Adapt the proof of Theorem 12.2.2 to find d'Alembert's solution of the initial-boundary value problem in Exercise 34.

Show answer

48. Use the result of Exercise 47 to show that the formal solution of the initial-boundary value problem in Exercise 34 is an actual solution if g is differentiable and f is twice differentiable on $[0, L]$ and

$$f(0) = f'_-(L) = g(0) = g'_-(L) = f''_+(0) = 0.$$

Hint: See Exercise 11.3*. 58 and apply Theorem 12.2.3 with* L *replaced by* $2L$*.*

49. Justify defining the formal solution of the initial-boundary value problem

$$\begin{gathered} u_{tt} = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L. \end{gathered}$$

to be

$$u(x,t) = \alpha_0 + \beta_0 t + \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{n\pi a t}{L} + \frac{L\beta_n}{n\pi a} \sin \frac{n\pi a t}{L} \right) \cos \frac{n\pi x}{L},$$

where

$$C_f(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{L} \quad \text{and} \quad C_g(x) = \beta_0 + \sum_{n=1}^{\infty} \beta_n \cos \frac{n\pi x}{L}$$

are the Fourier cosine series of f and g on $[0,L]$; that is,

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\,dx, \quad \beta_0 = \frac{1}{L} \int_0^L g(x)\,dx,$$

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx, \quad \text{and} \quad \beta_n = \frac{2}{L} \int_0^L g(x) \cos \frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \dots.$$

*In Exercises 50-59 use Exercise 49 to solve the initial-boundary value problem. In some of these exercises Theorem 11.3.5**(a)** will simplify the computation of the coefficients in the Fourier cosine series.*

50. $u_{tt} = 5u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(2, t) = 0, \quad t > 0,$
$u(x, 0) = 2x^2(3 - x), \quad u_t(x, 0) = 0, \quad 0 \le x \le 2$

Show answer

51. $u_{tt} = 5u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(2, t) = 0, \quad t > 0,$
$u(x, 0) = 0, \quad u_t(x, 0) = 2x^2(3 - x), \quad 0 \le x \le 2$

Show answer

52. $u_{tt} = 4u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \quad t > 0,$
$u(x, 0) = x^3(3x - 4\pi), \quad u_t(x, 0) = 0, \quad 0 \le x \le \pi$

Show answer

53. $u_{tt} = 7u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t > 0,$
$u(x, 0) = 3x^2(x^2 - 2), \quad u_t(x, 0) = 0, \quad 0 \le x \le 1$

Show answer

54. C $u_{tt} = 4u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \quad t > 0,$
$u(x, 0) = 0, \quad u_t(x, 0) = x^3(3x - 4\pi), \quad 0 \le x \le \pi$

Show answer

55. $u_{tt} = 7u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t > 0,$
$u(x, 0) = 0, \quad u_t(x, 0) = 3x^2(x^2 - 2), \quad 0 \le x \le 1$

Show answer

56. $u_{tt} = 16u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \quad t > 0,$
$u(x, 0) = x^2(x - \pi)^2, \quad u_t(x, 0) = 0, \quad 0 \le x \le \pi$

Show answer

57. C $u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t > 0,$
$u(x, 0) = x^2(3x^2 - 8x + 6), \quad u_t(x, 0) = 0, \quad 0 \le x \le 1$

Show answer

58. $u_{tt} = 16u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x^2(x-\pi)^2, \quad 0 \le x \le \pi$

Show answer

59. C $u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x^2(3x^2 - 8x + 6), \quad 0 \le x \le 1$

Show answer

60. Adapt the proof of Theorem 12.2.2 to find d'Alembert's solution of the initial-boundary value problem in Exercise 49.

Show answer

61. Use the result of Exercise 60 to show that the formal solution of the initial-boundary value problem in Exercise 49 is an actual solution if g is differentiable and f is twice differentiable on $[0, L]$ and

$$f'_+(0) = f'_-(L) = g'_+(0) = g'_-(L) = 0.$$

62. Suppose λ and μ are constants and either $p_n(x) = \cos n\lambda x$ or $p_n(x) = \sin n\lambda x$, while either $q_n(t) = \cos n\mu t$ or $q_n(t) = \sin n\mu t$ for $n = 1, 2, 3, \dots$. Let

$$u(x,t) = \sum_{n=1}^{\infty} k_n p_n(x) q_n(t), \tag{A}$$

where $\{k_n\}_{n=1}^{\infty}$ are constants.

a. Show that if $\sum_{n=1}^{\infty} |k_n|$ converges then $u(x,t)$ converges for all (x,t).

b. Use Theorem 12.1.2 to show that if $\sum_{n=1}^{\infty} n|k_n|$ converges then (A) can be differentiated term by term with respect to x and t for all (x,t); that is,

$$u_x(x,t) = \sum_{n=1}^{\infty} k_n p_n'(x) q_n(t)$$

and

$$u_t(x,t) = \sum_{n=1}^{\infty} k_n p_n(x) q_n'(t).$$

c. Suppose $\sum_{n=1}^{\infty} n^2|k_n|$ converges. Show that

$$u_{xx}(x,y) = \sum_{n=1}^{\infty} k_n p_n''(x) q_n(t)$$

and

$$u_{tt}(x,y) = \sum_{n=1}^{\infty} k_n p_n(x) q_n''(t)$$

d. Suppose $\sum_{n=1}^{\infty} n^2|\alpha_n|$ and $\sum_{n=1}^{\infty} n|\beta_n|$ both converge. Show that the formal solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{n\pi at}{L} + \frac{\beta_n L}{n\pi a} \sin \frac{n\pi at}{L} \right) \sin \frac{n\pi x}{L}$$

of Equation 12.2.1 satisfies $u_{tt} = a^2 u_{xx}$ for all (x,t).

This conclusion also applies to the formal solutions defined in Exercises 17, 34, and 49.

63. Suppose g is differentiable and f is twice differentiable on $(-\infty, \infty)$, and let

$$u_0(x,t) = \frac{f(x+at)+f(x-at)}{2} \quad \text{and} \quad u_1(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(u)\,du.$$

a. Show that

$$\frac{\partial^2 u_0}{\partial t^2} = a^2 \frac{\partial^2 u_0}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

and

$$u_0(x,0) = f(x), \quad \frac{\partial u_0}{\partial t}(x,0) = 0, \quad -\infty < x < \infty.$$

b. Show that

$$\frac{\partial^2 u_1}{\partial t^2} = a^2 \frac{\partial^2 u_1}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

and

$$u_1(x,0) = 0, \quad \frac{\partial u_1}{\partial t}(x,0) = g(x), \quad -\infty < x < \infty.$$

c. Solve

$$u_{tt} = a^2 u_{xx}, \quad -\infty < t < \infty, \quad t > 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty.$$

Show answer

In Exercises 64-68 use the result of Exercise 63 to find a solution of

$$u_{tt} = a^2 u_{xx}, \quad -\infty < x < \infty$$

that satisfies the given initial conditions.

64. $u(x,0) = x, \quad u_t(x,0) = 4ax, \quad -\infty < x < \infty$

Show answer

65. $u(x,0) = x^2, \quad u_t(x,0) = 1, \quad -\infty < x < \infty$

Show answer

66. $u(x,0) = \sin x, \quad u_t(x,0) = a\cos x, \quad -\infty < x < \infty$

Show answer

67. $u(x,0) = x^3, \quad u_t(x,0) = 6x^2, \quad -\infty < x < \infty$

Show answer

68. $u(x,0) = x\sin x, \quad u_t(x,0) = \sin x, \quad -\infty < x < \infty$

Show answer

12.3 Laplace's Equation in Rectangular Coordinates

The temperature $u = u(x,y,t)$ in a two-dimensional plate satisfies the two-dimensional heat equation

$$u_t = a^2(u_{xx} + u_{yy}), \tag{12.3.1}$$

where (x,y) varies over the interior of the plate and $t > 0$. To find a solution of (12.3.1), it's necessary to specify the initial temperature $u(x,y,0)$ and conditions that must be satisfied on the boundary. However, as $t \to \infty$, the influence of the initial condition decays, so

$$\lim_{t\to\infty} u_t(x,y,t) = 0$$

and the temperature approaches a steady state distribution $u = u(x,y)$ that satisfies

$$u_{xx} + u_{yy} = 0. \tag{12.3.2}$$

This is Laplace's equation (http://www-history.mcs.st-and.ac.uk/Mathematicians/Laplace.html). This equation also arises in applications to fluid mechanics and potential theory; in fact, it is also called the potential equation. We seek solutions of (12.3.2) in a region R that satisfy specified conditions – called boundary conditions – on the boundary of R. For example, we may require u to assume prescribed values on the boundary. This is called a Dirichlet condition (http://www-history.mcs.st-and.ac.uk/Mathematicians/Dirichlet.html), and the problem is called a Dirichlet problem (http://www-history.mcs.st-and.ac.uk/Mathematicians/Dirichlet.html). Or, we may require the normal derivative of u at each point (x,y) on the boundary to assume prescribed values. This is called a Neumann condition (http://www-history.mcs.st-and.ac.uk/Mathematicians/Neumann_Carl.html), and the problem is called a Neumann problem (http://www-history.mcs.st-and.ac.uk/Mathematicians/Neumann_Carl.html). In some problems we impose Dirichlet conditions on part of the boundary and Neumann conditions on the rest. Then we say that the boundary conditions and the problem are mixed.

Solving boundary value problems for (12.3.2) over general regions is beyond the scope of this book, so we consider only very simple regions. We begin by considering the rectangular region shown in Figure 12.3.1.

Figure 12.3.1. *A rectangular region and its boundary*

The possible boundary conditions for this region can be written as

$$\begin{array}{rcccll}(1-\alpha)u(x,0) & + & \alpha u_y(x,0) & = & f_0(x), & 0\le x\le a,\\ (1-\beta)u(x,b) & + & \beta u_y(x,b) & = & f_1(x), & 0\le x\le a,\\ (1-\gamma)u(0,y) & + & \gamma u_x(0,y) & = & g_0(y), & 0\le y\le b,\\ (1-\delta)u(a,y) & + & \delta u_x(a,y) & = & g_1(y), & 0\le y\le b,\end{array}$$

where α, β, γ, and δ can each be either 0 or 1; thus, there are 16 possibilities. Let BVP $(\alpha,\beta,\gamma,\delta)(f_0,f_1,g_0,g_1)$ denote the problem of finding a solution of (12.3.2) that satisfies these conditions. This is a Dirichlet problem if

$$\alpha=\beta=\gamma=\delta=0$$

(Figure 12.3.2), or a Neumann problem if

$$\alpha=\beta=\gamma=\delta=1$$

(Figure 12.3.3). The other 14 problems are mixed.

Figure 12.3.2. *A Dirichlet problem*

Figure 12.3.3. *A Neumann problem*

For given $(\alpha,\beta,\gamma,\delta)$, the sum of solutions of

$$\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(f_0,0,0,0),\quad \mathrm{BVP}(\alpha,\beta,\gamma,\delta)(0,f_1,0,0),$$

$$\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(0,0,g_0,0),\quad \text{and}\quad \mathrm{BVP}(\alpha,\beta,\gamma,\delta)(0,0,0,g_1)$$

is a solution of

$$\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(f_0,f_1,g_0,g_1).$$

Therefore we concentrate on problems where only one of the functions f_0, f_1, g_0, g_2 isn't identically zero. There are 64 (count them!) problems of this form. Each has homogeneous boundary conditions on three sides of the rectangle, and a nonhomogeneous boundary condition on the fourth. We use separation of variables to find infinitely many functions that satisfy Laplace's equation and the three homogeneous boundary conditions in the open rectangle. We then use these solutions as building blocks to construct a formal solution of Laplace's equation that also satisfies the nonhomogeneous boundary condition. Since it's not feasible to consider all 64 cases, we'll restrict our attention in the text to just four. Others are discussed in the exercises.

If $v(x,y)=X(x)Y(y)$ then

$$v_{xx}+v_{yy}=X''Y+XY''=0$$

for all (x,y) if and only if

$$\frac{X''}{X}=-\frac{Y''}{Y}=k$$

for all (x,y), where k is a separation constant. This equation is equivalent to

$$X''-kX=0,\quad Y''+kY=0. \tag{12.3.3}$$

From here, the strategy depends upon the boundary conditions. We illustrate this by examples.

Example 12.3.1

Define the formal solution of

$$\begin{gathered}u_{xx}+u_{yy}=0,\quad 0<x<a,\quad 0<y<b,\\ u(x,0)=f(x),\quad u(x,b)=0,\quad 0\le x\le a,\\ u(0,y)=0,\quad u(a,y)=0,\quad 0\le y\le b\end{gathered} \tag{12.3.4}$$

(Figure 12.3.4).

Figure 12.3.4. *The boundary value problem (12.3.4)*

Solution The boundary conditions in (12.3.4) require products $v(x, y) = X(x)Y(y)$ such that $X(0) = X(a)$ $= Y(b) = 0$; hence, we let $k = -\lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(a) = 0 \tag{12.3.5}$$

and

$$Y'' - \lambda Y = 0, \quad Y(b) = 0. \tag{12.3.6}$$

From Theorem 11.1.2, the eigenvalues of (12.3.5) are $\lambda_n = n^2\pi^2/a^2$, with associated eigenfunctions

$$X_n = \sin\frac{n\pi x}{a}, \quad n = 1, 2, 3, \dots .$$

Substituting $\lambda = n^2\pi^2/a^2$ into (12.3.6) yields

$$Y'' - (n^2\pi^2/a^2)Y = 0, \quad Y(b) = 0,$$

so we could take

$$Y_n = \sinh\frac{n\pi(b-y)}{a}; \tag{12.3.7}$$

however, because of the nonhomogeneous Dirichlet condition at $y = 0$, it's better to require that $Y_n(0)$ $= 1$, which can be achieved by dividing the right side of (12.3.7) by its value at $y = 0$; thus, we take

$$Y_n = \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}.$$

Then

$$v_n(x, y) = X_n(x)Y_n(y) = \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\sin\frac{n\pi x}{a},$$

so $v_n(x, 0) = \sin n\pi x/a$ and v_n satisfies (12.3.4) with $f(x) = \sin n\pi x/a$. More generally, if α_1, …, α_m are arbitrary constants then

$$u_m(x,y)=\sum_{n=1}^{m}\alpha_n\frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\sin\frac{n\pi x}{a}$$

satisfies (12.3.4) with

$$f(x)=\sum_{n=1}^{m}\alpha_n\sin\frac{n\pi x}{L}.$$

Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$, we define the formal solution of (12.3.4) to be

$$u(x,y)=\sum_{n=1}^{\infty}\alpha_n\frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\sin\frac{n\pi x}{a}, \tag{12.3.8}$$

where

$$S(x)=\sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi x}{a}$$

is the Fourier sine series of f on $[0,a]$; that is,

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If $y<b$ then

$$\frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\approx e^{-n\pi y/a} \tag{12.3.9}$$

for large n, so the series in (12.3.8) converges if $0<y<b$; moreover, since also

$$\frac{\cosh n\pi(b-y)/a}{\sinh n\pi b/a}\approx e^{-n\pi y/a}$$

for large n, Theorem 12.1.2 applied twice with $z=x$ and twice with $z=t$, shows that u_{xx} and u_{yy} can be obtained by differentiating u term by term if $0<y<b$. (Exercise 37). Therefore u satisfies Laplace's equation in the interior of the rectangle in Figure 12.3.4. Moreover, the series in (12.3.8) also converges on the boundary of the rectangle, and satisfies the three homogeneous boundary conditions conditions

in (12.3.4). Therefore, since $u(x,0)=S(x)$ for $0\le x\le L$, u is an actual solution of (12.3.5) if and only if $S(x)=f(x)$ for $0\le x\le a$. From Theorem 11.3.2, this is true if f is continuous and piecewise smooth on $[0,L]$, and $f(0)=f(L)=0$.

Example 12.3.2

Solve (12.3.4) with $f(x)=x(x^2-3ax+2a^2)$.

Solution From Example 11.3.6,

$$S(x)=\frac{12a^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{a}.$$

Therefore

$$u(x,y)=\frac{12a^3}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh n\pi(b-y)/a}{n^3\sinh n\pi b/a}\sin\frac{n\pi x}{a}. \tag{12.3.10}$$

To compute approximate values of $u(x,y)$, we must use partial sums of the form

$$u_m(x,y)=\frac{12a^3}{\pi^3}\sum_{n=1}^{m}\frac{\sinh n\pi(b-y)/a}{n^3\sinh n\pi b/a}\sin\frac{n\pi x}{a}.$$

Because of (12.3.9), small values of m provide sufficient accuracy for most applications if $0<y<b$. Moreover, the n^3 in the denominator in (12.3.10) ensures that this is also true for $y=0$. For graphing purposes, we chose $a=2$, $b=1$, and $m=10$. Figure 12.3.5 shows the surface

$$u=u(x,y),\quad 0\le x\le 2,\quad 0\le y\le 1,$$

while Figure 12.3.6 shows the curves

$$u=u(x,0.1k),\quad 0\le x\le 2,\quad k=0,1,\dots,10.$$

Figure 12.3.5.

Figure 12.3.6.

Example 12.3.3

Define the formal solution of

$$\begin{gathered} u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b, \\ u(x,0)=0, \quad u_y(x,b)=f(x), \quad 0\le x\le a, \\ u_x(0,y)=0, \quad u_x(a,y)=0, \quad 0\le y\le b \end{gathered} \tag{12.3.11}$$

(Figure 12.3.7).

Figure 12.3.7. *The boundary value problem (12.3.11)*

Solution The boundary conditions in (12.3.11) require products $v(x,y)=X(x)Y(y)$ such that $X'(0)$ $=X'(a)=Y(0)=0$; hence, we let $k=-\lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X''+\lambda X=0, \quad X'(0)=0, \quad X'(a)=0 \tag{12.3.12}$$

and

$$Y''-\lambda Y=0, \quad Y(0)=0. \tag{12.3.13}$$

From Theorem 11.1.3, the eigenvalues of (12.3.12) are $\lambda=0$, with associated eigenfunction $X_0=1$, and $\lambda_n=n^2\pi^2/a^2$, with associated eigenfunctions

$$X_n=\cos\frac{n\pi x}{a}, \quad n=1,2,3,\dots.$$

Since $Y_0=y$ satisfies (12.3.13) with $\lambda=0$, we take $v_0(x,y)=X_0(x)Y_0(y)=y$. Substituting $\lambda=n^2\pi^2/a^2$ into (12.3.13) yields

$$Y''-(n^2\pi^2/a^2)Y=0, \quad Y(0)=0,$$

so we could take

$$Y_n=\sinh\frac{n\pi y}{a}. \tag{12.3.14}$$

However, because of the nonhomogeneous Neumann condition at $y = b$, it's better to require that $Y_n'(b) = 1$, which can be achieved by dividing the right side of (12.3.14) by the value of its derivative at $y = b$; thus,

$$Y_n = \frac{a \sinh n\pi y/a}{n\pi \cosh n\pi b/a}.$$

Then

$$v_n(x,y) = X_n(x)Y_n(y) = \frac{a \sinh n\pi y/a}{n\pi \cosh n\pi b/a} \cos \frac{n\pi x}{a},$$

so

$$\frac{\partial v_n}{\partial y}(x,b) = \cos \frac{n\pi x}{a}.$$

Therefore v_n satisfies (12.3.11) with $f(x) = \cos n\pi x/a$. More generally, if α_0, …, α_m are arbitrary constants then

$$u_m(x,y) = \alpha_0 y + \frac{a}{\pi} \sum_{n=1}^{m} \alpha_n \frac{\sinh n\pi y/a}{n \cosh n\pi b/a} \cos \frac{n\pi x}{a}$$

satisfies (12.3.11) with

$$f(x) = \alpha_0 + \sum_{n=1}^{m} \alpha_n \cos \frac{n\pi x}{L}.$$

Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$ we define the formal solution of (12.3.11) to be

$$u(x,y) = \alpha_0 y + \frac{a}{\pi} \sum_{n=1}^{\infty} \alpha_n \frac{\sinh n\pi y/a}{n \cosh n\pi b/a} \cos \frac{n\pi x}{a},$$

where

$$C(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{a}$$

is the Fourier cosine series of f on $[0,a]$; that is,

$$\alpha_0 = \frac{1}{a}\int_0^a f(x)\,dx \quad \text{and} \quad \alpha_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx, \quad n = 1, 2, 3, \dots.$$

Example 12.3.4

Solve (12.3.11) with $f(x) = x$.

Solution From Example 11.3.1,

$$C(x) = \backslash\text{dst}\frac{a}{2} - \frac{4a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{a}.$$

Therefore

$$u(x,y) = \backslash\text{dst}\frac{ay}{2} - \frac{4a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh(2n-1)\pi y/a}{(2n-1)^3\cosh(2n-1)\pi b/a}\cos\frac{(2n-1)\pi x}{a}. \tag{12.3.15}$$

For graphing purposes, we chose $a = 2$, $b = 1$, and retained the terms through $n = 10$ in (12.3.15). Figure 12.3.8 shows the surface

$$u = u(x,y), \quad 0 \le x \le 2, \quad 0 \le y \le 1,$$

while Figure 12.3.9 shows the curves

$$u = u(x, .1k), \quad 0 \le x \le 2, \quad k = 0, 1, \dots, 10.$$

Figure 12.3.8.

Figure 12.3.9.

Example 12.3.5

Define the formal solution of

$$\begin{array}{c}u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b, \\ u(x,0)=0, \quad u_y(x,b)=0, \quad 0\le x\le a, \\ u(0,y)=g(y), \quad u_x(a,y)=0, \quad 0\le y\le b\end{array} \tag{12.3.16}$$

(Figure 12.3.10).

Figure 12.3.10. *The boundary value problem (12.3.16)*

Solution The boundary conditions in (12.3.16) require products $v(x,y)=X(x)Y(y)$ such that $Y(0)=Y'(b)=X'(a)=0$; hence, we let $k=\lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X''-\lambda X=0, \qquad X'(a)=0 \tag{12.3.17}$$

and

$$Y''+\lambda Y=0, \quad Y(0)=0, \quad Y'(b)=0. \tag{12.3.18}$$

From Theorem 11.1.4, the eigenvalues of (12.3.18) are $\lambda_n=(2n-1)^2\pi^2/4b^2$, with associated eigenfunctions

$$Y_n=\sin\frac{(2n-1)\pi y}{2b}, \quad n=1,2,3,\dots.$$

Substituting $\lambda=(2n-1)^2\pi^2/4b^2$ into (12.3.17) yields

$$X''-((2n-1)^2\pi^2/4b^2)X=0, \quad X'(a)=0,$$

so we could take

$$X_n=\cosh\frac{(2n-1)\pi(x-a)}{2b}. \tag{12.3.19}$$

However, because of the nonhomogeneous Dirichlet condition at $x=0$, it's better to require that $X_n(0)=1$, which can be achieved by dividing the right side of (12.3.19) by its value at $x=0$; thus,

$$X_n=\frac{\cosh(2n-1)\pi(x-a)/2b}{\cosh(2n-1)\pi a/2b}.$$

Then

$$v_n(x,y) = X_n(x)Y_n(y) = \frac{\cosh(2n-1)\pi(x-a)/2b}{\cosh(2n-1)\pi a/2b} \sin\frac{(2n-1)\pi y}{2b},$$

so

$$v_n(0,y) = \sin\frac{(2n-1)\pi y}{2b}.$$

Therefore v_n satisfies (12.3.16) with $g(y) = \sin(2n-1)\pi y/2b$. More generally, if α_1, …, α_m are arbitrary constants then

$$u_m(x,y) = \sum_{n=1}^{m} \alpha_n \frac{\cosh(2n-1)\pi(x-a)/2b}{\cosh(2n-1)\pi a/2b} \sin\frac{(2n-1)\pi y}{2b}$$

satisfies (12.3.16) with

$$g(y) = \sum_{n=1}^{m} \alpha_n \sin\frac{(2n-1)\pi y}{2b}.$$

Thus, if g is an arbitrary piecewise smooth function on $[0,b]$, we define the formal solution of (12.3.16) to be

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n \frac{\cosh(2n-1)\pi(x-a)/2b}{\cosh(2n-1)\pi a/2b} \sin\frac{(2n-1)\pi y}{2b},$$

where

$$S_M(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{(2n-1)\pi y}{2b}$$

is the mixed Fourier sine series of g on $[0,b]$; that is,

$$\alpha_n = \frac{2}{b}\int_0^b g(y)\sin\frac{(2n-1)\pi y}{2b}\,dy.$$

Example 12.3.6

Solve (12.3.16) with $g(y)=y(2y^2-9by+12b^2)$.

Solution From Example 11.3.8,

$$S_M(y)=\backslash\text{dst}\frac{96b^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3+(-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi y}{2b}.$$

Therefore

$$u(x,y)=\backslash\text{dst}\frac{96b^3}{\pi^3}\sum_{n=1}^{\infty}\frac{\cosh(2n-1)\pi(x-a)/2b}{(2n-1)^3\cosh(2n-1)\pi a/2b}\left[3+(-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi y}{2b}.$$

Example 12.3.7

Define the formal solution of

$$\begin{gathered}u_{xx}+u_{yy}=0,\quad 0<x<a,\quad 0<y<b,\\ u_y(x,0)=0,\quad u(x,b)=0,\quad 0\le x\le a,\\ u_x(0,y)=0,\quad u_x(a,y)=g(y),\quad 0\le y\le b\end{gathered}\tag{12.3.20}$$

(Figure 12.3.11).

Figure 12.3.11. *The boundary value problem (12.3.20)*

Solution The boundary conditions in (12.3.20) require products $v(x,y)=X(x)Y(y)$ such that $Y'(0)=Y(b)$ $=X'(0)=0$; hence, we let $k=\lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X''-\lambda X=0,\qquad X'(0)=0\tag{12.3.21}$$

and

$$Y''+\lambda Y=0,\quad Y'(0)=0,\quad Y(b)=0.\tag{12.3.22}$$

From Theorem 11.1.4, the eigenvalues of (12.3.22) are $\lambda_n=(2n-1)^2\pi^2/4b^2$, with associated eigenfunctions

$$Y_n = \cos\frac{(2n-1)\pi y}{2b}, \quad n = 1, 2, 3, \dots .$$

Substituting $\lambda = (2n-1)^2\pi^2/4b^2$ into (12.3.21) yields

$$X'' - ((2n-1)^2\pi^2/4b^2)X = 0, \quad X'(0) = 0,$$

so we could take

$$X_n = \cosh\frac{(2n-1)\pi x}{2b}. \tag{12.3.23}$$

However, because of the nonhomogeneous Neumann condition at $x = a$, it's better to require that $X_n'(a) = 1$, which can be achieved by dividing the right side of (12.3.23) by the value of its derivative at $x = a$; thus,

$$X_n = \frac{2b\cosh(2n-1)\pi x/2b}{(2n-1)\pi\sinh(2n-1)\pi a/2b}.$$

Then

$$v_n(x,y) = X_n(x)Y_n(y) = \frac{2b\cosh(2n-1)\pi x/2b}{(2n-1)\pi\sinh(2n-1)\pi a/2b}\cos\frac{(2n-1)\pi y}{2b},$$

so

$$\frac{\partial v_n}{\partial x}(a,y) = \cos\frac{(2n-1)\pi y}{2b}.$$

Therefore v_n satisfies (12.3.20) with $g(y) = \cos(2n-1)\pi y/2b$. More generally, if α_1, …, α_m are arbitrary constants then

$$u_m(x,y) = \frac{2b}{\pi}\sum_{n=1}^{m}\alpha_n\frac{\cosh(2n-1)\pi x/2b}{(2n-1)\sinh(2n-1)\pi a/2b}\cos\frac{(2n-1)\pi y}{2b}$$

satisfies (12.3.20) with

$$g(y) = \sum_{n=1}^{\infty}\alpha_n\cos\frac{(2n-1)\pi y}{2b}.$$

Therefore, if g is an arbitrary piecewise smooth function on $[0, b]$, we define the formal solution of (12.3.20) to be

$$u(x,y)=\frac{2b}{\pi}\sum_{n=1}^{\infty}\alpha_n\frac{\cosh(2n-1)\pi x/2b}{(2n-1)\sinh(2n-1)\pi a/2b}\cos\frac{(2n-1)\pi y}{2b},$$

where

$$C_M(y)=\sum_{n=1}^{\infty}\alpha_n\cos\frac{(2n-1)\pi y}{2b}$$

is the mixed Fourier cosine series of g on $[0, b]$; that is,

$$\alpha_n=\frac{2}{b}\int_0^b g(y)\cos\frac{(2n-1)\pi y}{2b}\,dy.$$

> **Example 12.3.8**
>
> Solve (12.3.20) with $g(y)=y-b$.

Solution From Example 11.3.3,

$$C_M(y)=-\frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi y}{2b}.$$

Therefore

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Laplace’s Equation for a Semi-Infinite Strip

We now seek solutions of Laplace’s equation on the semi-infinite strip

$$S:\{0<x<a,\quad y>0\}$$

(Figure 12.3.12) that satisfy homogeneous boundary conditions at $x=0$ and $x=a$, and a nonhomogeneous Dirichlet or Neumann condition at $y=0$. An example of such a problem is

$$\begin{gathered} u_{xx}+u_{yy}=0, \quad 0<x<a, \quad y>0, \\ u(x,0)=f(x), \quad 0\le x\le a, \\ u(0,y)=0, \quad u(a,y)=0, \quad y>0, \end{gathered} \tag{12.3.24}$$

Figure 12.3.12. *A boundary value problem on a semi-infinite strip*

The boundary conditions in this problem are not sufficient to determine u, for if $u_0=u_0(x,y)$ is a solution and K is a constant then

$$u_1(x,y)=u_0(x,y)+K\sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}.$$

is also a solution. (Verify.) However, if we also require — on physical grounds — that the solution remain bounded for all (x,y) in S then $K=0$ and this difficulty is eliminated.

Example 12.3.9

Define the bounded formal solution of (12.3.24).

Solution Proceeding as in the solution of Example 12.3.1, we find that the building block functions are of the form

$$v_n(x,y)=Y_n(y)\sin\frac{n\pi x}{a},$$

where

$$Y_n''-(n^2\pi^2/a^2)Y_n=0.$$

Therefore

$$Y_n=c_1e^{n\pi y/a}+c_2e^{-n\pi y/a}$$

where c_1 and c_2 are constants. Although the boundary conditions in (12.3.24) don't restrict c_1, and c_2, we must set $c_1=0$ to ensure that Y_n is bounded. Letting $c_2=1$ yields

$$v_n(x,y)=e^{-n\pi y/a}\sin\frac{n\pi x}{a},$$

and we define the bounded formal solution of (12.3.24) to be

$$u(x,y)=\sum_{n=1}^{\infty} b_n e^{-n\pi y/a}\sin\frac{n\pi x}{a},$$

where

$$S(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}$$

is the Fourier sine series of f on $[0,a]$. ∎

See Exercises 29-34 for other boundary value problems on a semi-infinite strip.

12.3 Exercises

In Exercises 1-16 apply the definition developed in Example 1 to solve the boundary value problem. (Use Theorem 11.3.5 where it applies.) Where indicated by C, graph the surface $u=u(x,y)$, $0\le x\le a$, $0\le y\le b$.

1. $u_{xx}+u_{yy}=0,\quad 0<x<1,\quad 0<y<1,$
$u(x,0)=x(1-x),\quad u(x,1)=0,\quad 0\le x\le 1,$
$u(0,y)=0,\quad u(1,y)=0,\quad 0\le y\le 1$

Show answer

2. $u_{xx}+u_{yy}=0,\quad 0<x<2,\quad 0<y<3,$
$u(x,0)=x^2(2-x),\quad u(x,3)=0,\quad 0\le x\le 2,$
$u(0,y)=0,\quad u(2,y)=0,\quad 0\le y\le 3$

Show answer

3. C $u_{xx}+u_{yy}=0,\quad 0<x<2,\quad 0<y<2,$
$u(x,0)=\left\{\begin{array}{cl} x, & 0\le x\le 1,\\ 2-x, & 1\le x\le 2,\end{array}\right.\qquad u(x,2)=0,\quad 0\le x\le 2,$
$u(0,y)=0,\quad u(2,y)=0,\quad 0\le y\le 2$

Show answer

4. $u_{xx}+u_{yy}=0,\quad 0<x<\pi,\quad 0<y<1,$
$u(x,0)=x\sin x,\quad u(x,\pi)=0,\quad 0\le x\le \pi,$
$u(0,y)=0,\quad u(\pi,y)=0,\quad 0\le y\le 1$

Show answer

5. $u_{xx}+u_{yy}=0,\quad 0<x<3,\quad 0<y<2,$
$u(x,0)=0,\quad u_y(x,2)=x^2,\quad 0\le x\le 3,$
$u_x(0,y)=0,\quad u_x(3,y)=0,\quad 0\le y\le 2$

Show answer

6. $u_{xx}+u_{yy}=0,\quad 0<x<1,\quad 0<y<2,$
$u(x,0)=0,\quad u_y(x,2)=1-x,\quad 0\le x\le 1,$
$u_x(0,y)=0,\quad u_x(1,y)=0,\quad 0\le y\le 2$

Show answer

7. $u_{xx}+u_{yy}=0,\quad 0<x<2,\quad 0<y<2,$
$u(x,0)=0,\quad u_y(x,2)=x^2-4,\quad 0\le x\le 2,$
$u_x(0,y)=0,\quad u_x(2,y)=0,\quad 0\le y\le 2$

Show answer

8. $u_{xx}+u_{yy}=0,\quad 0<x<1,\quad 0<y<1,$
$u(x,0)=0,\quad u_y(x,1)=(x-1)^2,\quad 0\le x\le 1,$
$u_x(0,y)=0,\quad u_x(1,y)=0,\quad 0\le y\le 1$

Show answer

9. C $u_{xx}+u_{yy}=0,\quad 0<x<3,\quad 0<y<2,$
$u(x,0)=0,\quad u_y(x,2)=0,\quad 0\le x\le 3,$
$u(0,y)=y(4-y),\quad u_x(3,y)=0,\quad 0\le y\le 2$

Show answer

10. $u_{xx}+u_{yy}=0,\quad 0<x<2,\quad 0<y<1,$
$u(x,0)=0,\quad u_y(x,1)=0,\quad 0\le x\le 2,$
$u(0,y)=y^2(3-2y),\quad u_x(2,y)=0,\quad 0\le y\le 1$

Show answer

11. $u_{xx}+u_{yy}=0,\quad 0<x<2,\quad 0<y<2,$
$u(x,0)=0,\quad u_y(x,2)=0,\quad 0\le x\le 2,$
$u(0,y)=(y-2)^3+8,\quad u_x(2,y)=0,\quad 0\le y\le 2$

Show answer

12. $u_{xx}+u_{yy}=0,\quad 0<x<3,\quad 0<y<1,$
$u(x,0)=0,\quad u_y(x,1)=0,\quad 0\le x\le 3,$
$u(0,y)=y(2y^2-9y+12),\quad u_x(3,y)=0,\quad 0\le y\le 1$

Show answer

13. C $u_{xx}+u_{yy}=0,\quad 0<x<1,\quad 0<y<\pi,$
$u_y(x,0)=0,\quad u(x,\pi)=0,\quad 0\le x\le 1,$
$u_x(0,y)=0,\quad u_x(1,y)=\sin y,\quad 0\le y\le \pi$

Show answer

14. $u_{xx}+u_{yy}=0,\quad 0<x<2,\quad 0<y<3,$
$u_y(x,0)=0,\quad u(x,3)=0,\quad 0\le x\le 2,$
$u_x(0,y)=0,\quad u_x(2,y)=y(3-y),\quad 0\le y\le 3$

Show answer

15. $u_{xx}+u_{yy}=0,\quad 0<x<1,\quad 0<y<\pi,$
$u_y(x,0)=0,\quad u(x,\pi)=0,\quad 0\le x\le 1,$
$u_x(0,y)=0,\quad u_x(1,y)=\pi^2-y^2,\quad 0\le y\le \pi$

Show answer

16. $u_{xx}+u_{yy}=0,\quad 0<x<1,\quad 0<y<1,$
$u_y(x,0)=0,\quad u(x,1)=0,\quad 0\le x\le 1,$
$u_x(0,y)=0,\quad u_x(1,y)=1-y^3,\quad 0\le y\le 1$

Show answer

In Exercises 17-28 define the formal solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b$$

that satisfies the given boundary conditions for general a, b, and f or g. Then solve the boundary value problem for the specified a, b, and f or g. (Use Theorem 11.3.5 where it applies.) Where indicated by C *, graph the surface $u = u(x, y)$, $0 \le x \le a$, $0 \le y \le b$.*

17. C $u(x,0)=0, \quad u(x,b)=f(x), \quad 0<x<a,$
$u(0,y)=0, \quad u(a,y)=0, \quad 0<y<b$
$a=3, \quad b=2, \quad f(x)=x(3-x)$

Show answer

18. $u(x,0)=f(x), \quad u(x,b)=0, \quad 0<x<a,$
$u_x(0,y)=0, \quad u_x(a,y)=0, \quad 0<y<b$
$a=2, \quad b=1, \quad f(x)=x^2(x-2)^2$

Show answer

19. $u(x,0)=f(x), \quad u(x,b)=0, \quad 0<x<a,$
$u_x(0,y)=0, \quad u(a,y)=0, \quad 0<y<b$
$a=1, \quad b=2, \quad f(x)=3x^3-4x^2+1$

Show answer

20. $u(x,0)=f(x), \quad u(x,b)=0, \quad 0<x<a,$
$u(0,y)=0, \quad u_x(a,y)=0, \quad 0<y<b$
$a=3, \quad b=2, \quad f(x)=x(6-x)$

Show answer

21. $u(x,0)=f(x), \quad u_y(x,b)=0, \quad 0<x<a,$
$u(0,y)=0, \quad u(a,y)=0, \quad 0<y<b$
$a=\pi, \quad b=2, \quad f(x)=x(\pi^2-x^2)$

Show answer

22. $u_y(x,0)=0, \quad u(x,b)=f(x), \quad 0<x<a,$
$u_x(0,y)=0, \quad u_x(a,y)=0, \quad 0<y<b$
$a=\pi, \quad b=1, \quad f(x)=x^2(x-\pi)^2$

Show answer

23. C $u_y(x,0)=f(x), \quad u(x,b)=0, \quad 0<x<a,$
$u(0,y)=0, \quad u(a,y)=0, \quad 0<y<b$
$a=\pi, \quad b=1, \quad f(x)=\begin{cases} x, & 0\le x\le \frac{\pi}{2}, \\ \pi-x, & \frac{\pi}{2}\le x\le \pi \end{cases}$

Show answer

24. $u(x,0)=0, \quad u(x,b)=0, \quad 0<x<a,$
$u_x(0,y)=0, \quad u(a,y)=g(y), \quad 0<y<b$
$a=1, \quad b=1, \quad g(y)=y(y^3-2y^2+1)$

Show answer

25. C $u_y(x,0)=0, \quad u(x,b)=0, \quad 0<x<a,$
$u_x(0,y)=0, \quad u(a,y)=g(y), \quad 0<y<b$
$a=2, \quad b=2, \quad g(y)=4-y^2$

Show answer

26. $u(x,0)=0, \quad u(x,b)=0, \quad 0<x<a,$
$u_x(0,y)=0, \quad u_x(a,y)=g(y), \quad 0<y<b$
$a=1,\ b=4,\ g(y)=\begin{cases} y, & 0\le y\le 2, \\ 4-y, & 2\le y\le 4 \end{cases}$

Show answer

27. $u(x,0)=0, \quad u_y(x,b)=0, \quad 0<x<a,$
$u_x(0,y)=g(y), \quad u_x(a,y)=0, \quad 0<y<b$
$a=1, \quad b=\pi, \quad g(y)=y^2(3\pi-2y)$

Show answer

28. $u_y(x,0)=0, \quad u_y(x,b)=0, \quad 0<x<a,$
$u_x(0,y)=g(y), \quad u(a,y)=0, \quad 0<y<b$
$a=2, \quad b=\pi, \quad g(y)=y$

Show answer

In Exercises 29-34 define the bounded formal solution of

$$u_{xx}+u_{yy}=0, \quad 0<x<a, \quad y>0$$

that satisfies the given boundary conditions for general a and f. Then solve the boundary value problem for the specified a and f.

29. $u(x,0)=f(x), \quad 0<x<a,$
$u_x(0,y)=0, \quad u_x(a,y)=0, \quad y>0$
$a=\pi \ f(x)=x^2(3\pi-2x)$

Show answer

30. $u(x,0)=f(x), \quad 0<x<a,$
$u_x(0,y)=0, \quad u(a,y)=0, \quad y>0$
$a=3, \quad f(x)=9-x^2$

Show answer

31. $u(x,0)=f(x), \quad 0<x<a,$
$u(0,y)=0, \quad u_x(a,y)=0, \quad y>0$
$a=\pi, \quad f(x)=x(2\pi-x)$

Show answer

32. $u_y(x,0)=f(x), \quad 0<x<a,$
$u(0,y)=0, \quad u(a,y)=0, \quad y>0$
$a=\pi, \quad f(x)=x^2(\pi-x)$

Show answer

33. $u_y(x,0)=f(x), \quad 0<x<a,$
$u_x(0,y)=0, \quad u(a,y)=0, \quad y>0$
$a=7, \quad f(x)=x(7-x)$

Show answer

34. $u_y(x,0)=f(x), \quad 0<x<a,$
$u(0,y)=0, \quad u_x(a,y)=0, \quad y>0$
$a=5, \quad f(x)=x(5-x)$

Show answer

35. Define the formal solution of the Dirichlet problem

$$\begin{gathered}u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b,\\ u(x,0)=f_0(x), \quad u(x,b)=f_1(x), \quad 0\le x\le a,\\ u(0,y)=g_0(y), \quad u(a,y)=g_1(y), \quad 0\le y\le b\end{gathered}$$

Show answer

36. Show that the Neumann Problem

$$\begin{gathered}u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b, \\ u_y(x,0)=f_0(x), \quad u_y(x,b)=f_1(x), \quad 0\le x\le a, \\ u_x(0,y)=g_0(y), \quad u_x(a,y)=g_1(y), \quad 0\le y\le b\end{gathered}$$

has no solution unless

$$\int_0^a f_0(x)\,dx=\int_0^a f_1(x)\,dx=\int_0^b g_0(y)\,dy=\int_0^b g_1(y)\,dy=0.$$

In this case it has infinitely many formal solutions. Find them.

Show answer

37. In this exercise take it as given that the infinite series $\sum_{n=1}^{\infty} n^p e^{-qn}$ converges for all p if $q > 0$, and, where appropriate, use the comparison test for absolute convergence of an infinite series.

Let

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a} \sin \frac{n\pi x}{a},$$

where

$$\alpha_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

and f is piecewise smooth on $[0,a]$.

a. Verify the approximations

$$\frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a} \approx e^{-n\pi y/a}, \quad y<b, \tag{A}$$

and

$$\frac{\cosh n\pi(b-y)/a}{\sinh n\pi b/a} \approx e^{-n\pi y/a}, \quad y<b \tag{B}$$

for large n.

b. Use (A) to show that u is defined for (x,y) such that $0<y<b$.

c. For fixed y in $(0,b)$, use (A) and Theorem 12.1.2 with $z=x$ to show that

$$u_x(x,y)=\frac{\pi}{a}\sum_{n=1}^{\infty} n\alpha_n \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\cos\frac{n\pi x}{a}, \quad -\infty<x<\infty.$$

d. Starting from the result of **(b)**, use (A) and Theorem 12.1.2 with $z=x$ to show that, for a fixed y in $(0,b)$,

$$u_{xx}(x,y)=-\frac{\pi^2}{a^2}\sum_{n=1}^{\infty} n^2\alpha_n \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\sin\frac{n\pi x}{a}, \quad -\infty<x<\infty.$$

e. For fixed but arbitrary x, use (B) and Theorem 12.1.2 with $z=y$ to show that

$$u_y(x,y)=-\frac{\pi}{a}\sum_{n=1}^{\infty} n\alpha_n \frac{\cosh n\pi(b-y)/a}{\sinh n\pi b/a}\sin\frac{n\pi x}{a}$$

if $0<y_0<y<b$, where y_0 is an arbitrary number in $(0,b)$. Then argue that since y_0 can be chosen arbitrarily small, the conclusion holds for all y in $(0,b)$.

f. Starting from the result of **(e)**, use (A) and Theorem 12.1.2 to show that

$$u_{yy}(x,y)=\frac{\pi^2}{a^2}\sum_{n=1}^{\infty} n^2\alpha_n \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\sin\frac{n\pi x}{a}, \quad 0<y<b.$$

g. Conclude that u satisfies Laplace's equation for all (x,y) such that $0<y<b$.

By repeatedly applying the arguments in **(c)**–**(f)**, it can be shown that u can be differentiated term by term any number of times with respect to x and/or y if $0<y<b$.

12.4 Laplace's Equation in Polar Coordinates

In Section 12.3 we solved boundary value problems for Laplace's equation over a rectangle with sides parallel to the x, y-axes. Now we'll consider boundary value problems for Laplace's equation over regions with boundaries best described in terms of polar coordinates. In this case it's appropriate to regard u as function of (r, θ) and write Laplace's equation in polar form as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \tag{12.4.1}$$

where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \cos^{-1}\frac{x}{r} = \sin^{-1}\frac{x}{r}.$$

We begin with the case where the region is a circular disk with radius ρ, centered at the origin; that is, we want to define a formal solution of the boundary value problem

$$\begin{aligned} \text{\dst} u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < \rho, \quad -\pi \le \theta < \pi, \\ u(\rho, \theta) = f(\theta), \quad -\pi \le \theta < \pi \end{aligned} \tag{12.4.2}$$

(Figure 12.4.1). Note that (12.4.2) imposes no restriction on $u(r, \theta)$ when $r = 0$. We'll address this question at the appropriate time.

Figure 12.4.1. *The boundary value problem (12.4.2)*

We first look for products $v(r, \theta) = R(r)\Theta(\theta)$ that satisfy (12.4.1). For this function,

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

for all (r, θ) with $r \ne 0$ if

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda,$$

where λ is a separation constant. (Verify.) This equation is equivalent to

$$\Theta'' + \lambda\Theta = 0$$

and

$$r^2R'' + rR' - \lambda R = 0. \tag{12.4.3}$$

Since (r,π) and $(r,-\pi)$ are the polar coordinates of the same point, we impose periodic boundary conditions on Θ; that is,

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi). \tag{12.4.4}$$

Since we don't want $R\Theta$ to be identically zero, λ must be an eigenvalue of (12.4.4) and Θ must be an associated eigenfunction. From Theorem 11.1.6, the eigenvalues of (12.4.4) are $\lambda_0 = 0$ with associated eigenfunctions $\Theta_0 = 1$ and, for $n = 1, 2, 3, \dots$, $\lambda_n = n^2$, with associated eigenfunction $\cos n\theta$ and $\sin n\theta$ therefore,

$$\Theta_n = \alpha_n \cos n\theta + \beta_n \sin n\theta$$

where α_n and β_n are constants.

Substituting $\lambda = 0$ into (12.4.3) yields the

$$r^2R'' + rR' = 0,$$

so

$$\frac{R_0''}{R_0'} = -\frac{1}{r},$$

$$R_0' = \frac{c_1}{r},$$

and

$$R_0 = c_2 + c_1 \ln r. \tag{12.4.5}$$

If $c_1 \neq 0$ then

$$\lim_{r\to 0+} |R_0(r)| = \infty,$$

which doesn't make sense if we interpret $u_0(r,\theta)=R_0(r)\Theta_0(\theta)=R_0(r)$ as the steady state temperature distribution in a disk whose boundary is maintained at the constant temperature $R_0(\rho)$. Therefore we now require R_0 to be bounded as $r\to 0+$. This implies that $c_1=0$, and we take $c_2=1$. Thus, $R_0=1$ and $v_0(r,\theta)=R_0(r)\Theta_0(\theta)=1$. Note that v_0 satisfies (12.4.2) with $f(\theta)=1$.

Substituting $\lambda=n^2$ into (12.4.3) yields the Euler equation

$$r^2R_n''+rR_n'-n^2R_n=0 \tag{12.4.6}$$

for R_n. The indicial polynomial of this equation is

$$s(s-1)+s-n^2=(s-n)(s+n),$$

so the general solution of (12.4.6) is

$$R_n=c_1r^n+c_2r^{-n}, \tag{12.4.7}$$

by Theorem 7.4.3. Consistent with our previous assumption on R_0, we now require R_n to be bounded as $r\to 0+$. This implies that $c_2=0$, and we choose $c_1=\rho^{-n}$. Then $R_n(r)=r^n/\rho^n$, so

$$v_n(r,\theta)=R_n(r)\Theta_n(\theta)=\frac{r^n}{\rho^n}(\alpha_n\cos n\theta+\sin n\theta).$$

Now v_n satisfies (12.4.2) with

$$f(\theta)=\alpha_n\cos n\theta+\beta_n\sin n\theta.$$

More generally, if α_0, α_1,…, α_m and β_1, β_2, …, β_m are arbitrary constants then

$$u_m(r,\theta)=\alpha_0+\sum_{n=1}^{m}\frac{r^n}{\rho^n}(\alpha_n\cos n\theta+\beta_n\sin n\theta)$$

satisfies (12.4.2) with

$$f(\theta)=\alpha_0+\sum_{n=1}^{m}(\alpha_n\cos n\theta+\beta_n\sin n\theta).$$

This motivates the next definition.

DEFINITION 12.4.1

The bounded formal solution of the boundary value problem (12.4.2) is

$$u(r,\theta)=\alpha_0+\sum_{n=1}^{\infty}\frac{r^n}{\rho^n}(\alpha_n\cos n\theta+\beta_n\sin n\theta), \tag{12.4.8}$$

where

$$F(\theta)=\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta)$$

is the Fourier series of f on $[-\pi,\pi]$; that is,

$$\alpha_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta,$$

and

$$\alpha_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta\quad\text{and}\quad\beta_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta,\quad n=1,2,3,\dots.$$

Since $\sum_{n=0}^{\infty}n^k(r/\rho)^n$ converges for every k if $0<r<\rho$, Theorem 12.1.2 can be used to show that if $0<r<\rho$ then (12.4.8) can be differentiated term by term any number of times with respect to both r and θ. Since the terms in (12.4.8) satisfy Laplace's equation if $r>0$, (12.4.8) satisfies Laplace's equation if $0<r<\rho$. Therefore, since $u(\rho,\theta)=F(\theta)$, u is an actual solution of (12.4.2) if and only if

$$F(\theta)=f(\theta),\quad -\pi\le\theta<\pi.$$

From Theorem 11.2.4, this is true if f is continuous and piecewise smooth on $[-\pi,\pi]$ and $f(-\pi)=f(\pi)$.

EXAMPLE 12.4.1

Find the bounded formal solution of (12.4.2) with $f(\theta)=\theta(\pi^2-\theta^2)$.

SOLUTION From Example 11.2.6,

$$\theta(\pi^2-\theta^2)=12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\theta,\quad -\pi\le\theta\le\pi,$$

so

$$u(r,\theta)=12\sum_{n=1}^{\infty}\frac{r^n}{\rho^n}\frac{(-1)^n}{n^3}\sin n\theta,\quad 0\le r\le\rho,\quad -\pi\le\theta\le\pi.$$

Example 12.4.2

Define the formal solution of

$$\begin{array}{c}\dst u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\quad \rho_0<r<\rho,\quad -\pi\le\theta<\pi,\\ u(\rho_0,\theta)=0,\quad u(\rho,\theta)=f(\theta),\quad -\pi\le\theta<\pi,\end{array}\tag{12.4.9}$$

where $0<\rho_0<\rho$ (Figure 12.4.2).

Figure 12.4.2. *The boundary value problem (12.4.9)*

Solution We use separation of variables exactly as before, except that now we choose the constants in (12.4.5) and (12.4.7) so that $R_n(\rho_0)=0$ for $n=0,\,1,\,2,\ldots$. In view of the nonhomogeneous Dirichlet condition on the boundary $r=\rho$, it's also convenient to require that $R_n(\rho)=1$ for $n=0,\,1,\,2,\ldots$. We leave it to you to verify that

$$R_0(r)=\frac{\ln r/\rho_0}{\ln\rho/\rho_0}\quad\text{and}\quad R_n=\frac{\rho_0^{-n}r^n-\rho_0^n r^{-n}}{\rho_0^{-n}\rho^n-\rho_0^n\rho^{-n}},\quad n=1,2,3,\ldots$$

satisfy these requirements. Therefore

$$v_0(\rho,\theta)=\frac{\ln r/\rho_0}{\ln\rho/\rho_0}$$

and

$$v_n(r,\theta)=\frac{\rho_0^{-n}r^n-\rho_0^n r^{-n}}{\rho_0^{-n}\rho^n-\rho_0^n\rho^{-n}}(\alpha_n\cos n\theta+\beta_n\sin n\theta),\quad n=1,2,3,\ldots,$$

where α_n and β_n are arbitrary constants.

If α_0, α_1,…, α_m and β_1, β_2, …, β_m are arbitrary constants then

$$u_m(r,\theta)=\alpha_0\frac{\ln r/\rho_0}{\ln\rho/\rho_0}+\sum_{n=1}^{m}\frac{\rho_0^{-n}r^n-\rho_0^n r^{-n}}{\rho_0^{-n}\rho^n-\rho_0^n\rho^{-n}}(\alpha_n\cos n\theta+\beta_n\sin n\theta)$$

satisfies (12.4.9), with

$$f(\theta)=\alpha_0+\sum_{n=1}^{m}(\alpha_n\cos n\theta+\beta_n\sin n\theta).$$

This motivates us to define the formal solution of (12.4.9) for general f to be

$$u(r,\theta)=\alpha_0\frac{\ln r/\rho_0}{\ln\rho/\rho_0}+\sum_{n=1}^{\infty}\frac{\rho_0^{-n}r^n-\rho_0^n r^{-n}}{\rho_0^{-n}\rho^n-\rho_0^n\rho^{-n}}(\alpha_n\cos n\theta+\beta_n\sin n\theta),$$

where

$$F(\theta)=\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta)$$

is the Fourier series of f on $[-\pi,\pi]$.

Example 12.4.3

Define the bounded formal solution of

$$\begin{aligned} \backslash\mathrm{dst}\, u_{rr}+\tfrac{1}{r}u_r+\tfrac{1}{r^2}u_{\theta\theta}=0,\quad 0<r<\rho,\quad 0<\theta<\gamma,\\ u(\rho,\theta)=f(\theta),\quad 0\le\theta\le\gamma,\\ u(r,0)=0,\quad u(r,\gamma)=0,\quad 0<r<\rho, \end{aligned}\tag{12.4.10}$$

where $0<\gamma<2\pi$ (Figure 12.4.3).

Figure 12.4.3. *The boundary value problem (12.4.10)*

Solution Now $v(r,\theta)=R(r)\Theta(\theta)$, where

$$r^2R'' + rR' - \lambda R = 0 \tag{12.4.11}$$

and

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta(\gamma) = 0. \tag{12.4.12}$$

From Theorem 11.1.2, the eigenvalues of (12.4.12) are $\lambda_n = n^2\pi^2/\gamma^2$, with associated eigenfunction $\Theta_n = \sin n\pi\theta/\gamma$, $n = 1$, 2, 3,…. Substituting $\lambda = n^2\pi^2/\gamma^2$ into (12.4.11) yields the Euler equation

$$r^2R'' + rR_n' - \frac{n^2\pi^2}{\gamma^2}R = 0.$$

The indicial polynomial of this equation is

$$s(s-1) + s - \frac{n^2\pi^2}{\gamma^2} = \left(s - \frac{n\pi}{\gamma}\right)\left(s + \frac{n\pi}{\gamma}\right),$$

so

$$R_n = c_1 r^{n\pi/\gamma} + c_2 r^{-n\pi/\gamma},$$

by Theorem 7.4.3. To obtain a solution that remains bounded as $r \to 0+$ we let $c_2 = 0$. Because of the Dirichlet condition at $r = \rho$, it’s convenient to have $r(\rho) = 1$; therefore we take $c_1 = \rho^{-n\pi/\gamma}$, so

$$R_n(r) = \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}}.$$

Now

$$v_n(r,\theta) = R_n(r)\Theta_n(\theta) = \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}}\sin\frac{n\pi\theta}{\gamma}$$

satisfies (12.4.10) with

$$f(\theta) = \sin\frac{n\pi\theta}{\gamma}.$$

More generally, if α_1, α_2, …, α_m and are arbitrary constants then

$$u_m(r,\theta)=\sum_{n=1}^{m}\alpha_n\frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}}\sin\frac{n\pi\theta}{\gamma}$$

satisfies (12.4.10) with

$$f(\theta)=\sum_{n=1}^{m}\alpha_n\sin\frac{n\pi\theta}{\gamma}.$$

This motivates us to define the bounded formal solution of (12.4.10) to be

$$u_m(r,\theta)=\sum_{n=1}^{\infty}\alpha_n\frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}}\sin\frac{n\pi\theta}{\gamma},$$

where

$$S(\theta)=\sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi\theta}{\gamma}$$

is the Fourier sine expansion of f on $[0,\gamma]$; that is,

$$\alpha_n=\frac{2}{\gamma}\int_0^{\gamma}f(\theta)\sin\frac{n\pi\theta}{\gamma}\,d\theta.$$

12.4 Exercises

1. Define the formal solution of

$$\backslash\mathrm{dst} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad \rho_0 < r < \rho, \quad -\pi \le \theta < \pi,$$
$$u(\rho_0, \theta) = f(\theta), \quad u(\rho, \theta) = 0, \quad -\pi \le \theta < \pi,$$

where $0 < \rho_0 < \rho$.

Show answer

2. Define the formal solution of

$$\backslash\mathrm{dst} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad \rho_0 < r < \rho, \quad 0 < \theta < \gamma,$$
$$u(\rho_0, \theta) = 0, \quad u(\rho, \theta) = f(\theta), \quad 0 \le \theta \le \gamma,$$
$$u(r, 0) = 0, \quad u(r, \gamma) = 0, \quad \rho_0 < r < \rho,$$

where $0 < \gamma < 2\pi$ and $0 < \rho_0 < \rho$.

Show answer

3. Define the formal solution of

$$\backslash\mathrm{dst} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad \rho_0 < r < \rho, \quad 0 < \theta < \gamma,$$
$$u(\rho_0, \theta) = 0, \quad u_r(\rho, \theta) = g(\theta), \quad 0 \le \theta \le \gamma,$$
$$u_\theta(r, 0) = 0, \quad u_\theta(r, \gamma) = 0, \quad \rho_0 < r < \rho,$$

where $0 < \gamma < 2\pi$ and $0 < \rho_0 < \rho$.

Show answer

4. Define the bounded formal solution of

$$\backslash\mathrm{dst} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < \rho, \quad 0 < \theta < \gamma,$$
$$u(\rho, \theta) = f(\theta), \quad 0 \le \theta \le \gamma,$$
$$u_\theta(r, 0) = 0, \quad u(r, \gamma) = 0, \quad 0 < r < \rho,$$

where $0 < \gamma < 2\pi$.

Show answer

5. Define the formal solution of

$$\backslash\mathrm{dst}u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, \quad \rho_0 < r < \rho, \quad 0 < \theta < \gamma,$$
$$u_r(\rho_0, \theta) = g(\theta), \quad u_r(\rho, \theta) = 0, \quad 0 \le \theta \le \gamma,$$
$$u(r, 0) = 0, \quad u_\theta(r, \gamma) = 0, \quad \rho_0 < r < \rho,$$

where $0 < \gamma < 2\pi$ and $0 < \rho_0 < \rho$.

Show answer

6. Define the bounded formal solution of

$$\backslash\mathrm{dst}u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < \rho, \quad 0 < \theta < \gamma,$$
$$u(\rho, \theta) = f(\theta), \quad 0 \le \theta \le \gamma,$$
$$u_\theta(r, 0) = 0, \quad u_\theta(r, \gamma) = 0, \quad 0 < r < \rho,$$

where $0 < \gamma < 2\pi$.

Show answer

7. Show that the Neumann problem

$$\backslash\mathrm{dst}u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < \rho, \quad -\pi \le \theta < \pi,$$
$$u_r(\rho, \theta) = f(\theta), \quad -\pi \le \theta < \pi$$

has no bounded formal solution unless $\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0$. In this case it has infinitely many solutions. Find those solutions.

Show answer

Chapter 12 Fourier Solutions of Partial Differential Equations

12.1 The Heat Equation

8 $u(x,t) = \dst\frac{8}{\pi^3}\sum_{n=1}^\infty \frac{1}{(2n-1)^3}e^{-(2n-1)^2\pi^2 t}\sin(2n-1)\pi x$

9 $u(x,t) = \dst\frac{4}{\pi}\sum_{n=1}^\infty \frac{1}{(2n-1)}e^{-9(2n-1)^2\pi^2 t/16}\sin\frac{(2n-1)\pi x}{4}$

10 $u(x,t) = \dst\frac{\pi}{2}e^{-3t}\sin x - \frac{16}{\pi}\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}e^{-12n^2 t}\sin 2nx$

11 $u(x,t) = \dst-\frac{32}{\pi^3}\sum_{n=1}^\infty \frac{(1+(-1)^n 2)}{n^3}e^{-9n^2\pi^2 t/4}\sin\frac{n\pi x}{2}$

12 $u(x,t) = \dst-\frac{324}{\pi^3}\sum_{n=1}^\infty \frac{(-1)^n}{n^3}e^{-4n^2\pi^2 t/9}\sin\frac{n\pi x}{3}$

13 $u(x,t) = \dst\frac{8}{\pi^2}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{(2n-1)^2}e^{-(2n-1)^2\pi^2 t}\sin\frac{(2n-1)\pi x}{2}$

14 $u(x,t) = \dst-\frac{720}{\pi^5}\sum_{n=1}^\infty \frac{(-1)^n}{n^5}e^{-7n^2\pi^2 t}\sin n\pi x$

15 $u(x,t) = \dst\frac{96}{\pi^5}\sum_{n=1}^\infty \frac{1}{(2n-1)^5}e^{-5(2n-1)^2\pi^2 t}\sin(2n-1)\pi x$

16 $u(x,t) = \dst-\frac{240}{\pi^5}\sum_{n=1}^\infty \frac{1+(-1)^n 2}{n^5}e^{-2n^2\pi^2 t}\sin n\pi x.$

17 $u(x,t) = \dst\frac{16}{3}+\frac{64}{\pi^2}\sum_{n=1}^\infty \frac{(-1)^n}{n^2}e^{-9\pi^2 n^2 t/16}\cos\frac{n\pi x}{4}$

18 $u(x,t) = \dst-\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}e^{-n^2\pi^2 t}\cos\frac{n\pi x}{2}$

19 $u(x,t) = \dst\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}e^{-36n^2\pi^2 t}\cos 2n\pi x$

20 $u(x,t) = \dst 4-\frac{384}{\pi^4}\sum_{n=1}^\infty \frac{1}{(2n-1)^4}e^{-3(2n-1)^2\pi^2 t/4}\cos\frac{(2n-1)\pi x}{2}$

21 $u(x,y) = \dst-\frac{28}{5}-\frac{576}{\pi^4}\sum_{n=1}^\infty \frac{(-1)^n}{n^4}e^{-5n^2\pi^2 t/2}\cos\frac{n\pi x}{\sqrt{2}}$

22 $u(x,t) = \dst-\frac{2}{5}-\frac{48}{\pi^4}\sum_{n=1}^\infty \frac{1+(-1)^n 2}{n^4}e^{-3n^2\pi^2 t}\cos n\pi x$

23 $u(x,t) = \dst\frac{3}{5}-\frac{48}{\pi^4}\sum_{n=1}^\infty \frac{2+(-1)^n}{n^4}e^{-n^2\pi^2 t}\cos n\pi x$

24 $\dst u(x,t) = \frac{\pi^4}{30}-3\sum_{n=1}^\infty \frac{1}{n^4}e^{-4n^2 t}\cos 2nx$

25 $u(x,t) = \dst\frac{8}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{(2n+1)(2n-3)}e^{-(2n-1)^2\pi^2 t/4}\sin\frac{(2n-1)\pi x}{2}$

26 $u(x,t) =$ \dst $8\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right] e^{-3(2n-1)^2t/4} \sin\frac{(2n-1)x}{2}$

27 $u(x,t) =$ \dst $\frac{128}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-5(2n-1)^2t/16} \sin\frac{(2n-1)\pi x}{4}$

28 $u(x,t) =$ \dst $-\frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[1 + (-1)^n \frac{4}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \sin\frac{(2n-1)\pi x}{2}$

29 $u(x,t) =$ \dst $\frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[1 + (-1)^n \frac{2}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \sin\frac{(2n-1)\pi x}{2}$

30 $u(x,t) =$ \dst $\frac{192}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^4} e^{-(2n-1)^2\pi^2t/4} \sin\frac{(2n-1)\pi x}{2}$

31 $u(x,t) =$ \dst $\frac{1536}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{3}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \sin\frac{(2n-1)\pi x}{2}$

32 $u(x,t) =$ \dst $\frac{384}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \sin\frac{(2n-1)\pi x}{2}$

33 $u(x,t) = -$\dst $64\sum_{n=1}^{\infty} \frac{e^{-3(2n-1)^2t/4}}{(2n-1)^3}\left[(-1)^n + \frac{3}{(2n-1)\pi}\right] \cos\frac{(2n-1)x}{2}$

34 $u(x,t) = -$\dst $\frac{16}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2t} \cos\frac{(2n-1)x}{4}$

35 $u(x,t) =$ \dst $-\frac{64}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}\left[1 - \frac{8}{(2n-1)^2\pi^2}\right] e^{-9(2n-1)^2\pi^2t/64} \cos\frac{(2n-1)\pi x}{8}$

36 $u(x,t) =$ \dst $\frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-3(2n-1)^2\pi^2t/4} \cos\frac{(2n-1)\pi x}{2}$

37 $u(x,t) = -$\dst $\frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[(-1)^n + \frac{2}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \cos\frac{(2n-1)\pi x}{2}$

38 $u(x,t) = -$\dst $\frac{32}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} e^{-7(2n-1)^2t/4} \cos\frac{(2n-1)x}{2}$

39 $u(x,t) =$ \dst $\frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{8}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \cos\frac{(2n-1)\pi x}{2}$

40 $u(x,t) =$ \dst $\frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \cos\frac{(2n-1)\pi x}{2}$

41 $u(x,t) = -$\dst $\frac{768}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 2}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \cos\frac{(2n-1)\pi x}{2}$

42 $u(x,t) = -$\dst $\frac{384}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4} \cos\frac{(2n-1)\pi x}{2}$

43 $u(x,t) =$ \dst $\frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2a^2t/L^2} \cos\frac{(2n-1)\pi x}{L}$

44 $u(x,t) =$ \dst $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 - \cos\frac{n\pi}{2}\right] e^{-n^2\pi^2a^2t/L^2} \sin\frac{n\pi x}{L}$

45 $u(x,t) =$ \dst $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\frac{(2n-1)\pi}{4} e^{-(2n-1)^2\pi^2a^2t/4L^2} \cos\frac{(2n-1)\pi x}{2L}$

46 $u(x,t) =$ \dst $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\left[1 - \cos\frac{(2n-1)\pi)}{4}\right] e^{-(2n-1)^2\pi^2a^2t/4L^2} \sin\frac{(2n-1)\pi x}{2L}$

48 $u(x,t) = 1 - x + x^3 + \backslash\mathrm{dst}\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{e^{-9\pi^2(2n-1)^2t/16}}{(2n-1)} \sin\frac{(2n-1)\pi x}{4}$

49 $u(x,t) = \backslash\mathrm{dst}1 + x + x^2 - \backslash\mathrm{dst}\frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\pi^2t}}{(2n-1)^3} \sin(2n-1)\pi x$

50 $u(x,t) = -1 - x + x^3 + \backslash\mathrm{dst}\frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-3(2n-1)^2\pi^2t/4} \cos\frac{(2n-1)\pi x}{2}$

51 $u(x,t) = x^2 - x - 2\backslash\mathrm{dst} - \frac{64}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}\left[1 - \frac{8}{(2n-1)^2\pi^2}\right] e^{-9(2n-1)^2\pi^2t/64} \cos\frac{(2n-1)\pi x}{8}$

52 $u(x,t) = \sin\pi x + \backslash\mathrm{dst}\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n-3)} e^{-(2n-1)^2\pi^2t/4} \sin\frac{(2n-1)\pi x}{2}$

53 $u(x,t) = x^3 - x + 3 + \backslash\mathrm{dst}\frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\pi^2t/4}}{(2n-1)^3} \sin\frac{(2n-1)\pi x}{2}$

12.2 The Wave Equation

1 $u(x,t)=$ \dst $\frac{4}{3\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\sin 3(2n-1)\pi t\sin(2n-1)\pi x$

2 $u(x,t)=$ \dst $\frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos 3(2n-1)\pi t\sin(2n-1)\pi x$

3 $u(x,t)=$ \dst $-\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{(1+(-1)^n2)}{n^3}\cos n\sqrt{7}\,\pi t\ \sin n\pi x$

4 $u(x,t)=$ \dst $\frac{8}{3\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin 3(2n-1)\pi t\sin(2n-1)\pi x$

5 $u(x,t)=$ \dst $-\frac{4}{\sqrt{7}\,\pi^4}\sum_{n=1}^{\infty}\frac{(1+(-1)^n2)}{n^4}\sin n\sqrt{7}\,\pi t\ \sin n\pi x$

6 $u(x,t)=$ \dst $\frac{324}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\frac{8n\pi t}{3}\sin\frac{n\pi x}{3}$

7 $u(x,t)=$ \dst $\frac{96}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\cos 2(2n-1)\pi t\sin(2n-1)\pi x$

8 $u(x,t)=$ \dst $\frac{243}{2\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\sin\frac{8n\pi t}{3}\sin\frac{n\pi x}{3}$

9 $u(x,t)=$ \dst $\frac{48}{\pi^6}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}\sin 2(2n-1)\pi t\sin(2n-1)\pi x.$

10 $u(x,t)=$ \dst $\frac{\pi}{2}\cos\sqrt{5}\,t\sin x-\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)^2}\cos 2n\sqrt{5}\,t\sin 2nx$

11 $u(x,t)=$ \dst $-\frac{240}{\pi^5}\sum_{n=1}^{\infty}\frac{1+(-1)^n2}{n^5}\cos n\pi t\sin n\pi x$

12 $u(x,t)=$ \dst $\frac{\pi}{2\sqrt{5}}\sin\sqrt{5}\,t\sin x-\frac{8}{\pi\sqrt{5}}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\sin 2n\sqrt{5}\,t\sin 2nx$

13 $u(x,t)=$ \dst $-\frac{240}{\pi^6}\sum_{n=1}^{\infty}\frac{1+(-1)^n2}{n^6}\sin n\pi t\sin n\pi x$

14 $u(x,t)=$ \dst $-\frac{720}{\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\cos 3n\pi t\sin n\pi x$

15 $u(x,t)=$ \dst $-\frac{240}{\pi^6}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^6}\sin 3n\pi t\sin n\pi x$

18 $u(x,t)=-$\dst $\frac{128}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{3(2n-1)\pi t}{4}\ \cos\frac{(2n-1)\pi x}{4}$

19 $u(x,t)=-$\dst $\frac{64}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n+\frac{3}{(2n-1)\pi}\right]\cos(2n-1)\pi t\ \cos\frac{(2n-1)\pi x}{2}$

20 $u(x,t)=-$\dst $\frac{512}{3\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\sin\frac{3(2n-1)\pi t}{4}\ \cos\frac{(2n-1)\pi x}{4}$

21 $u(x,t)=-$\dst $\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[(-1)^n+\frac{3}{(2n-1)\pi}\right]\sin(2n-1)\pi t\ \cos\frac{(2n-1)\pi x}{2}$

22 $u(x,t)=$ \dst $\frac{96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n3+\frac{4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\sqrt{5}\,\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

23 $u(x,t) = -\backslash\mathrm{dst}96\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[(-1)^n + \frac{2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\sqrt{3}\,t}{2}\cos\frac{(2n-1)x}{2}$

24 $u(x,t) = \backslash\mathrm{dst}\frac{192}{\pi^4\sqrt{5}}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\sqrt{5}\,\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

25 $u(x,t) = -\backslash\mathrm{dst}\frac{192}{\sqrt{3}}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{2}{(2n-1)\pi}\right]\sin\frac{(2n-1)\sqrt{3}t}{2}\sin\frac{(2n-1)x}{2}$

26 $u(x,t) = -\backslash\mathrm{dst}\frac{384}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right]\cos\frac{3(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

27 $u(x,t) = \backslash\mathrm{dst}\frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{8}{(2n-1)\pi}\right]\cos\frac{(2n-1)\sqrt{7}\,\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

28 $u(x,t) = -\backslash\mathrm{dst}\frac{768}{3\pi^5}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^5}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right]\sin\frac{3(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

29 $u(x,t) = \backslash\mathrm{dst}\frac{192}{\pi^4\sqrt{7}}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[(-1)^n 5 + \frac{8}{(2n-1)\pi}\right]\sin\frac{(2n-1)\sqrt{7}\,\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

30 $u(x,t) = -\backslash\mathrm{dst}\frac{768}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

31 $u(x,t) = -\backslash\mathrm{dst}\frac{1536}{\pi^5}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^5}\left[1 + \frac{(-1)^n 2}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

32 $u(x,t) = \backslash\mathrm{dst}\frac{1}{2}\left[C_{Mf}(x+at) + C_{Mf}(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} C_{Mg}(\tau)\,d\tau$

35 $u(x,t) = \backslash\mathrm{dst}\frac{32}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\cos 4(2n-1)t\,\sin\frac{(2n-1)x}{2}$

36 $u(x,t) = \backslash\mathrm{dst} - \frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\cos\frac{3(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}$

37 $u(x,t) = \backslash\mathrm{dst}\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\sin 4(2n-1)t\,\sin\frac{(2n-1)x}{2}$

38 $u(x,t) = \backslash\mathrm{dst} - \frac{64}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{3(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}$

39 $u(x,t) = \backslash\mathrm{dst}\frac{96}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{2}{(2n-1)\pi}\right]\cos\frac{3(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}$

40 $u(x,t) = \backslash\mathrm{dst}\frac{192}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^4}\cos\frac{(2n-1)\sqrt{3}\,t}{2}\sin\frac{(2n-1)x}{2}$

41 $u(x,t) = \backslash\mathrm{dst}\frac{64}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[1 + (-1)^n\frac{2}{(2n-1)\pi}\right]\sin\frac{3(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}$

42 $u(x,t) = \backslash\mathrm{dst}\frac{384}{\sqrt{3}\,\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^5}\sin\frac{(2n-1)\sqrt{3}\,t}{2}\sin\frac{(2n-1)x}{2}$

43 $u(x,t) = \backslash\mathrm{dst}\frac{1536}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{3}{(2n-1)\pi}\right]\cos\frac{(2n-1)\sqrt{5}\,\pi t}{2}\sin\frac{(2n-1)\pi x}{2}$

44 $u(x,t) = \backslash\mathrm{dst}\frac{384}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\cos(2n-1)\pi t\sin\frac{(2n-1)\pi x}{2}$

45 $u(x,t) = \backslash\mathrm{dst}\frac{3072}{\sqrt{5}\,\pi^5}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^5}\left[(-1)^n + \frac{3}{(2n-1)\pi}\right]\sin\frac{(2n-1)\sqrt{5}\,\pi t}{2}\sin\frac{(2n-1)\pi x}{2}$

46 $u(x,t) =$ \dst $\frac{384}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\left[(-1)^n+\frac{4}{(2n-1)\pi}\right]\sin(2n-1)\pi t\sin\frac{(2n-1)\pi x}{2}$

47 \dst $u(x,t)=\frac{1}{2}[S_{Mf}(x+at)+S_{Mf}(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}S_{Mg}(\tau)\,d\tau$

50 $u(x,t) =$ \dst $4-\frac{768}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\frac{\sqrt{5}(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

51 $u(x,t) =$ \dst $4t-\frac{1536}{\sqrt{5}\,\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin\frac{\sqrt{5}(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}$

52 $u(x,t) =$ \dst $-\frac{2\pi^4}{5}-48\sum_{n=1}^{\infty}\frac{1+(-1)^n2}{n^4}\cos 2nt\cos nx$

53 $u(x,t) =$ \dst $-\frac{7}{5}-\frac{144}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos n\sqrt{7}\,\pi t\cos n\pi x$

54 $u(x,t) =$ \dst $-\frac{2\pi^4 t}{5}-24\sum_{n=1}^{\infty}\frac{1+(-1)^n2}{n^5}\sin 2nt\cos nx$

55 $u(x,t) =$ \dst $-\frac{7t}{5}-\frac{144}{\pi^5\sqrt{7}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\sin n\sqrt{7}\,\pi t\cos n\pi x$

56 $u(x,t) =$ \dst $\frac{\pi^4}{30}-3\sum_{n=1}^{\infty}\frac{1}{n^4}\cos 8nt\cos 2nx$

57 $u(x,t) =$ \dst $\frac{3}{5}-\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{2+(-1)^n}{n^4}\cos n\pi t\cos n\pi x$

58 $u(x,t) =$ \dst $\frac{\pi^4 t}{30}-\frac{3}{8}\sum_{n=1}^{\infty}\frac{1}{n^5}\sin 8nt\cos 2nx$

59 $u(x,t) =$ \dst $\frac{3t}{5}-\frac{48}{\pi^5}\sum_{n=1}^{\infty}\frac{2+(-1)^n}{n^5}\sin n\pi t\cos n\pi x$

60 $u(x,t)=\frac{1}{2}[C_f(x+at)+C_f(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}C_g(\tau)\,d\tau$

63 **(c)** $u(x,t) =$ \dst $\frac{f(x+at)+f(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}g(u)\,du$

64 $u(x,t)=x(1+4at)$

65 $u(x,t)=x^2+a^2t^2+t$

66 $u(x,t)=\sin(x+at)$

67 $u(x,t)=x^3+6tx^2+3a^2t^2x+2a^2t^3$

68 $u(x,t) =$ \dst $x\sin x\cos at+at\cos x\sin at+\frac{\sin x\sin at}{a}$

12.3 Laplace's Equation in Rectangular Coordinates

1 $u(x,y)=\backslash\mathrm{dst}\,\frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh(2n-1)\pi(1-y)}{(2n-1)^3\sinh(2n-1)\pi}\sin(2n-1)\pi x$

2 $u(x,y)=\backslash\mathrm{dst}-\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{(1+(-1)^n 2)\sinh n\pi(3-y)/2}{n^3\sinh 3n\pi/2}\sin\frac{n\pi x}{2}$

3 $u(x,y)=\backslash\mathrm{dst}\,\frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sinh(2n-1)\pi(1-y/2)}{(2n-1)^2\sinh(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}$

4 $u(x,y)=\backslash\mathrm{dst}\,\frac{\pi}{2}\frac{\sinh(1-y)}{\sinh 1}\sin x-\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n\sinh 2n(1-y)}{(4n^2-1)^2\sinh 2n}\sin 2nx$

5 $u(x,y)=\backslash\mathrm{dst}3y+\frac{108}{\pi^3}\sum_{n=1}^{\infty}(-1)^n\frac{\sinh n\pi y/3}{n^3\cosh 2n\pi/3}\cos\frac{n\pi x}{3}$

6 $u(x,y)=\backslash\mathrm{dst}\,\frac{y}{2}+\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh(2n-1)\pi y}{(2n-1)^3\cosh 2(2n-1)\pi}\cos(2n-1)\pi x$

7 $u(x,y)=\backslash\mathrm{dst}-\frac{8y}{3}+\frac{32}{\pi^3}\sum_{n=1}^{\infty}(-1)^n\frac{\sinh n\pi y/2}{n^3\cosh n\pi}\cos\frac{n\pi x}{2}$

8 $u(x,y)=\backslash\mathrm{dst}\,\frac{y}{3}+\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh n\pi y}{n^3\cosh n\pi}\cos n\pi x$

9 $u(x,y)=\backslash\mathrm{dst}\,\frac{128}{\pi^3}\sum_{n=1}^{\infty}\frac{\cosh(2n-1)\pi(x-3)/4}{(2n-1)^3\cosh 3(2n-1)\pi/4}\sin\frac{(2n-1)\pi y}{4}$

10 $u(x,y)=\backslash\mathrm{dst}-\frac{96}{\pi^3}\sum_{n=1}^{\infty}\left[1+(-1)^n\frac{4}{(2n-1)\pi}\right]\frac{\cosh(2n-1)\pi(x-2)/2}{(2n-1)^3\cosh(2n-1)\pi}\sin\frac{(2n-1)\pi y}{2}$

11 $u(x,y)=\backslash\mathrm{dst}\,\frac{768}{\pi^3}\sum_{n=1}^{\infty}\left[1+(-1)^n\frac{2}{(2n-1)\pi}\right]\frac{\cosh(2n-1)\pi(x-2)/4}{(2n-1)^3\cosh(2n-1)\pi/2}\sin\frac{(2n-1)\pi y}{4}$

12 $u(x,y)=\backslash\mathrm{dst}\,\frac{96}{\pi^3}\sum_{n=1}^{\infty}\left[3+(-1)^n\frac{4}{(2n-1)\pi}\right]\frac{\cosh(2n-1)\pi(x-3)/2}{(2n-1)^3\cosh 3(2n-1)\pi/2}\sin\frac{(2n-1)\pi y}{2}$

13 $u(x,y)=-\backslash\mathrm{dst}\,\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{\cosh(2n-1)x/2}{(2n-3)(2n+1)(2n-1)\sinh(2n-1)/2}\cos\frac{(2n-1)y}{2}$

14 $u(x,y)=-\backslash\mathrm{dst}\,\frac{432}{\pi^3}\sum_{n=1}^{\infty}\left[1+\frac{4(-1)^n}{(2n-1)\pi}\right]\frac{\cosh(2n-1)\pi x/6}{(2n-1)^3\sinh(2n-1)\pi/3}\cos\frac{(2n-1)\pi y}{6}$

15 $u(x,y)=-\backslash\mathrm{dst}\,\frac{64}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\cosh(2n-1)x/2}{(2n-1)^4\sinh(2n-1)/2}\cos\frac{(2n-1)y}{2}.$

16 $u(x,y)=-\backslash\mathrm{dst}\,\frac{192}{\pi^4}\sum_{n=1}^{\infty}\frac{\cosh(2n-1)\pi x/2}{(2n-1)^4\sinh(2n-1)\pi/2}\left[(-1)^n+\frac{2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi y}{2}$

17 $\backslash\mathrm{dst}u(x,y)=\sum_{n=1}^{\infty}\alpha_n\frac{\sinh n\pi y/a}{\sinh n\pi b/a}\sin\frac{n\pi x}{a},\quad \alpha_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$

$u(x,y)$
$=\backslash\mathrm{dst}\,\frac{72}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh(2n-1)\pi y/3}{(2n-1)^3\sinh 2(2n-1)\pi/3}\sin\frac{(2n-1)\pi x}{3}$

18 \dst$u(x,y)=\alpha_0(1-y/b)+\sum_{n=1}^\infty \alpha_n\frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\cos\frac{n\pi x}{a}$, \dst$\alpha_0=\frac{1}{a}\int_0^a f(x)\,dx$,

\dst$u(x,y)=\frac{8(1-y)}{15}-\frac{48}{\pi^4}\sum_{n=1}^\infty \frac{1}{n^4}\frac{\sinh n\pi(1-y)}{\sinh n\pi}\cos n\pi x$

\dst$\alpha_n$
$=\frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}$
$dx,\quad n$
≥ 1

19 \dst$u(x,y)=\sum_{n=1}^\infty \alpha_n\frac{\sinh(2n-1)\pi(b-y)/2a}{\sinh(2n-1)\pi b/2a}\cos\frac{(2n-1)\pi x}{2a}$,

$u(x,y)=$ \dst$\frac{288}{\pi^3}\sum_{n=1}^\infty \frac{\sinh(2n-1)\pi(2-y)/6}{(2n-1)^3\sinh(2n-1)\pi/3}\sin\frac{(2n-1)\pi x}{6}$

\dst$\alpha_n$
$=\frac{2}{a}\int_0^a f(x)\cos\frac{(2n-1)\pi x}{2a}$
dx

20 \dst$u(x,y)=\sum_{n=1}^\infty \alpha_n\frac{\sinh(2n-1)\pi(b-y)/2a}{\sinh(2n-1)\pi b/2a}\sin\frac{(2n-1)\pi x}{2a}$,

$u(x,y)=$ \dst$\frac{32}{\pi^3}\sum_{n=1}^\infty \left[(-1)^n5+\frac{18}{(2n-1)\pi}\right]\frac{\sinh(2n-1)\pi(2-y)/2}{(2n-1)^3\sinh(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}$.

\dst$\alpha_n$
$=\frac{2}{a}\int_0^a f(x)\sin\frac{(2n-1)\pi x}{2a}$
dx

21 \dst$u(x,y)=\sum_{n=1}^\infty \alpha_n\frac{\cosh n\pi(y-b)/a}{\cosh n\pi b/a}\sin\frac{n\pi x}{a}$, \dst$\alpha_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$

$u(x,y)$
$=$ \dst
$-12\sum_{n=1}^\infty (-1)^n\frac{\cosh n(y-2)}{n^3\cosh 2n}\sin nx$

22 \dst$u(x,y)=\alpha_0+\sum_{n=1}^\infty \alpha_n\frac{\cosh n\pi y/a}{\cosh n\pi b/a}\cos\frac{n\pi x}{a}$, \dst$\alpha_0=\frac{1}{a}\int_0^a f(x)\,dx$,

\dst$u(x,y)=\frac{\pi^4}{30}-3\sum_{n=1}^\infty \frac{1}{n^4}\frac{\cosh 2ny}{\cos 2n}\cos 2nx$

\dst$\alpha_n$
$=\frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}$
$dx,\quad n$
≥ 1

23 \dst$u(x,y)=\frac{a}{\pi}\sum_{n=1}^\infty \alpha_n \frac{\sinh n\pi(y-b)/a}{n\cosh n\pi b/a}\sin\frac{n\pi x}{a}$, \dst$\alpha_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$

$u(x,y)$
$=$ \dst$\frac{4}{\pi}\sum_{n=1}^\infty (-1)^{n+1}\frac{\sinh(2n-1)(y-1)}{(2n-1)^3\cosh(2n-1)}\sin(2n$
$-1)x$

24 \dst$u(x,y)=\sum_{n=1}^\infty \alpha_n\frac{\cosh n\pi x/b}{\cosh n\pi a/b}\sin\frac{n\pi y}{b}$, \dst$\alpha_n=\frac{2}{b}\int_0^b g(y)\sin\frac{n\pi y}{b}\,dy$

$u(x,y)$
$=$ \dst$\frac{96}{\pi^5}\sum_{n=1}^\infty \frac{\cosh(2n-1)\pi x}{(2n-1)^5\cosh(2n-1)\pi}\sin(2n$
$-1)\pi y.$

25 \dst$u(x,y)=\sum_{n=1}^\infty \alpha_n\frac{\cosh(2n-1)\pi x/2b}{\cosh(2n-1)\pi a/2b}\cos\frac{(2n-1)\pi y}{2b}$,

$u(x,y)=-$\dst$\frac{128}{\pi^3}\sum_{n=1}^\infty (-1)^n\frac{\cosh(2n-1)\pi x/4}{(2n-1)^3\cosh(2n-1)\pi/2}\cos\frac{(2n-1)\pi y}{4}.$

\dst$\alpha_n$
$=\frac{2}{b}\int_0^b g(y)\cos\frac{(2n-1)\pi y}{2b}$
dy

26 \dst$u(x,y)=\frac{b}{\pi}\sum_{n=1}^\infty \alpha_n\frac{\cosh n\pi x/b}{n\sinh n\pi a/b}\sin\frac{n\pi y}{b}$, \dst$\alpha_n=\frac{2}{b}\int_0^b g(y)\sin\frac{n\pi y}{b}\,dy$

$u(x,y)$
$=$ \dst$\frac{64}{\pi^3}\sum_{n=1}^\infty (-1)^{n+1}\frac{\cosh(2n-1)\pi x/4}{(2n-1)^3\sinh(2n-1)\pi/4}\sin\frac{(2n-1)\pi y}{4}$

27 \dst$u(x,y)=-\frac{2b}{\pi}\sum_{n=1}^\infty \alpha_n\frac{\cosh(2n-1)\pi(x-a)/2b}{(2n-1)\sinh(2n-1)\pi a/2b}\sin\frac{(2n-1)y}{2b}$,

$u(x,y)=$ \dst$192\sum_{n=1}^\infty \left[1+(-1)^n\frac{4}{(2n-1)\pi}\right]\frac{\cosh(2n-1)(x-1)/2}{(2n-1)^4\sinh(2n-1)/2}\sin\frac{(2n-1)y}{2}.$

\dst$\alpha_n$
$=\frac{2}{b}\int_0^b g(y)\sin\frac{(2n-1)\pi y}{2b}$
dy

28 \dst$u(x,y)=\alpha_0(x-a)+\frac{b}{\pi}\sum_{n=1}^\infty \alpha_n\frac{\sinh n\pi(x-a)/b}{n\cosh n\pi a/b}\cos\frac{n\pi y}{b}$, \dst$\alpha_0=\frac{1}{b}\int_0^b g(y)\cos\frac{n\pi y}{b}\,dy$,

$u(x,y)=$ \dst$\frac{\pi(x-2)}{2}-\frac{4}{\pi}\sum_{n=1}^\infty \frac{\sinh(2n-1)(x-2)}{(2n-1)^3\cosh 2(2n-1)}\cos(2n-1)y.$

\dst$\alpha_n$
$=\frac{2}{b}\int_0^b g(y)\cos\frac{n\pi y}{b}$
dy

29 $u(x,y) = \dst\alpha_0 + \sum_{n=1}^{\infty} \alpha_n e^{-n\pi y/a} \cos \frac{n\pi x}{a}, \quad \alpha_0 = \dst\frac{1}{a}\int_0^a f(x)\,dx,$

$$u(x,y) = \frac{\pi^3}{2} - \frac{48}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} e^{-(2n-1)y}\cos(2n-1)x$$

α_n
$= \dst\frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}$
$dx, \quad n$
≥ 1

30 $u(x,y) = \dst\sum_{n=1}^{\infty} \alpha_n e^{-(2n-1)\pi y/2a} \cos\frac{(2n-1)\pi x}{2a}, \quad \alpha_n = \dst\frac{2}{a}\int_0^a f(x)\cos\frac{(2n-1)\pi x}{2a}\,dx$

$u(x,y)$
$= -\dst\frac{288}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3} e^{-(2n-1)\pi y/6}\cos\frac{(2n-1)\pi x}{6}$

31 $u(x,y) = \dst\sum_{n=1}^{\infty} \alpha_n e^{-(2n-1)\pi y/2a} \sin\frac{(2n-1)\pi x}{2a}, \quad \alpha_n = \dst\frac{2}{a}\int_0^a f(x)\sin\frac{(2n-1)\pi x}{2a}\,dx$

$u(x,y)$
$= \dst\frac{32}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} e^{-(2n-1)y/2}\sin\frac{(2n-1)x}{2}.$

32 $u(x,y) = -\dst\frac{a}{\pi}\sum_{n=1}^{\infty}\frac{\alpha_n}{n} e^{-n\pi y/a}\sin\frac{n\pi x}{a}, \quad \alpha_n = \dst\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$

$u(x)$
$= \dst 4\sum_{n=1}^{\infty}\frac{(1+(-1)^n 2)}{n^4} e^{-ny}\sin nx$

33 $u(x,y) = -\dst\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{\alpha_n}{2n-1} e^{-(2n-1)\pi y/2a}\cos\frac{(2n-1)\pi x}{2a}, \quad \alpha_n = \dst\frac{2}{a}\int_0^a f(x)\cos\frac{(2n-1)\pi x}{2a}\,dx$

$u(x,y)$
$= \dst\frac{5488}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1 + \frac{4(-1)^n}{(2n-1)\pi}\right] e^{-(2n-1)\pi y/14}\cos\frac{(2n-1)\pi x}{14}$

34 $u(x,y) = \dst - \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{\alpha_n}{2n-1} e^{-(2n-1)\pi y/2a}\sin\frac{(2n-1)\pi x}{2a}, \quad \alpha_n = \dst\frac{2}{a}\int_0^a f(x)\sin\frac{(2n-1)\pi x}{2a}\,dx$

$u(x,y)$
$= -\dst\frac{2000}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right] e^{-(2n-1)\pi y/10}\sin\frac{(2n-1)\pi x}{10}$

35 Missing dimension or its units for \hspace

36 Missing dimension or its units for \hspace

12.4 Laplace's Equation in Polar Coordinates

1 $u(r,\theta)=\backslash\text{dst}\,\alpha_0\frac{\ln r/\rho}{\ln\rho_0/\rho}+\sum_{n=1}^{\infty}\frac{r^n\rho^{-n}-\rho^n r^{-n}}{\rho_0^n\rho^{-n}-\rho^n\rho_0^{-n}}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\ \backslash\text{dst}\,\alpha_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta,$

and

$\backslash\text{dst}\,\alpha_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\cos n\theta\, d\theta,$

$\backslash\text{dst}\,\beta_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\sin n\theta\, d\theta,\ n=1,\,2,\,3,\,\dots$

2 $\backslash\text{dst}\,u(r,\theta)=\sum_{n=1}^{\infty}\alpha_n\backslash\text{dst}\,\frac{\rho_0^{-n\pi/\gamma}r^{n\pi/\gamma}-\rho_0^{n\pi/\gamma}r^{-n\pi/\gamma}}{\rho_0^{-n\pi/\gamma}\rho^{n\pi/\gamma}-\rho_0^{n\pi/\gamma}\rho^{-n\pi/\gamma}}\sin\frac{n\pi\theta}{\gamma}$

$\alpha_n=\backslash\text{dst}\,\frac{1}{\gamma}\int_0^{\gamma}f(\theta)\sin\frac{n\pi\theta}{\gamma}\, d\theta,\ n=1,\,2,\,3,\dots$

3 $\backslash\text{dst}\,u(r,\theta)=\rho\alpha_0\backslash\text{dst}\ln\frac{r}{\rho_0}+\frac{\rho\gamma}{\pi}\sum_{n=1}^{\infty}\frac{\alpha_n}{n}\backslash\text{dst}\,\frac{\rho_0^{-n\pi/\gamma}r^{n\pi/\gamma}-\rho_0^{n\pi/\gamma}r^{-n\pi/\gamma}}{\rho_0^{-n\pi/\gamma}\rho^{n\pi/\gamma}+\rho_0^{n\pi/\gamma}\rho^{-n\pi/\gamma}}\backslash\text{dst}\cos\frac{n\pi\theta}{\gamma}$

$\alpha_0=\backslash\text{dst}\,\frac{1}{\gamma}\int_0^{\gamma}f(\theta)\, d\theta,$

$\alpha_n=\backslash\text{dst}\,\frac{2}{\gamma}\int_0^{\gamma}f(\theta)\cos\frac{n\pi\theta}{\gamma}\, d\theta,\ n=1,\,2,\,3,\dots$

4 $u(r,\theta)=\dst\sum_{n=1}^\infty \alpha_n\dst\frac{r^{(2n-1)\pi/2\gamma}}{\rho^{(2n-1)\pi/2\gamma}}\cos\frac{(2n-1)\pi\theta}{2\gamma}$

α_n
$=\dst\frac{2}{\gamma}\int_0^\gamma f(\theta)\cos\frac{(2n-1)\pi\theta}{2\gamma}$
$d\theta,\ n$
$=1,\ 2,$
$3,\dots$

5 $u(r,\theta)=\dst\frac{2\gamma\rho_0}{\pi}\sum_{n=1}^\infty\frac{\alpha_n}{2n-1}\frac{\rho^{-(2n-1)\pi/2\gamma}r^{(2n-1)\pi/2\gamma}+\rho^{(2n-1)\pi/2\gamma}r^{-(2n-1)\pi/2\gamma}}{\rho^{-(2n-1)\pi/2\gamma}\rho_0^{(2n-1)\pi/2\gamma}-\rho^{(2n-1)\pi/2\gamma}\rho_0^{-(2n-1)\pi/2\gamma}}\dst\sin\frac{(2n-1)\pi\theta}{2\gamma},$

α_n
$=\dst\frac{2}{\gamma}\int_0^\gamma g(\theta)\sin\frac{(2n-1)\pi\theta}{2\gamma}$
$d\theta,\quad n$
$=1,\ 2,$
$3,\dots$

6 $\dst u(r,\theta)=\alpha_0+\sum_{n=1}^\infty\alpha_n\dst\frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}}\cos\frac{n\pi\theta}{\gamma}\qquad \alpha_0=\dst\frac{1}{\gamma}\int_0^\gamma f(\theta)\,d\theta,$

α_n
$=\dst\frac{2}{\gamma}\int_0^\gamma f(\theta)\cos\frac{n\pi\theta}{\gamma}$
$d\theta,\ n$
$=1,\ 2,$
$3,\dots$

7 $\dst v_n(r,\theta)=\frac{r^n}{n\rho^{n-1}}(\alpha_n\cos n\theta+\sin n\theta)$

$$\dst\beta_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta,\ n=1,\ 2,\ 3,\dots$$

$u(r,\theta)$
$=\dst c+\sum_{n=1}^\infty\frac{r^n}{n\rho^{n-1}}(\alpha_n\cos n\theta+\beta_n\sin n\theta)$
α_n
$=\dst\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta,$