11 Boundary Value Problems and Fourier Expansions

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IN THIS CHAPTER we develop series representations of functions that will be used to solve partial differential equations in Chapter 12.

SECTION 11.1 deals with five boundary value problems for the differential equation

$$y'' + \lambda y = 0.$$

They are related to problems in partial differential equations that will be discussed in Chapter 12. We define what is meant by eigenvalues and eigenfunctions of the boundary value problems, and show that the eigenfunctions have a property called orthogonality.

SECTION 11.2 introduces Fourier series (http://www-history.mcs.st-and.ac.uk/Mathematicians/Fourier.html), which are expansions of given functions in term of sines and cosines.

SECTION 11.3 deals with expansions of functions in terms of the eigenfunctions of four of the eigenvalue problems discussed in Section 11.1. They are all related to the Fourier series discussed in Section 11.2.

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Answers to selected exercises in this chapter

11.1 Eigenvalue Problems for $y'' + \lambda y = 0$

In Chapter 12 we'll study partial differential equations that arise in problems of heat conduction, wave propagation, and potential theory. The purpose of this chapter is to develop tools required to solve these equations. In this section we consider the following problems, where λ is a real number and $L > 0$:

Problem 1: $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$

Problem 2: $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$

Problem 3: $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$

Problem 4: $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$

Problem 5: $y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L)$

In each problem the conditions following the differential equation are called boundary conditions. Note that the boundary conditions in Problem 5, unlike those in Problems 1-4, don't require that y or y' be zero at the boundary points, but only that y have the same value at $x = \pm L$, and that y' have the same value at $x = \pm L$. We say that the boundary conditions in Problem 5 are periodic.

Obviously, $y \equiv 0$ (the trivial solution) is a solution of Problems 1-5 for any value of λ. For most values of λ, there are no other solutions. The interesting question is this:

For what values of λ does the problem have nontrivial solutions, and what are they?

A value of λ for which the problem has a nontrivial solution is an eigenvalue of the problem, and the nontrivial solutions are λ- eigenfunctions, or eigenfunctions associated with λ. Note that a nonzero constant multiple of a λ-eigenfunction is again a λ-eigenfunction.

Problems 1-5 are called eigenvalue problems. Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions. We'll take it as given here that all the eigenvalues of Problems 1-5 are real numbers. This is proved in a more general setting in Section 13.2.

Theorem 11.1.1

Problems 1–5 have no negative eigenvalues. Moreover, $\lambda = 0$ is an eigenvalue of Problems 2 and $5,$ with associated eigenfunction $y_0 = 1,$ but $\lambda = 0$ isn't an eigenvalue of Problems $1,$ $3,$ or $4.$

Proof We consider Problems 1-4, and leave Problem 5 to you (Exercise 1). If $y'' + \lambda y = 0$, then $y(y'' + \lambda y) = 0$, so

$$\int_0^L y(x)(y''(x) + \lambda y(x))\, dx = 0;$$

therefore,

$$\lambda \int_0^L y^2(x)\, dx = -\int_0^L y(x) y''(x)\, dx. \tag{11.1.1}$$

Integration by parts yields

$$\begin{aligned} \backslash\mathrm{dst} \int_0^L y(x)y''(x)\, dx &= \backslash\mathrm{dst}\, y(x)y'(x) \backslash\mathrm{lims} 0 L - \int_0^L (y'(x))^2\, dx \\ &= \backslash\mathrm{dst}\, y(L)y'(L) - y(0)y'(0) - \int_0^L (y'(x))^2\, dx. \end{aligned} \tag{11.1.2}$$

However, if y satisfies any of the boundary conditions of Problems 1-4, then

$$y(L)y'(L) - y(0)y'(0) = 0;$$

hence, (11.1.1) and (11.1.2) imply that

$$\lambda \int_0^L y^2(x)\,dx = \int_0^L (y'(x))^2\,dx.$$

If $y \not\equiv 0$, then $\int_0^L y^2(x)\,dx > 0$. Therefore $\lambda \geq 0$ and, if $\lambda = 0$, then $y'(x) = 0$ for all x in $(0, L)$ (why?), and y is constant on $(0, L)$. Any constant function satisfies the boundary conditions of Problem 2, so $\lambda = 0$ is an eigenvalue of Problem 2 and any nonzero constant function is an associated eigenfunction. However, the only constant function that satisfies the boundary conditions of Problems 1, 3, or 4 is $y \equiv 0$. Therefore $\lambda = 0$ isn't an eigenvalue of any of these problems.

Example 11.1.1 *(Problem 1)*

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0. \tag{11.1.3}$$

Solution From Theorem 11.1.1, any eigenvalues of (11.1.3) must be positive. If y satisfies (11.1.3) with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin\sqrt{\lambda}\,x$. Now the boundary condition $y(L) = 0$ implies that $c_2 \sin\sqrt{\lambda}\,L = 0$. To make $c_2 \sin\sqrt{\lambda}\,L = 0$ with $c_2 \neq 0$, we must choose $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. Therefore $\lambda_n = n^2\pi^2/L^2$ is an eigenvalue and

$$y_n = \sin\frac{n\pi x}{L}$$

is an associated eigenfunction. ∎

For future reference, we state the result of Example 11.1.1 as a theorem.

Theorem 11.1.2

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots .$$

There are no other eigenvalues.

We leave it to you to prove the next theorem about Problem 2 by an argument like that of Example 11.1.1 (Exercise 17).

Theorem 11.1.3

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$, and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \cos\frac{n\pi x}{L}, n = 1, 2, 3 \dots .$$

There are no other eigenvalues.

Example 11.1.2 *(Problem 3)*

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0. \tag{11.1.4}$$

Solution From Theorem 11.1.1, any eigenvalues of (11.1.4) must be positive. If y satisfies (11.1.4) with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x,$$

where c_1 and c_2 are constants. The boundary condition $y(0)=0$ implies that $c_1=0$. Therefore $y = c_2 \sin\sqrt{\lambda}\,x$. Hence, $y' = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$ and the boundary condition $y'(L)=0$ implies that $c_2\cos\sqrt{\lambda}\,L = 0$. To make $c_2\cos\sqrt{\lambda}\,L=0$ with $c_2\neq 0$ we must choose

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2L},$$

where n is a positive integer. Then $\lambda_n = (2n-1)^2\pi^2/4L^2$ is an eigenvalue and

$$y_n = \sin\frac{(2n-1)\pi x}{2L}$$

is an associated eigenfunction. ∎

For future reference, we state the result of Example 11.1.2 as a theorem.

Theorem 11.1.4

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$, with associated eigenfunctions

$$y_n = \sin\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \dots.$$

There are no other eigenvalues.

We leave it to you to prove the next theorem about Problem 4 by an argument like that of Example 11.1.2 (Exercise 18).

Theorem 11.1.5

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$, with associated eigenfunctions

$$y_n = \cos\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \ldots .$$

There are no other eigenvalues.

Example 11.1.3 *(Problem 5)*

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L). \tag{11.1.5}$$

Solution From Theorem 11.1.1, $\lambda = 0$ is an eigenvalue of (11.1.5) with associated eigenfunction $y_0 = 1$, and any other eigenvalues must be positive. If y satisfies (11.1.5) with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x, \tag{11.1.6}$$

where c_1 and c_2 are constants. The boundary condition $y(-L) = y(L)$ implies that

$$c_1 \cos(-\sqrt{\lambda}\,L) + c_2 \sin(-\sqrt{\lambda}\,L) = c_1 \cos\sqrt{\lambda}\,L + c_2 \sin\sqrt{\lambda}\,L. \tag{11.1.7}$$

Since

$$\cos(-\sqrt{\lambda}\,L) = \cos\sqrt{\lambda}\,L \quad \text{and} \quad \sin(-\sqrt{\lambda}\,L) = -\sin\sqrt{\lambda}\,L, \tag{11.1.8}$$

(11.1.7) implies that

$$c_2 \sin\sqrt{\lambda}\,L = 0. \tag{11.1.9}$$

Differentiating (11.1.6) yields

$$y' = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x\right).$$

The boundary condition $y'(-L) = y'(L)$ implies that

$$-c_1 \sin(-\sqrt{\lambda}\,L) + c_2 \cos(-\sqrt{\lambda}\,L) = -c_1 \sin\sqrt{\lambda}\,L + c_2 \cos\sqrt{\lambda}\,L,$$

and (11.1.8) implies that

$$c_1 \sin \sqrt{\lambda}\, L = 0. \tag{11.1.10}$$

Eqns. (11.1.9) and (11.1.10) imply that $c_1 = c_2 = 0$ unless $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. In this case (11.1.9) and (11.1.10) both hold for arbitrary c_1 and c_2. The eigenvalue determined in this way is $\lambda_n = n^2\pi^2/L^2$, and each such eigenvalue has the linearly independent associated eigenfunctions

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For future reference we state the result of Example 11.1.3 as a theorem.

Theorem 11.1.6

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L),$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$ and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_{1n} = \cos \frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots.$$

There are no other eigenvalues.

Orthogonality

We say that two integrable functions f and g are orthogonal on an interval $[a, b]$ if

$$\int_a^b f(x)g(x)\, dx = 0.$$

More generally, we say that the functions ϕ_1, ϕ_2, …, ϕ_n, … (finitely or infinitely many) are orthogonal on $[a, b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\,dx = 0 \quad \text{whenever} \quad i \neq j.$$

The importance of orthogonality will become clear when we study Fourier series in the next two sections.

Example 11.1.4

Show that the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L},\dots,\cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L},\dots \tag{11.1.11}$$

of Problem 5 are orthogonal on $[-L, L]$.

Solution We must show that

$$\int_{-L}^{L} f(x)g(x)\,dx = 0 \tag{11.1.12}$$

whenever f and g are distinct functions from (11.1.11). If r is any nonzero integer, then

$$\int_{-L}^{L} \cos\frac{r\pi x}{L}\,dx = \frac{L}{r\pi}\sin\frac{r\pi x}{L}\Big|_{-L}^{L} = 0. \tag{11.1.13}$$

and

$$\int_{-L}^{L} \sin\frac{r\pi x}{L}\,dx = -\frac{L}{r\pi}\cos\frac{r\pi x}{L}\Big|_{-L}^{L} = 0.$$

Therefore (11.1.12) holds if $f \equiv 1$ and g is any other function in (11.1.11).

If $f(x) = \cos m\pi x/L$ and $g(x) = \cos n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^{L} f(x)g(x)\,dx = \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx. \tag{11.1.14}$$

To evaluate this integral, we use the identity

$$\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

with $A=m\pi x/L$ and $B=n\pi x/L$. Then (11.1.14) becomes

$$\int_{-L}^{L}f(x)g(x)\,dx=\frac{1}{2}\left[\int_{-L}^{L}\cos\frac{(m-n)\pi x}{L}\,dx+\int_{-L}^{L}\cos\frac{(m+n)\pi x}{L}\,dx\right].$$

Since $m-n$ and $m+n$ are both nonzero integers, (11.1.13) implies that the integrals on the right are both zero. Therefore (11.1.12) is true in this case.

If $f(x)=\sin m\pi x/L$ and $g(x)=\sin n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^{L}f(x)g(x)\,dx=\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx. \tag{11.1.15}$$

To evaluate this integral, we use the identity

$$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

with $A=m\pi x/L$ and $B=n\pi x/L$. Then (11.1.15) becomes

$$\int_{-L}^{L}f(x)g(x)\,dx=\frac{1}{2}\left[\int_{-L}^{L}\cos\frac{(m-n)\pi x}{L}\,dx-\int_{-L}^{L}\cos\frac{(m+n)\pi x}{L}\,dx\right]=0.$$

If $f(x)=\sin m\pi x/L$ and $g(x)=\cos n\pi x/L$ where m and n are positive integers (not necessarily distinct), then

$$\int_{-L}^{L}f(x)g(x)\,dx=\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx=0$$

because the integrand is an odd function and the limits are symmetric about $x=0$. ∎

Exercises 19-22 ask you to verify that the eigenfunctions of Problems 1-4 are orthogonal on $[0,L]$. However, this also follows from a general theorem that we'll prove in Chapter 13.

11.1 Exercises

1. Prove that $\lambda = 0$ is an eigenvalue of Problem 5 with associated eigenfunction $y_0 = 1$, and that any other eigenvalues must be positive. *Hint: See the proof of Theorem 11.1.1.*

In Exercises 2-16 solve the eigenvalue problem.

2. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$

Show answer

3. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$

Show answer

4. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$

Show answer

5. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0$

Show answer

6. $y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi)$

Show answer

7. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0$

Show answer

8. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$

Show answer

9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0$

Show answer

10. $y'' + \lambda y = 0, \quad y(-1) = y(1), \quad y'(-1) = y'(1)$

Show answer

11. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$

Show answer

12. $y'' + \lambda y = 0, \quad y(-2) = y(2), \quad y'(-2) = y'(2)$

Show answer

13. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2) = 0$

Show answer

14. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$

Show answer

15. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1/2) = 0$

Show answer

16. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(5) = 0$

Show answer

17. Prove Theorem 11.1.3.

18. Prove Theorem 11.1.5.

19. Verify that the eigenfunctions

$$\sin\frac{\pi x}{L},\ \sin\frac{2\pi x}{L}, \dots,\ \sin\frac{n\pi x}{L}, \dots$$

of Problem 1 are orthogonal on $[0, L]$.

20. Verify that the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L}, \dots,\ \cos\frac{n\pi x}{L}, \dots$$

of Problem 2 are orthogonal on $[0, L]$.

21. Verify that the eigenfunctions

$$\sin\frac{\pi x}{2L},\ \sin\frac{3\pi x}{2L}, \dots,\ \sin\frac{(2n-1)\pi x}{2L}, \dots$$

of Problem 3 are orthogonal on $[0, L]$.

22. Verify that the eigenfunctions

$$\cos\frac{\pi x}{2L},\ \cos\frac{3\pi x}{2L}, \dots,\ \cos\frac{(2n-1)\pi x}{2L}, \dots$$

of Problem 4 are orthogonal on $[0, L]$.

In Exercises 23-26 solve the eigenvalue problem.

23. $y'' + \lambda y = 0, \quad y(0) = 0, \quad \dst \int_0^L y(x)\, dx = 0$

Show answer

24. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad \dst \int_0^L y(x)\, dx = 0$

Show answer

25. $y'' + \lambda y = 0, \quad y(L) = 0, \quad \dst \int_0^L y(x)\, dx = 0$

Show answer

26. $y'' + \lambda y = 0, \quad y'(L) = 0, \quad \dst \int_0^L y(x)\, dx = 0$

Show answer

11.2 Fourier Expansions I

In Example 11.1.4 and Exercises 11.1. 4–11.1. 22 we saw that the eigenfunctions of Problem 5 are orthogonal on $[-L,L]$ and the eigenfunctions of Problems 1–4 are orthogonal on $[0,L]$. In this section and the next we introduce some series expansions in terms of these eigenfunctions. We'll use these expansions to solve partial differential equations in Chapter 12.

Theorem 11.2.1

Suppose the functions ϕ_1, ϕ_2, ϕ_3, …, are orthogonal on $[a,b]$ and

$$\int_a^b \phi_n^2(x)\, dx \ne 0, \quad n = 1, 2, 3, \dots. \tag{11.2.1}$$

Let c_1, c_2, c_3,… be constants such that the partial sums $f_N(x) = \sum_{m=1}^N c_m\phi_m(x)$ satisfy the inequalities

$$|f_N(x)| \le M, \quad a \le x \le b, \quad N = 1, 2, 3, \dots$$

for some constant $M < \infty$. Suppose also that the series

$$f(x) = \sum_{m=1}^\infty c_m\phi_m(x) \tag{11.2.2}$$

converges and is integrable on $[a,b]$. Then

$$c_n = \frac{\backslash\text{dst} \int_a^b f(x)\phi_n(x)\,dx}{\backslash\text{dst} \int_a^b \phi_n^2(x)\,dx}, \quad n = 1, 2, 3, \dots. \tag{11.2.3}$$

Proof Multiplying (11.2.2) by ϕ_n and integrating yields

$$\int_a^b f(x)\phi_n(x)\,dx = \int_a^b \phi_n(x)\left(\sum_{m=1}^{\infty} c_m\phi_m(x)\right)\,dx. \tag{11.2.4}$$

It can be shown that the boundedness of the partial sums $\{f_N\}_{N=1}^{\infty}$ and the integrability of f allow us to interchange the operations of integration and summation on the right of (11.2.4), and rewrite (11.2.4) as

$$\int_a^b f(x)\phi_n(x)\,dx = \sum_{m=1}^{\infty} c_m \int_a^b \phi_n(x)\phi_m(x)\,dx. \tag{11.2.5}$$

(This isn't easy to prove.) Since

$$\int_a^b \phi_n(x)\phi_m(x)\,dx = 0 \quad \text{if} \quad m \neq n,$$

(11.2.5) reduces to

$$\int_a^b f(x)\phi_n(x)\,dx = c_n \int_a^b \phi_n^2(x)\,dx.$$

Now (11.2.1) implies (11.2.3). ∎

Theorem 11.2.1 motivates the next definition.

Definition 11.2.2

Suppose ϕ_1, ϕ_2, …, ϕ_n,… are orthogonal on $[a,b]$ and $\int_a^b \phi_n^2(x)\,dx \neq 0$, $n = 1$, 2, 3, …. Let f be integrable on $[a,b]$, and define

$$c_n = \frac{\backslash\text{dst} \int_a^b f(x)\phi_n(x)\,dx}{\backslash\text{dst} \int_a^b \phi_n^2(x)\,dx}, \quad n = 1, 2, 3, \dots. \tag{11.2.6}$$

> Then the infinite series $\sum_{n=1}^{\infty} c_n\phi_n(x)$ is called the Fourier expansion of f in terms of the orthogonal set $\{\phi_n\}_{n=1}^{\infty}$, and c_1, c_2, …, c_n, … are called the Fourier coefficients of f with respect to $\{\phi_n\}_{n=1}^{\infty}$. We indicate the relationship between f and its Fourier expansion by
>
> $$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \quad a \le x \le b. \tag{11.2.7}$$

You may wonder why we don't write

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad a \le x \le b,$$

rather than (11.2.7). Unfortunately, this isn't always true. The series on the right may diverge for some or all values of x in $[a,b]$, or it may converge to $f(x)$ for some values of x and not for others. So, for now, we'll just think of the series as being associated with f because of the definition of the coefficients $\{c_n\}$, and we'll indicate this association informally as in (11.2.7).

Fourier Series

We'll now study Fourier expansions in terms of the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L},\ \dots,\ \cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L},\ \dots.$$

of Problem 5. If f is integrable on $[-L,L]$, its Fourier expansion in terms of these functions is called the Fourier series of f on $[-L,L]$. Since

$$\int_{-L}^{L} 1^2\,dx = 2L,$$

$$\int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1+\cos\frac{2n\pi x}{L}\right)dx = \frac{1}{2}\left(x+\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\backslash\text{lims}{-LL} = L,$$

and

$$\int_{-L}^{L}\sin^2\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1-\cos\frac{2n\pi x}{L}\right)dx = \frac{1}{2}\left(x-\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right),\backslash\text{lims}{-LL} = L,$$

we see from (11.2.6) that the Fourier series of f on $[-L,L]$ is

$$a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right),$$

where

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx,$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx,\quad\text{and}\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx,\ n\geq 1.$$

Note that a_0 is the average value of f on $[-L,L]$, while a_n and b_n (for $n\geq 1$) are twice the average values of

$$f(x)\cos\frac{n\pi x}{L}\quad\text{and}\quad f(x)\sin\frac{n\pi x}{L}$$

on $[-L,L]$, respectively.

Convergence of Fourier Series

The question of convergence of Fourier series for arbitrary integrable functions is beyond the scope of this book. However, we can state a theorem that settles this question for most functions that arise in applications.

Definition 11.2.3

A function f is said to be piecewise smooth on $[a,b]$ if:

a. f has at most finitely many points of discontinuity in (a,b);

b. f' exists and is continuous except possibly at finitely many points in (a,b);

c. $f(x_0+)=\lim_{x\to x_0+}f(x)$ and $f'(x_0+)=\lim_{x\to x_0+}f'(x)$ exist if $a\leq x_0<b$;

d. $f(x_0-)=\lim_{x\to x_0-}f(x)$ and $f'(x_0-)=\lim_{x\to x_0-}f'(x)$ exist if $a<x_0\leq b$.

Since f and f' are required to be continuous at all but finitely many points in $[a,b]$, $f(x_0+)=f(x_0-)$ and $f'(x_0+)=f'(x_0-)$ for all but finitely many values of x_0 in (a,b). Recall from Section 8.1 that f is said to have a jump discontinuity at x_0 if $f(x_0+)\neq f(x_0-)$.

The next theorem gives sufficient conditions for convergence of a Fourier series. The proof is beyond the scope of this book.

Theorem 11.2.4

If f is piecewise smooth on $[-L,L]$, then the Fourier series

$$F(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right) \tag{11.2.8}$$

of f on $[-L,L]$ converges for all x in $[-L,L]$; moreover,

$$F(x)=\begin{cases} f(x) & \text{if } -L<x<L \text{ and } f \text{ is continuous at } x\\ \backslash\text{dst}\,\frac{f(x-)+f(x+)}{2} & \text{if } -L<x<L \text{ and } f \text{ is discontinuous at } x\\ \backslash\text{dst}\,\frac{f(-L+)+f(L-)}{2} & \text{if } x=L \text{ or } x=-L.\end{cases}$$

Since $f(x+)=f(x-)$ if f is continuous at x, we can also say that

$$F(x)=\begin{cases} \backslash\text{dst}\,\frac{f(x+)+f(x-)}{2} & \text{if } \ -L<x<L,\\ \backslash\text{dst}\,\frac{f(L-)+f(-L+)}{2} & \text{if } x=\pm L.\end{cases}$$

Note that F is itself piecewise smooth on $[-L,L]$, and $F(x)=f(x)$ at all points in the open interval $(-L,L)$ where f is continuous. Since the series in (11.2.8) converges to $F(x)$ for all x in $[-L,L]$, you may be tempted to infer that the error

$$E_N(x)=\left|F(x)-a_0-\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\right|$$

can be made as small as we please for all x in $[-L,L]$ by choosing N sufficiently large. However, this isn't true if f has a discontinuity somewhere in $(-L,L)$, or if $f(-L+)\neq f(L-)$. Here's the situation in this case.

If f has a jump discontinuity at a point α in $(-L,L)$, there will be sequences of points $\{u_N\}$ and $\{v_N\}$ in $(-L,\alpha)$ and (α,L), respectively, such that

$$\lim_{N\to\infty} u_N = \lim_{N\to\infty} v_N = \alpha$$

and

$$E_N(u_N) \approx .09|f(\alpha-) - f(\alpha+)| \quad \text{and} \quad E_N(v_N) \approx .09|f(\alpha-) - f(\alpha+)|.$$

Thus, the maximum value of the error $E_N(x)$ near α does not approach zero as $N \to \infty$, but just occurs closer and closer to $($and on both sides of$)$ α, and is essentially independent of N.

If $f(-L+) \neq f(L-)$, then there will be sequences of points $\{u_N\}$ and $\{v_N\}$ in $(-L, L)$ such that

$$\lim_{N\to\infty} u_N = -L, \quad \lim_{N\to\infty} v_N = L,$$

$$E_N(u_N) \approx .09|f(-L+) - f(L-)| \quad \text{and} \quad E_N(v_N) \approx .09|f(-L+) - f(L-)|.$$

This is the Gibbs phenomenon (http://www-history.mcs.st-and.ac.uk/Mathematicians/Gibbs.html). Having been alerted to it, you may see it in Figures 11.2.2–11.2.4, below; however, we'll give a specific example at the end of this section.

Example 11.2.1

Find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0, \\ \frac{1}{2}, & 0 < x < 2 \end{cases}$$

on $[-2, 2]$ (Figure 11.2.1). Determine the sum of the Fourier series for $-2 \leq x \leq 2$.

Figure 11.2.1.

Solution Note that wen't bothered to define $f(-2)$, $f(0)$, and $f(2)$. No matter how they may be defined, f is piecewise smooth on $[-2, 2]$, and the coefficients in the Fourier series

$$F(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$

are not affected by them. In any case, Theorem 11.2.4 implies that $F(x)=f(x)$ in $(-2,0)$ and $(0,2)$, where f is continuous, while

$$F(-2)=F(2)=\frac{f(-2+)+f(2-)}{2}=\frac{1}{2}\left(2+\frac{1}{2}\right)=\frac{5}{4}$$

and

$$F(0)=\frac{f(0-)+f(0+)}{2}=\frac{1}{2}\left(0+\frac{1}{2}\right)=\frac{1}{4}.$$

To summarize,

$$F(x)=\begin{cases}\frac{5}{4}, & x=-2\\ -x, & -2<x<0,\\ \frac{1}{4}, & x=0,\\ \frac{1}{2}, & 0<x<2,\\ \frac{5}{4}, & x=2.\end{cases}$$

We compute the Fourier coefficients as follows:

$$a_0=\frac{1}{4}\int_{-2}^{2} f(x)\,dx=\frac{1}{4}\left[\int_{-2}^{0}(-x)\,dx+\int_{0}^{2}\frac{1}{2}\,dx\right]=\frac{3}{4}.$$

If $n\ge 1$, then

$$\begin{aligned}a_n&=\frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx=\frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx+\int_{0}^{2}\frac{1}{2}\cos\frac{n\pi x}{2}\,dx\right]\\&=\frac{2}{n^2\pi^2}(\cos n\pi-1),\end{aligned}$$

and

$$\begin{aligned}b_n&=\frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx=\frac{1}{2}\left[\int_{-2}^{0}(-x)\sin\frac{n\pi x}{2}\,dx+\int_{0}^{2}\frac{1}{2}\sin\frac{n\pi x}{2}\,dx\right]\\&=\frac{1}{2n\pi}(1+3\cos n\pi).\end{aligned}$$

Therefore

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi - 1}{n^2}\cos\frac{n\pi x}{2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1+3\cos n\pi}{n}\sin\frac{n\pi x}{2}.$$

Figure 11.2.2 shows how the partial sum

$$F_m(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{m}\frac{\cos n\pi - 1}{n^2}\cos\frac{n\pi x}{2} + \frac{1}{2\pi}\sum_{n=1}^{m}\frac{1+3\cos n\pi}{n}\sin\frac{n\pi x}{2}$$

approximates $f(x)$ for $m=5$ (dotted curve), $m=10$ (dashed curve), and $m=15$ (solid curve).

Figure 11.2.2.

Even and Odd Functions

Computing the Fourier coefficients of a function f can be tedious; however, the computation can often be simplified by exploiting symmetries in f or some of its terms. To focus on this, we recall some concepts that you studied in calculus. Let u and v be defined on $[-L,L]$ and suppose that

$$u(-x) = u(x) \quad \text{and} \quad v(-x) = -v(x), \quad -L \le x \le L.$$

Then we say that u is an even function and v is an odd function. Note that:

- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.

Example 11.2.2

The functions $u(x)=\cos\omega x$ and $u(x)=x^2$ are even, while $v(x)=\sin\omega x$ and $v(x)=x^3$ are odd. The function $w(x)=e^x$ is neither even nor odd.

You learned parts **(a)** and **(b)** of the next theorem in calculus, and the other parts follow from them (Exercise 1).

Theorem 11.2.5

Suppose u is even and v is odd on $[-L,L]$. Then:

$$\text{\parta}\quad \int_{-L}^{L} u(x)\,dx = 2\int_0^L u(x)\,dx, \quad \text{\partb}\quad \int_{-L}^{L} v(x)\,dx = 0,$$

$$\text{\partc}\quad \int_{-L}^{L} u(x)\cos\frac{n\pi x}{L}\,dx = 2\int_0^L u(x)\cos\frac{n\pi x}{L}\,dx,$$

$$\text{\partd}\quad \int_{-L}^{L} v(x)\sin\frac{n\pi x}{L}\,dx = 2\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx,$$

$$\text{\parte}\quad \int_{-L}^{L} u(x)\sin\frac{n\pi x}{L}\,dx = 0 \quad\text{and}\quad \text{\partf}\quad \int_{-L}^{L} v(x)\cos\frac{n\pi x}{L}\,dx = 0.$$

Example 11.2.3

Find the Fourier series of $f(x) = x^2 - x$ on $[-2, 2]$, and determine its sum for $-2 \le x \le 2$.

Solution Since $L = 2$,

$$F(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2} + b_n\sin\frac{n\pi x}{2}\right)$$

where

$$a_0 = \frac{1}{4}\int_{-2}^{2}(x^2 - x)\,dx, \tag{11.2.9}$$

$$a_n = \frac{1}{2}\int_{-2}^{2}(x^2 - x)\cos\frac{n\pi x}{2}\,dx, \quad n = 1, 2, 3, \dots, \tag{11.2.10}$$

and

$$b_n = \frac{1}{2}\int_{-2}^{2}(x^2 - x)\sin\frac{n\pi x}{2}\,dx, \quad n = 1, 2, 3, \dots. \tag{11.2.11}$$

We simplify the evaluation of these integrals by using Theorem 11.2.5 with $u(x) = x^2$ and $v(x) = x$; thus, from (11.2.9),

$$a_0 = \frac{1}{2}\int_0^2 x^2\,dx = \frac{x^3}{6}\backslash\mathrm{lims}02 = \frac{4}{3}.$$

From (11.2.10),

$$\begin{aligned} a_n &= \int_0^2 x^2 \cos\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}\left[x^2 \sin\frac{n\pi x}{2}\backslash\mathrm{lims}02 - 2\int_0^2 x\sin\frac{n\pi x}{2}\,dx\right] \\ &= \frac{8}{n^2\pi^2}\left[x\cos\frac{n\pi x}{2}\backslash\mathrm{lims}02 - \int_0^2 \cos\frac{n\pi x}{2}\,dx\right] \\ &= \frac{8}{n^2\pi^2}\left[2\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\backslash\mathrm{lims}02\right] = (-1)^n\frac{16}{n^2\pi^2}. \end{aligned}$$

From (11.2.11),

$$\begin{aligned} b_n &= -\int_0^2 x\sin\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}\left[x\cos\frac{n\pi x}{2}\backslash\mathrm{lims}02 - \int_0^2 \cos\frac{n\pi x}{2}\,dx\right] \\ &= \frac{2}{n\pi}\left[2\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\backslash\mathrm{lims}02\right] = (-1)^n\frac{4}{n\pi}. \end{aligned}$$

Therefore

$$F(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}.$$

Theorem 11.2.4 implies that

$$F(x) = \left\{\begin{array}{cc} 4, & x = -2, \\ x^2 - x, & -2 < x < 2, \\ 4, & x = 2. \end{array}\right.$$

Figure 11.2.3 shows how the partial sum

$$F_m(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{m}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2} + \frac{4}{\pi}\sum_{n=1}^{m}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

approximates $f(x)$ for $m = 5$ (dotted curve), $m = 10$ (dashed curve), and $m = 15$ (solid curve). ∎

Figure 11.2.3. *Approximation of $f(x) = x^2 - x$ by partial sums of its Fourier series on $[-2, 2]$*

Theorem 11.2.5 ilmplies the next theorem follows.

Theorem 11.2.6

Suppose f is integrable on $[-L, L]$.

a. If f is even, the Fourier series of f on $[-L, L]$ is

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n \geq 1.$$

b. If f is odd, the Fourier series of f on $[-L, L]$ is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Example 11.2.4

Find the Fourier series of $f(x) = x$ on $[-\pi, \pi]$, and determine its sum for $-\pi \leq x \leq \pi$.

Solution Since f is odd and $L = \pi$,

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = -\frac{2}{n\pi}\left[x\cos nx\backslash\text{lims}0\pi - \int_0^\pi \cos nx\,dx\right] \\ &= -\frac{2}{n}\cos n\pi + \frac{2}{n^2\pi}\sin nx\backslash\text{lims}0\pi = (-1)^{n+1}\frac{2}{n}. \end{aligned}$$

Therefore

$$F(x) = -2\sum_{n=1}^\infty \frac{(-1)^n}{n}\sin nx.$$

Theorem 11.2.4 implies that

$$F(x) = \left\{\begin{array}{cc} 0, & x=-\pi, \\ x, & -\pi<x<\pi, \\ 0, & x=\pi. \end{array}\right.$$

Figure 11.2.4 shows how the partial sum

$$F_m(x) = -2\sum_{n=1}^m \frac{(-1)^n}{n}\sin nx$$

approximates $f(x)$ for $m=5$ (dotted curve), $m=10$ (dashed curve), and $m=15$ (solid curve).

Figure 11.2.4. *Approximation of $f(x)=x$ by partial sums of its Fourier series on $[-\pi,\pi]$*

EXAMPLE 11.2.5

Find the Fourier series of $f(x)=|x|$ on $[-\pi,\pi]$ and determine its sum for $-\pi\le x\le\pi$.

SOLUTION Since f is even and $L=\pi$,

$$F(x) = a_0 + \sum_{n=1}^\infty a_n\cos nx.$$

Since $f(x)=x$ if $x\ge 0$,

$$a_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{x^2}{2\pi}\backslash\mathrm{lims}0\pi = \frac{\pi}{2}$$

and, if $n \geq 1$,

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{n\pi}\left[x\sin nx\backslash\mathrm{lims}0\pi - \int_0^\pi \sin nx\,dx\right] \\ &= \frac{2}{n^2\pi}\cos nx\backslash\mathrm{lims}0\pi = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1].\end{aligned}$$

Therefore

$$F(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=0}\frac{(-1)^n - 1}{n^2}\cos nx. \tag{11.2.12}$$

However, since

$$(-1)^n - 1 = \begin{cases} 0 & \text{if } n = 2m, \\ -2 & \text{if } n = 2m+1, \end{cases}$$

the terms in (11.2.12) for which $n = 2m$ are all zeros. Therefore we only to include the terms for which $n = 2m+1$; that is, we can rewrite (11.2.12) as

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^\infty \frac{1}{(2m+1)^2}\cos(2m+1)x.$$

However, since the name of the index of summation doesn't matter, we prefer to replace m by n, and write

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^\infty \frac{1}{(2n+1)^2}\cos(2n+1)x.$$

Since $|x|$ is continuous for all x and $|-\pi| = |\pi|$, Theorem 11.2.4 implies that $F(x) = |x|$ for all x in $[-\pi, \pi]$.

Example 11.2.6

Find the Fourier series of $f(x) = x(x^2 - L^2)$ on $[-L, L]$, and determine its sum for $-L \leq x \leq L$.

Solution Since f is odd,

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x(x^2-L^2)\sin\frac{n\pi x}{L}\,dx \\ &= -\frac{2}{n\pi}\left[x(x^2-L^2)\cos\frac{n\pi x}{L}\backslash\mathrm{lims}0L - \int_0^L (3x^2-L^2)\cos\frac{n\pi x}{L}\,dx\right] \\ &= \frac{2L}{n^2\pi^2}\left[(3x^2-L^2)\sin\frac{n\pi x}{L}\backslash\mathrm{lims}0L - 6\int_0^L x\sin\frac{n\pi x}{L}\,dx\right] \\ &= \frac{12L^2}{n^3\pi^3}\left[x\cos\frac{n\pi x}{L}\backslash\mathrm{lims}0L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right] = (-1)^n\frac{12L^3}{n^3\pi^3}. \end{aligned}$$

Therefore

$$F(x) = \frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{L}.$$

Theorem 11.2.4 implies that $F(x) = x(x^2-L^2)$ for all x in $[-L, L]$.

Example 11.2.7 *(Gibbs Phenomenon)*

The Fourier series of

$$f(x) = \begin{cases} 0, & -1 < x < -\frac{1}{2}, \\ 1, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

on $[-1, 1]$ is

$$F(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)\pi x.$$

(Verify.) According to Theorem 11.2.4,

$$F(x)=\begin{cases} 0, & -1\le x<-\frac{1}{2},\\ \frac{1}{2}, & x=-\frac{1}{2},\\ 1, & -\frac{1}{2}<x<\frac{1}{2},\\ \frac{1}{2}, & x=\frac{1}{2},\\ 0, & \frac{1}{2}<x\le 1;\end{cases}$$

thus, F (as well as f) has unit jump discontinuities at $x=\pm\frac{1}{2}$. Figures 11.2.5-11.2.7 show the graphs of $y=f(x)$ and

$$y=F_{2N-1}(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)\pi x$$

for $N=10$, 20, and 30. You can see that although F_{2N-1} approximates F (and therefore f) well on larger intervals as N increases, the maximum absolute values of the errors remain approximately equal to $.09$, but occur closer to the discontinuities $x=\pm\frac{1}{2}$ as N increases.

Figure 11.2.5. *The Gibbs Phenomenon: Example 11.2.7,* $N=10$

Figure 11.2.6. *The Gibbs Phenomenon: Example 11.2.7,* $N=20$

Figure 11.2.7. *The Gibbs Phenomenon: Example 11.2.7,* $N=30$

Using Technology

The computation of Fourier coefficients will be tedious in many of the exercises in this chapter and the next. To learn the technique, we recommend that you do some exercises in each section “by hand,” perhaps using the table of integrals at the front of the book. However, we encourage you to use your favorite symbolic computation software in the more difficult problems.

11.2 Exercises

1. Prove Theorem 11.1.5.

In Exercises 2-16 find the Fourier series of f on $[-L,L]$ and determine its sum for $-L\le x\le L$. Where indicated by C *, graph f and*

$$F_m(x) = a_0 + \sum_{n=1}^{m} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

on the same axes for various values of m*.*

2. C $L=1$; $f(x)=2-x$

Show answer

3. $L=\pi$; $f(x)=2x-3x^2$

Show answer

4. $L=1$; $f(x)=1-3x^2$

Show answer

5. $L=\pi$; $f(x)=|\sin x|$

Show answer

6. C $L=\pi$; $f(x)=x\cos x$

Show answer

7. $L=\pi$; $f(x)=|x|\cos x$

Show answer

8. C $L=\pi$; $f(x)=x\sin x$

Show answer

9. $L=\pi$; $f(x)=|x|\sin x$

Show answer

10. $L=1$; $f(x)=\begin{cases} 0, & -1<x<\frac{1}{2}, \\ \cos\pi x, & -\frac{1}{2}<x<\frac{1}{2}, \\ 0, & \frac{1}{2}<x<1 \end{cases}$

Show answer

11. $L=1$; $f(x)=\begin{cases} 0, & -1<x<\frac{1}{2}, \\ x\cos\pi x, & -\frac{1}{2}<x<\frac{1}{2}, \\ 0, & \frac{1}{2}<x<1 \end{cases}$

Show answer

12. $L=1; \quad f(x)=\begin{cases} 0, & -1<x<\frac{1}{2}, \\ \sin\pi x, & -\frac{1}{2}<x<\frac{1}{2}, \\ 0, & \frac{1}{2}<x<1 \end{cases}$

Show answer

13. $L=1; \quad f(x)=\begin{cases} 0, & -1<x<\frac{1}{2}, \\ |\sin\pi x|, & -\frac{1}{2}<x<\frac{1}{2}, \\ 0, & \frac{1}{2}<x<1 \end{cases}$

Show answer

14. $L=1; \quad f(x)=\begin{cases} 0, & -1<x<\frac{1}{2}, \\ x\sin\pi x, & -\frac{1}{2}<x<\frac{1}{2}, \\ 0, & \frac{1}{2}<x<1 \end{cases}$

Show answer

15. C $L=4; \quad f(x)=\begin{cases} 0, & -4<x<0, \\ x, & 0<x<4 \end{cases}$

Show answer

16. C $L=1; \quad f(x)=\begin{cases} x^2, & -1<x<0, \\ 1-x^2, & 0<x<1 \end{cases}$

Show answer

17. L Verify the Gibbs phenomenon for $f(x)=\begin{cases} 2, & -2<x<-1, \\ 1, & -1<x<1, \\ -1, & 1<x<2. \end{cases}$

Show answer

18. L Verify the Gibbs phenomenon for $f(x)=\begin{cases} 2, & -3<x<-2, \\ 3, & -2<x<2, \\ 1, & 2<x<3. \end{cases}$

Show answer

19. Deduce from Example 11.2.5 that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

20. a. Find the Fourier series of $f(x)=e^x$ on $[-\pi,\pi]$.

b. Deduce from **(a)** that

$$\sum_{n=0}^{\infty}\frac{1}{n^2+1}=\frac{\pi\coth\pi-1}{2}.$$

Show answer

21. Find the Fourier series of $f(x)=(x-\pi)\cos x$ on $[-\pi,\pi]$.

Show answer

22. Find the Fourier series of $f(x)=(x-\pi)\sin x$ on $[-\pi,\pi]$.

Show answer

23. Find the Fourier series of $f(x)=\sin kx$ ($k\neq$ integer) on $[-\pi,\pi]$.

Show answer

24. Find the Fourier series of $f(x)=\cos kx$ ($k\neq$ integer) on $[-\pi,\pi]$.

Show answer

25. a. Suppose g' is continuous on $[a,b]$ and $\omega\neq 0$. Use integration by parts to show that there's a constant M such that

$$\left|\int_a^b g(x)\cos\omega x\,dx\right|\le\frac{M}{\omega}\quad\text{and}\quad\left|\int_a^b g(x)\sin\omega x\,dx\right|\le\frac{M}{\omega},\quad\omega>0.$$

b. Show that the conclusion of **(a)** also holds if g is piecewise smooth on $[a,b]$. (This is a special case of Riemann's Lemma (http://www-history.mcs.st-and.ac.uk/Mathematicians/Riemann.html).

c. We say that a sequence $\{\alpha_n\}_{n=1}^{\infty}$ is of order n^{-k} and write $\alpha_n=O(1/n^k)$ if there's a constant M such that

$$|\alpha_n|<\frac{M}{n^k},\quad n=1,2,3,\dots.$$

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be the Fourier coefficients of a piecewise smooth function. Conclude from **(b)** that $a_n=O(1/n)$ and $b_n=O(1/n)$.

26. a. Suppose $f(-L)=f(L)$, $f'(-L)=f'(L)$, f' is continuous, and f'' is piecewise continuous on $[-L,L]$. Use Theorem 11.2.4 and integration by parts to show that

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right),\quad -L\le x\le L,$$

with

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx,$$

$$a_n=-\frac{L}{n^2\pi^2}\int_{-L}^{L}f''(x)\cos\frac{n\pi x}{L}\,dx,\quad \text{and}\quad b_n=-\frac{L}{n^2\pi^2}\int_{-L}^{L}f''(x)\sin\frac{n\pi x}{L}\,dx,\ n\ge 1.$$

b. Show that if, in addition to the assumptions in **(a)**, f'' is continuous and f''' is piecewise continuous on $[-L,L]$, then

$$a_n=\frac{L^2}{n^3\pi^3}\int_{-L}^{L}f'''(x)\sin\frac{n\pi x}{L}\,dx.$$

27. Show that if f is integrable on $[-L,L]$ and

$$f(x+L)=f(x),\quad -L<x<0$$

(Figure 11.2.8), then the Fourier series of f on $[-L,L]$ has the form

$$A_0+\sum_{n=1}^{\infty}\left(A_n\cos\frac{2n\pi}{L}+B_n\sin\frac{2n\pi}{L}\right)$$

where

$$A_0=\frac{1}{L}\int_0^L f(x)\,dx,$$

and

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx,\quad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx,\quad n=1,2,3,\dots.$$

Figure 11.2.8. *$y=f(x)$, where $f(x+L)=f(x),\ -L<x<0$*

Figure 11.2.9. *$y=f(x)$, where $f(x+L)=-f(x),\ -L<x<0$*

28. Show that if f is integrable on $[-L,L]$ and

$$f(x+L)=-f(x),\quad -L<x<0$$

(Figure 11.2.9), then the Fourier series of f on $[-L,L]$ has the form

$$\sum_{n=1}^{\infty}\left(A_n\cos\frac{(2n-1)\pi x}{L}+B_n\sin\frac{(2n-1)\pi x}{L}\right),$$

where

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{L}\,dx\quad\text{and}\quad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{L}\,dx,\quad n=1,2,3,\dots.$$

29. Suppose ϕ_1, ϕ_2, …, ϕ_m are orthogonal on $[a,b]$ and

$$\int_a^b \phi_n^2(x)\,dx \neq 0, \quad n = 1, 2, \dots, m.$$

If a_1, a_2, …, a_m are arbitrary real numbers, define

$$P_m = a_1\phi_1 + a_2\phi_2 + \cdots + a_m\phi_m.$$

Let

$$F_m = c_1\phi_1 + c_2\phi_2 + \cdots + c_m\phi_m,$$

where

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx};$$

that is, c_1, c_2, …, c_m are Fourier coefficients of f.

a. Show that

$$\int_a^b (f(x) - F_m(x))\phi_n(x)\,dx = 0, \quad n = 1, 2, \dots, m.$$

b. Show that

$$\int_a^b (f(x) - F_m(x))^2\,dx \le \int_a^b (f(x) - P_m(x))^2\,dx,$$

with equality if and only if $a_n = c_n$, $n = 1, 2, \dots, m$.

c. Show that

$$\int_a^b (f(x) - F_m(x))^2\,dx = \int_a^b f^2(x)\,dx - \sum_{n=1}^m c_n^2 \int_a^b \phi_n^2\,dx.$$

d. Conclude from **(c)** that

$$\sum_{n=1}^m c_n^2 \int_a^b \phi_n^2(x)\,dx \le \int_a^b f^2(x)\,dx.$$

30. If A_0, A_1, …, A_m and B_1, B_2, …, B_m are arbitrary constants we say that

$$P_m(x) = A_0 + \sum_{n=1}^{m} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right)$$

is a trigonometric polynomial of degree $\leq m$.

Now let

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

be the Fourier series of an integrable function f on $[-L, L]$, and let

$$F_m(x) = a_0 + \sum_{n=1}^{m} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

a. Conclude from Exercise 29**(b)** that

$$\int_{-L}^{L} (f(x) - F_m(x))^2 \, dx \leq \int_{-L}^{L} (f(x) - P_m(x))^2 \, dx,$$

with equality if and only if $A_n = a_n$, $n = 0, 1, \ldots, m$, and $B_n = b_n$, $n = 1, 2, \ldots, m$.

b. Conclude from Exercise 29**(d)** that

$$2a_0^2 + \sum_{n=1}^{m} (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^{L} f^2(x) \, dx$$

for every $m \geq 0$.

c. Conclude from **(b)** that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

11.3 Fourier Expansions II

In this section we discuss Fourier expansions in terms of the eigenfunctions of Problems 1-4 for Section 11.1.

Fourier Cosine Series

From Exercise 11.1. 20, the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\dots,\ \cos\frac{n\pi x}{L},\dots$$

of the boundary value problem

$$y''+\lambda y=0,\quad y'(0)=0,\quad y'(L)=0 \tag{11.3.1}$$

(Problem 2) are orthogonal on $[0,L]$. If f is integrable on $[0,L]$ then the Fourier expansion of f in terms of these functions is called the Fourier cosine series of f on $[0,L]$. This series is

$$a_0+\sum_{n=1}^\infty a_n\cos\frac{n\pi x}{L},$$

where

$$a_0=\frac{\backslash\mathrm{dst}\int_0^L f(x)\,dx}{\backslash\mathrm{dst}\int_0^L dx}=\frac{1}{L}\int_0^L f(x)\,dx$$

and

$$a_n=\frac{\backslash\mathrm{dst}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}{\backslash\mathrm{dst}\int_0^L \cos^2\frac{n\pi x}{L}\,dx}=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\quad n=1,2,3,\dots.$$

Comparing this definition with Theorem 6**(a)** shows that the Fourier cosine series of f on $[0,L]$ is the Fourier series of the function

$$f_1(x)=\left\{\begin{array}{cl} f(-x), & -L<x<0,\\ f(x), & 0\le x\le L,\end{array}\right.$$

obtained by extending f over $[-L,L]$ as an even function (Figure 11.3.1).

Figure 11.3.1.

Applying Theorem 11.2.4 to f_1 yields the next theorem.

THEOREM 11.3.1

If f is piecewise smooth on $[0,L]$, then the Fourier cosine series

> $$C(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$
>
> of f on $[0,L]$, with
>
> $$a_0=\frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n=1,2,3,\dots,$$
>
> converges for all x in $[0,L]$; moreover,
>
> $$C(x)=\begin{cases} f(0+) & \text{if } x=0 \\ f(x) & \text{if } 0<x<L \text{ and } f \text{ is continuous at } x \\ \backslash\mathrm{dst}\frac{f(x-)+f(x+)}{2} & \text{if } 0<x<L \text{ and } f \text{ is discontinuous at } x \\ f(L-) & \text{if } x=L. \end{cases}$$

> **Example 11.3.1**
>
> Find the Fourier cosine series of $f(x)=x$ on $[0,L]$.

Solution The coefficients are

$$a_0=\frac{1}{L}\int_0^L x\,dx=\frac{1}{L}\frac{x^2}{2}\backslash\mathrm{lims}0L=\frac{L}{2}$$

and, if $n\ge 1$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx=\frac{2}{n\pi}\left[x\sin\frac{n\pi x}{L}\backslash\mathrm{lims}0L-\int_0^L \sin\frac{n\pi x}{L}\,dx\right] \\ &= -\frac{2}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx=\frac{2L}{n^2\pi^2}\cos\frac{n\pi x}{L}\backslash\mathrm{lims}0L=\frac{2L}{n^2\pi^2}[(-1)^n-1] \\ &= \begin{cases} \backslash\mathrm{dst}-\frac{4L}{(2m-1)^2\pi^2} & \text{if } n=2m-1, \\ 0 & \text{if } n=2m. \end{cases} \end{aligned}$$

Therefore

$$C(x)=\backslash\mathrm{dst}\frac{L}{2}-\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}.$$

Theorem 11.3.1 implies that

$$C(x)=x, \quad 0\le x\le L.$$

Fourier Sine Series

From Exercise 11.1. 19, the eigenfunctions

$$\sin\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\dots,\ \sin\frac{n\pi x}{L},\dots$$

of the boundary value problem

$$y''+\lambda y=0, \quad y(0)=0, \quad y(L)=0$$

(Problem 1) are orthogonal on $[0,L]$. If f is integrable on $[0,L]$ then the Fourier expansion of f in terms of these functions is called the Fourier sine series of f on $[0,L]$. This series is

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where

$$b_n=\frac{\backslash\text{dst}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}{\backslash\text{dst}\int_0^L \sin^2\frac{n\pi x}{L}\,dx}=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,3,\dots.$$

Comparing this definition with Theorem 6**(b)** shows that the Fourier sine series of f on $[0,L]$ is the Fourier series of the function

$$f_2(x)=\begin{cases} -f(-x), & -L<x<0,\\ f(x), & 0\le x\le L,\end{cases}$$

obtained by extending f over $[-L,L]$ as an odd function (Figure 11.3.2).

Figure 11.3.2.

Applying Theorem 11.2.4 to f_2 yields the next theorem.

Theorem 11.3.2

If f is piecewise smooth on $[0, L]$, then the Fourier sine series

$$S(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

of f on $[0, L]$, with

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

converges for all x in $[0, L]$; moreover,

$$S(x)=\begin{cases} 0 & \text{if } x=0 \\ f(x) & \text{if } 0<x<L \text{ and } f \text{ is continuous at } x \\ \dst\frac{f(x-)+f(x+)}{2} & \text{if } 0<x<L \text{ and } f \text{ is discontinuous at } x \\ 0 & \text{if } x=L. \end{cases}$$

Example 11.3.2

Find the Fourier sine series of $f(x)=x$ on $[0, L]$.

Solution The coefficients are

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx = -\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{L}\lims0L-\int_0^L \cos\frac{n\pi x}{L}\,dx\right] \\ &= (-1)^{n+1}\frac{2L}{n\pi}+\frac{2L}{n^2\pi^2}\sin\frac{n\pi x}{L}\lims0L=(-1)^{n+1}\frac{2L}{n\pi}. \end{aligned}$$

Therefore

$$S(x)=\dst-\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{L}.$$

Theorem 11.3.2 implies that

$$S(x)=\left\{\begin{array}{ll} x, & 0\le x<L,\\ 0, & x=L.\end{array}\right.$$

Mixed Fourier Cosine Series

From Exercise 11.1. 22, the eigenfunctions

$$\cos\frac{\pi x}{2L},\ \cos\frac{3\pi x}{2L},\dots,\ \cos\frac{(2n-1)\pi x}{2L},\dots$$

of the boundary value problem

$$y''+\lambda y=0,\quad y'(0)=0,\quad y(L)=0 \tag{11.3.2}$$

(Problem 4) are orthogonal on $[0,L]$. If f is integrable on $[0,L]$ then the Fourier expansion of f in terms of these functions is

$$\sum_{n=1}^\infty c_n\cos\frac{(2n-1)\pi x}{2L},$$

where

$$c_n=\frac{\backslash\mathrm{dst}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx}{\backslash\mathrm{dst}\int_0^L \cos^2\frac{(2n-1)\pi x}{L}\,dx}=\frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.$$

We’ll call this expansion the mixed Fourier cosine series of f on $[0,L]$, because the boundary conditions of (11.3.2) are “mixed” in that they require y to be zero at one boundary point and y' to be zero at the other. By contrast, the “ordinary” Fourier cosine series is associated with (11.3.1), where the boundary conditions require that y' be zero at both endpoints.

It can be shown (Exercise 57) that the mixed Fourier cosine series of f on $[0,L]$ is simply the restriction to $[0,L]$ of the Fourier cosine series of

$$f_3(x)=\left\{\begin{array}{ll} f(x), & 0\le x\le L,\\ -f(2L-x), & L<x\le 2L\end{array}\right.$$

on $[0,2L]$ (Figure 11.3.3).

Figure 11.3.3.

Applying Theorem 11.3.1 with f replaced by f_3 and L replaced by $2L$ yields the next theorem.

Theorem 11.3.3

If f is piecewise smooth on $[0,L]$, then the mixed Fourier cosine series

$$C_M(x)=\sum_{n=1}^\infty c_n\cos\frac{(2n-1)\pi x}{2L}$$

of f on $[0,L]$, with

$$c_n=\frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx,$$

converges for all x in $[0,L]$; moreover,

$$C_M(x)=\left\{\begin{array}{cl} f(0+) & \text{if } x=0\\ f(x) & \text{if } 0<x<L \text{ and } f \text{ is continuous at } x\\ \dst\frac{f(x-)+f(x+)}{2} & \text{if } 0<x<L \text{ and } f \text{ is discontinuous at } x\\ 0 & \text{if } x=L.\end{array}\right.$$

Example 11.3.3

Find the mixed Fourier cosine series of $f(x)=x-L$ on $[0,L]$.

Solution The coefficients are

$$\begin{aligned} c_n&=\frac{2}{L}\int_0^L(x-L)\cos\frac{(2n-1)\pi x}{2L}\,dx\\ &=\frac{4}{(2n-1)\pi}\left[(x-L)\sin\frac{(2n-1)\pi x}{2L}\lims0L-\int_0^L\sin\frac{(2n-1)\pi x}{2L}\,dx\right]\\ &=\frac{8L}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi x}{2L}\lims0L=-\frac{8L}{(2n-1)^2\pi^2}.\end{aligned}$$

Therefore

$$C_M(x) = -\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2L}.$$

Theorem 11.3.3 implies that

$$C_M(x) = x - L, \quad 0 \le x \le L.$$

Mixed Fourier Sine Series

From Exercise 11.1. 21, the eigenfunctions

$$\sin\frac{\pi x}{2L},\ \sin\frac{3\pi x}{2L}, \dots,\ \sin\frac{(2n-1)\pi x}{2L}, \dots$$

of the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

(Problem 3) are orthogonal on $[0, L]$. If f is integrable on $[0, L]$, then the Fourier expansion of f in terms of these functions is

$$\sum_{n=1}^{\infty} d_n \sin\frac{(2n-1)\pi x}{2L},$$

where

$$d_n = \frac{\dst \int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx}{\dst \int_0^L \sin^2\frac{(2n-1)\pi x}{2L}\,dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx.$$

We'll call this expansion the mixed Fourier sine series of f on $[0, L]$.

It can be shown (Exercise 58) that the mixed Fourier sine series of f on $[0, L]$ is simply the restriction to $[0, L]$ of the Fourier sine series of

$$f_4(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(2L - x), & L < x \le 2L, \end{cases}$$

on $[0, 2L]$ (Figure 11.3.4).

Figure 11.3.4.

Applying Theorem 11.3.2 with f replaced by f_4 and L replaced by $2L$ yields the next theorem.

Theorem 11.3.4

If f is piecewise smooth on $[0, L]$, then the mixed Fourier sine series

$$S_M(x) = \sum_{n=1}^{\infty} d_n \sin \frac{(2n-1)\pi x}{2L}$$

of f on $[0, L]$, with

$$d_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx,$$

converges for all x in $[0, L]$; moreover,

$$S_M(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) & \text{if } 0 < x < L \text{ and } f \text{ is continuous at } x \\ \text{\dst} \frac{f(x-)+f(x+)}{2} & \text{if } 0 < x < L \text{ and } f \text{ is discontinuous at } x \\ f(L-) & \text{if } x = L. \end{cases}$$

Example 11.3.4

Find the mixed Fourier sine series of $f(x) = x$ on $[0, L]$.

Solution The coefficients are

$$\begin{aligned} d_n &= \frac{2}{L} \int_0^L x \sin \frac{(2n-1)\pi x}{2L}\, dx \\ &= -\frac{4}{(2n-1)\pi} \left[x \cos \frac{(2n-1)\pi x}{2L} \text{\lims} 0L - \int_0^L \cos \frac{(2n-1)\pi x}{2L}\, dx \right] \\ &= \frac{4}{(2n-1)\pi} \int_0^L \cos \frac{(2n-1)\pi x}{2L}\, dx \\ &= \frac{8L}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi x}{2L} \text{\lims} 0L = (-1)^{n+1} \frac{8L}{(2n-1)^2\pi^2}. \end{aligned}$$

Therefore

$$S_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2L}.$$

Theorem 11.3.4 implies that

$$S_M(x) = x, \quad 0 \le x \le L.$$

A Useful Observation

In applications involving expansions in terms of the eigenfunctions of Problems 1-4, the functions being expanded are often polynomials that satisfy the boundary conditions of the problem under consideration. In this case the next theorem presents an efficient way to obtain the coefficients in the expansion.

Theorem 11.3.5

(a) If $f'(0) = f'(L) = 0$, f'' is continuous, and f''' is piecewise continuous on $[0, L]$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \le x \le L, \tag{11.3.3}$$

with

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx \quad \text{and} \quad a_n = \frac{2L^2}{n^3\pi^3} \int_0^L f'''(x) \sin \frac{n\pi x}{L}\, dx, \quad n \ge 1. \tag{11.3.4}$$

Now suppose f' is continuous and f'' is piecewise continuous on $[0, L]$.

(b) If $f(0) = f(L) = 0$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 \le x \le L,$$

with

$$b_n = -\frac{2L}{n^2\pi^2}\int_0^L f''(x)\sin\frac{n\pi x}{L}\,dx. \tag{11.3.5}$$

(c) If $f'(0) = f(L) = 0$, then

$$f(x) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2L}, \quad 0 \le x \le L,$$

with

$$c_n = -\frac{8L}{(2n-1)^2\pi^2}\int_0^L f''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx. \tag{11.3.6}$$

(d) If $f(0) = f'(L) = 0$, then

$$f(x) = \sum_{n=1}^{\infty} d_n \sin\frac{(2n-1)\pi x}{2L}, \quad 0 \le x \le L,$$

with

$$d_n = -\frac{8L}{(2n-1)^2\pi^2}\int_0^L f''(x)\sin\frac{(2n-1)\pi x}{2L}\,dx. \tag{11.3.7}$$

Proof We'll prove **(a)** and leave the rest to you (Exercises 35, 42, and 50). Since f is continuous on $[0, L]$, Theorem 11.3.1 implies (11.3.3) with a_0, a_1, a_2,... as defined in Theorem 11.3.1. We already know that a_0 is as in (11.3.4). If $n \ge 1$, integrating twice by parts yields

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \\ &= \frac{2}{n\pi}\left[f(x)\sin\frac{n\pi x}{L}\lims0L - \int_0^L f'(x)\sin\frac{n\pi x}{L}\,dx\right] \\ &= -\frac{2}{n\pi}\int_0^L f'(x)\sin\frac{n\pi x}{L}\,dx \text{ (since } \sin 0 = \sin n\pi = 0) \\ &= \frac{2L}{n^2\pi^2}\left[f'(x)\cos\frac{n\pi x}{L}\lims0L - \int_0^L f''(x)\cos\frac{n\pi x}{L}\right]dx \\ &= -\frac{2L}{n^2\pi^2}\int_0^L f''(x)\cos\frac{n\pi x}{L}\,dx \text{ (since } f'(0) = f'(L) = 0) \\ &= -\frac{2L^2}{n^3\pi^3}\left[f''(x)\sin\frac{n\pi x}{L}\lims0L - \int_0^L f'''(x)\sin\frac{n\pi x}{L}\,dx\right] \\ &= \frac{2L^2}{n^3\pi^3}\int_0^L f'''(x)\sin\frac{n\pi x}{L}\,dx \text{ (since } \sin 0 = \sin n\pi = 0). \end{aligned}$$

(By an argument similar to one used in the proof of Theorem 8.3.1, the last integration by parts is legitimate in the case where f''' is undefined at finitely many points in $[0, L]$, so long as it's piecewise continuous on $[0, L]$.) This completes the proof.

Example 11.3.5

Find the Fourier cosine expansion of $f(x) = x^2(3L - 2x)$ on $[0, L]$.

Solution Here

$$a_0 = \frac{1}{L}\int_0^L (3Lx^2 - 2x^3)\,dx = \frac{1}{L}\left(Lx^3 - \frac{x^4}{2}\right)\lims0L = \frac{L^3}{2}$$

and

$$a_n = \frac{2}{L}\int_0^L (3Lx^2 - 2x^3)\cos\frac{n\pi x}{L}\,dx, \quad n \geq 1.$$

Evaluating this integral directly is laborious. However, since $f'(x) = 6Lx - 6x^2$, we see that $f'(0) = f'(L) = 0$. Since $f'''(x) = -12$, we see from (11.3.4) that if $n \geq 1$ then

$$\begin{aligned} a_n &= -\frac{24L^2}{n^3\pi^3}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{24L^3}{n^4\pi^4}\cos\frac{n\pi x}{L}\lims0L = \frac{24L^3}{n^4\pi^4}[(-1)^n - 1] \\ &= \begin{cases} \dst -\frac{48L^3}{(2m-1)^4\pi^4} & \text{if } n = 2m-1, \\ 0 & \text{if } n = 2m. \end{cases} \end{aligned}$$

Therefore

$$C(x)=\frac{L^3}{2}-\frac{48L^3}{\pi^4}\sum_{n=1}^\infty\frac{1}{(2n-1)^4}\cos\frac{(2n-1)\pi x}{L}.$$

Example 11.3.6

Find the Fourier sine expansion of $f(x)=x(x^2-3Lx+2L^2)$ on $[0,L]$.

Solution Since $f(0)=f(L)=0$ and $f''(x)=6(x-L)$, we see from (11.3.5) that

$$\begin{aligned} b_n &= -\frac{12L}{n^2\pi^2}\int_0^L(x-L)\sin\frac{n\pi x}{L}\,dx\\ &= \frac{12L^2}{n^3\pi^3}\left[(x-L)\cos\frac{n\pi x}{L}\backslash\mathrm{lims}0L-\int_0^L\cos\frac{n\pi x}{L}\,dx\right]\\ &= \frac{12L^2}{n^3\pi^3}\left[L-\frac{L}{n\pi}\sin\frac{n\pi x}{L}\backslash\mathrm{lims}0L\right]=\frac{12L^3}{n^3\pi^3}. \end{aligned}$$

Therefore

$$S(x)=\backslash\mathrm{dst}\frac{12L^3}{\pi^3}\sum_{n=1}^\infty\frac{1}{n^3}\sin\frac{n\pi x}{L}.$$

Example 11.3.7

Find the mixed Fourier cosine expansion of $f(x)=3x^3-4Lx^2+L^3$ on $[0,L]$.

Solution Since $f'(0)=f(L)=0$ and $f''(x)=2(9x-4L)$, we see from (11.3.6) that

$$\begin{aligned} c_n &= -\frac{16L}{(2n-1)^2\pi^2}\int_0^L(9x-4L)\cos\frac{(2n-1)\pi x}{2L}\,dx\\ &= -\frac{32L^2}{(2n-1)^3\pi^3}\left[(9x-4L)\sin\frac{(2n-1)\pi x}{2L}\backslash\mathrm{lims}0L-9\int_0^L\sin\frac{(2n-1)\pi x}{2L}\right]dx\\ &= -\frac{32L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5L+\frac{18L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\backslash\mathrm{lims}0L\right]\\ &= \frac{32L^3}{(2n-1)^3\pi^3}\left[(-1)^n5+\frac{18}{(2n-1)\pi}\right]. \end{aligned}$$

Therefore

$$C_M(x) = \dst\frac{32L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n5+\frac{18}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}.$$

Example 11.3.8

Find the mixed Fourier sine expansion of

$$f(x) = x(2x^2-9Lx+12L^2)$$

on $[0,L]$.

Solution Since $f(0)=f'(L)=0$, and $f''(x)=6(2x-3L)$, we see from (11.3.7) that

$$\begin{aligned} d_n &= -\dst\frac{48L}{(2n-1)^2\pi^2}\int_0^L(2x-3L)\sin\frac{(2n-1)\pi x}{2L}\,dx \\ &= \dst\frac{96L^2}{(2n-1)^3\pi^3}\left[(2x-3L)\cos\frac{(2n-1)\pi x}{2L}\lims0L - 2\int_0^L\cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\ &= \dst\frac{96L^2}{(2n-1)^3\pi^3}\left[3L-\frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\lims0L\right] \\ &= \dst\frac{96L^3}{(2n-1)^3\pi^3}\left[3+(-1)^n\frac{4}{(2n-1)\pi}\right]. \end{aligned}$$

Therefore

$$S_M(x) = \dst\frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3+(-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}.$$

11.3 Exercises

In exercises marked by C graph f and some partial sums of the required series. If the interval is $[0,L]$, choose a specific value of L for the graph.

In Exercises 1-10 find the Fourier cosine series.

1. $f(x) = x^2$; $[0, L]$

Show answer

2. C $f(x) = 1 - x$; $[0, 1]$

Show answer

3. C $f(x) = x^2 - 2Lx$; $[0, L]$

Show answer

4. $f(x) = \sin kx$ ($k \neq$ integer); $[0, \pi]$

Show answer

5. C $f(x) = \begin{cases} 1, & 0 \leq x \leq \frac{L}{2} \\ 0, & \frac{L}{2} < x < L; \end{cases}$ $[0, L]$

Show answer

6. $f(x) = x^2 - L^2$; $[0, L]$

Show answer

7. $f(x) = (x - 1)^2$; $[0, 1]$

Show answer

8. $f(x) = e^x$; $[0, \pi]$

Show answer

9. C $f(x) = x(L - x)$; $[0, L]$

Show answer

10. C $f(x) = x(x - 2L)$; $[0, L]$

Show answer

In Exercises 11-17 find the Fourier sine series.

11. C $f(x)=1; \quad [0,L]$

Show answer

12. C $f(x)=1-x; \quad [0,1]$

Show answer

13. $f(x)=\cos kx$ ($k\ne$ integer); $\quad [0,\pi]$

Show answer

14. C $f(x)=\begin{cases} 1, & 0\le x\le \frac{L}{2} \\ 0, & \frac{L}{2}<x<L; \end{cases} \quad [0,L]$

Show answer

15. C $f(x)=\begin{cases} x, & 0\le x\le \frac{L}{2}, \\ L-x, & \frac{L}{2}\le x\le L; \end{cases} \quad [0,L]$.

Show answer

16. C $f(x)=x\sin x; \quad [0,\pi]$

Show answer

17. $f(x)=e^x; \quad [0,\pi]$

Show answer

In Exercises 18-24 find the mixed Fourier cosine series.

18. C $f(x) = 1$; $[0, L]$

Show answer

19. $f(x) = x^2$; $[0, L]$

Show answer

20. C $f(x) = x$; $[0, 1]$

Show answer

21. C $f(x) = \begin{cases} 1, & 0 \le x \le \frac{L}{2} \\ 0, & \frac{L}{2} < x < L; \end{cases}$ $[0, L]$

Show answer

22. $f(x) = \cos x$; $[0, \pi]$

Show answer

23. $f(x) = \sin x$; $[0, \pi]$

Show answer

24. C $f(x) = x(L - x)$; $[0, L]$

Show answer

In Exercises 25-30 find the mixed Fourier sine series.

25. C $f(x)=1; \quad [0,L]$

Show answer

26. $f(x)=x^2; \quad [0,L]$

Show answer

27. C $f(x)=\begin{cases} 1, & 0\le x\le \frac{L}{2} \\ 0, & \frac{L}{2}<x<L; \end{cases} \quad [0,L]$

Show answer

28. $f(x)=\cos x; \quad [0,\pi]$

Show answer

29. $f(x)=\sin x; \quad [0,\pi]$

Show answer

30. C $f(x)=x(L-x); \quad [0,L]$.

Show answer

*In Exercises 31-34 use Theorem 11.3.5**(a)** to find the Fourier cosine series of f on $[0,L]$.*

31. $f(x)=3x^2(x^2-2L^2)$

Show answer

32. $f(x)=x^3(3x-4L)$

Show answer

33. $f(x)=x^2(3x^2-8Lx+6L^2)$

Show answer

34. $f(x)=x^2(x-L)^2$

Show answer

35.

a. Prove Theorem 11.3.5**(b)**.

b. In addition to the assumptions of Theorem 11.3.5**(b)**, suppose $f''(0)=f''(L)=0$, f''' is continuous, and $f^{(4)}$ is piecewise continuous on $[0,L]$. Show that

$$b_n=\frac{2L^3}{n^4\pi^4}\int_0^L f^{(4)}(x)\sin\frac{n\pi x}{L}\,dx, \quad n\ge 1.$$

*In Exercises 36-41 use Theorem 11.3.5**(b)** or, where applicable, Exercise 11.1. 35**(b)**, to find the Fourier sine series of f on $[0, L]$.*

36. C $f(x) = x(L - x)$

Show answer

37. C $f(x) = x^2(L - x)$

Show answer

38. $f(x) = x(L^2 - x^2)$

Show answer

39. $f(x) = x(x^3 - 2Lx^2 + L^3)$

Show answer

40. $f(x) = x(3x^4 - 10L^2x^2 + 7L^4)$

Show answer

41. $f(x) = x(3x^4 - 5Lx^3 + 2L^4)$

Show answer

42.

a. Prove Theorem 11.3.5**(c)**.

b. In addition to the assumptions of Theorem 11.3.5**(c)**, suppose $f''(L) = 0$, f'' is continuous, and f''' is piecewise continuous on $[0, L]$. Show that

$$c_n = \frac{16L^2}{(2n-1)^3\pi^3} \int_0^L f'''(x) \sin \frac{(2n-1)\pi x}{2L}\, dx, \quad n \geq 1.$$

*In Exercises 43-49 use Theorem 11.3.5**(c)** or, where applicable, Exercise 11.1. 42**(b)**, to find the mixed Fourier cosine series of f on $[0, L]$.*

43. C $f(x)=x^2(L-x)$

Show answer

44. $f(x)=L^2-x^2$

Show answer

45. $f(x)=L^3-x^3$

Show answer

46. $f(x)=2x^3+3Lx^2-5L^3$

Show answer

47. $f(x)=4x^3+3Lx^2-7L^3$

Show answer

48. $f(x)=x^4-2Lx^3+L^4$

Show answer

49. $f(x)=x^4-4Lx^3+6L^2x^2-3L^4$

Show answer

50.

a. Prove Theorem 11.3.5**(d)**.

b. In addition to the assumptions of Theorem 11.3.5**(d)**, suppose $f''(0)=0$, f'' is continuous, and f''' is piecewise continuous on $[0,L]$. Show that

$$d_n=-\frac{16L^2}{(2n-1)^3\pi^3}\int_0^L f'''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx,\quad n\ge 1.$$

*In Exercises 51-56 use Theorem 11.3.5**(d)** or, where applicable, Exercise 50**(b)**, to find the mixed Fourier sine series of the f on $[0,L]$.*

51. $f(x) = x(2L - x)$

Show answer

52. $f(x) = x^2(3L - 2x)$

Show answer

53. $f(x) = (x - L)^3 + L^3$

Show answer

54. $f(x) = x(x^2 - 3L^2)$

Show answer

55. $f(x) = x^3(3x - 4L)$

Show answer

56. $f(x) = x(x^3 - 2Lx^2 + 2L^3)$

Show answer

57. Show that the mixed Fourier cosine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier cosine series of

$$f_3(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(2L - x), & L < x \le 2L \end{cases}$$

on $[0, 2L]$. Use this to prove Theorem 11.3.3.

58. Show that the mixed Fourier sine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier sine series of

$$f_4(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(2L - x), & L < x \le 2L \end{cases}$$

on $[0, 2L]$. Use this to prove Theorem 11.3.4.

59. Show that the Fourier sine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier sine series of

$$f_3(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(2L - x), & L < x \le 2L \end{cases}$$

on $[0, 2L]$.

60. Show that the Fourier cosine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier cosine series of

$$f_4(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(2L - x), & L < x \le 2L \end{cases}$$

on $[0, 2L]$.

Chapter 11 Boundary Value Problems and Fourier Expansions

11.1 Eigenvalue Problems for $y'' + \lambda y = 0$

2 $\lambda_n = n^2$, $y_n = \sin nx$, $n = 1, 2, 3, \ldots$

3 $\lambda_0 = 0$, $y_0 = 1$; $\lambda_n = n^2$, $y_n = \cos nx$, $n = 1, 2, 3, \ldots$

4 $\lambda_n = \backslash\mathrm{dst}\frac{(2n-1)^2}{4}$, $y_n = \backslash\mathrm{dst}\sin\frac{(2n-1)x}{2}$, $n = 1,2,3,\ldots,$

5 $\lambda_n = \backslash\mathrm{dst}\frac{(2n-1)^2}{4}$, $y_n = \backslash\mathrm{dst}\cos\frac{(2n-1)x}{2}$, $n = 1, 2, 3, \ldots$

6 $\lambda_0 = 0$, $y_0 = 1$, $\lambda_n = n^2$, $y_{1n} = \cos nx$, $y_{2n} = \sin nx$, $n = 1, 2, 3, \ldots$

7 $\lambda_n = n^2\pi^2$, $y_n = \cos n\pi x$, $n = 1, 2, 3, \ldots$

8 $\lambda_n = \backslash\mathrm{dst}\frac{(2n-1)^2\pi^2}{4}$, $y_n = \backslash\mathrm{dst}\cos\frac{(2n-1)\pi x}{2}$, $n = 1, 2, 3, \ldots$

9 $\lambda_n = n^2\pi^2$, $y_n = \sin n\pi x$, $n = 1, 2, 3, \ldots$

10 $\lambda_0 = 0$, $y_0 = 1$, $\lambda_n = n^2\pi^2$, $y_{1n} = \cos n\pi x$, $y_{2n} = \sin n\pi x$, $n = 1, 2, 3, \ldots$

11 $\lambda_n = \backslash\mathrm{dst}\frac{(2n-1)^2\pi^2}{4}$, $y_n = \backslash\mathrm{dst}\sin\frac{(2n-1)\pi x}{2}$, $n = 1, 2, 3, \ldots$

12 $\lambda_0 = 0$, $y_0 = 1$, $\lambda_n = \backslash\mathrm{dst}\frac{n^2\pi^2}{4}$, $y_{1n} = \backslash\mathrm{dst}\cos\frac{n\pi x}{2}$, $y_{2n} = \backslash\mathrm{dst}\sin\frac{n\pi x}{2}$, $n = 1, 2, 3, \ldots$

13 $\lambda_n = \backslash\mathrm{dst}\frac{n^2\pi^2}{4}$, $y_n = \backslash\mathrm{dst}\sin\frac{n\pi x}{2}$, $n = 1, 2, 3, \ldots$

14 $\lambda_n = \backslash\mathrm{dst}\frac{(2n-1)^2\pi^2}{36}$, $y_n = \backslash\mathrm{dst}\cos\frac{(2n-1)\pi x}{6}$, $n = 1, 2, 3, \ldots$

15 $\lambda_n = (2n-1)^2\pi^2$, $y_n = \sin(2n-1)\pi x$, $n = 1, 2, 3, \ldots$

16 $\lambda_n = \backslash\mathrm{dst}\frac{n^2\pi^2}{25}$, $y_n = \backslash\mathrm{dst}\cos\frac{n\pi x}{5}$, $n = 1, 2, 3, \ldots$

23 $\lambda_n = 4n^2\pi^2/L^2$ $y_n = \backslash\mathrm{dst}\sin\frac{2n\pi x}{L}$, $n = 1, 2, 3, \ldots$

24 $\lambda_n = n^2\pi^2/L^2$ $y_n = \backslash\mathrm{dst}\cos\frac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$

25 $\lambda_n = 4n^2\pi^2/L^2$ $y_n = \backslash\mathrm{dst}\sin\frac{2n\pi x}{L}$, $n = 1, 2, 3, \ldots$

26 $\lambda_n = n^2\pi^2/L^2$ $y_n =$ \dst $\cos\frac{n\pi x}{L}$, $n = 1,\ 2,\ 3,\ \ldots.$

11.2 Fourier Expansions I

2 $F(x) = \dst 2+\frac{2}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n}\sin n\pi x$; $\quad F(x)=\left\{\begin{array}{cc} 2, & x=-1,\\ 2-x, & -1<x<1,\\ 2, & x=1\end{array}\right.$

3 $F(x) = \dst -\pi^2-12\sum_{n=1}^\infty \frac{(-1)^n}{n^2}\cos nx-4\sum_{n=1}^\infty \frac{(-1)^n}{n}\sin nx$;

$F(x)$
$=\left\{\begin{array}{cc} -3\pi^2, & x=-\pi,\\ 2x-3x^2, & -\pi<x<\pi,\\ -3\pi^2, & x=\pi\end{array}\right.$

4 $F(x) = -\dst \frac{12}{\pi^2}\sum_{n=1}^\infty (-1)^n\frac{\cos n\pi x}{n^2}$; $\quad F(x)=1-3x^2 \quad -1\le x\le 1$

5 $F(x) = \dst \frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^\infty \frac{1}{4n^2-1}\cos 2nx$; $\quad F(x)=|\sin x|,\ -\pi\le x\le\pi$

6 $F(x) = \dst -\frac{1}{2}\sin x+2\sum_{n=2}^\infty (-1)^n\frac{n}{n^2-1}\sin nx$;; $\quad F(x)=x\cos x,\ -\pi\le x\le\pi$

7 $F(x) = \dst -\frac{2}{\pi}+\frac{\pi}{2}\cos x-\frac{4}{\pi}\sum_{n=1}^\infty \frac{4n^2+1}{(4n^2-1)^2}\cos 2nx$;

$F(x)$
$=|x|\cos x,$
$-\pi\le x$
$\le\pi$

8 $F(x) = \dst 1-\frac{1}{2}\cos x-2\sum_{n=2}^\infty \frac{(-1)^n}{n^2-1}\cos nx$; $\quad F(x)=x\sin x,\ -\pi\le x\le\pi$

9 $F(x) = \dst \frac{\pi}{2}\sin x-\frac{16}{\pi}\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}\sin 2nx$; $\quad F(x)=|x|\sin x,\ -\pi\le x\le\pi$

10 $F(x) = \dst \frac{1}{\pi}+\frac{1}{2}\cos \pi x-\frac{2}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{4n^2-1}\cos 2n\pi x$; $\quad F(x)=f(x),\ -1\le x\le 1$

11 $F(x) = \dst \frac{1}{4\pi}\sin \pi x-\frac{8}{\pi^2}\sum_{n=1}^\infty (-1)^n\frac{n}{(4n^2-1)^2}\sin 2n\pi x$;

$\dst -\frac{1}{4\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n(n+1)}\sin(2n+1)\pi x$

$F(x)$
$=f(x),$
$-1\le x$
≤ 1

12 $F(x) =$ \dst $\frac{1}{2}\sin\pi x - \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{n}{4n^2-1}\sin 2n\pi x$; $F(x) = \begin{cases} 0, & -1 \le x < \frac{1}{2}, \\ -\frac{1}{2}, & x = -\frac{1}{2}, \\ \sin\pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1 \end{cases}$

13 $F(x) =$ \dst $\frac{1}{\pi} + \frac{1}{\pi}\cos\pi x - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1}{n^2-1}\left(1 - n\sin\frac{n\pi}{2}\right)\cos n\pi x$;

$$F(x) = \begin{cases} 0, & -1 \le x < \frac{1}{2}, \\ \frac{1}{2}, & x = -1, \\ |\sin\pi x|, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2}, & x = 1, \\ 0, & \frac{1}{2} < x \le 1 \end{cases}$$

14 $F(x) =$ \dst $\frac{1}{\pi^2} + \frac{1}{4\pi}\cos\pi x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{4n^2+1}{(4n^2-1)^2}\cos 2n\pi x$

\dst $+\frac{1}{4\pi}\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}\cos(2n+1)\pi x$;

$$F(x) = \begin{cases} 0, & -1 \le x < \frac{1}{2}, \\ \frac{1}{4}, & x = -\frac{1}{2}, \\ x\sin\pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{4}, & x = \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1, \end{cases}$$

15 \dst $F(x) = 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{4} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{4}$;

$$F(x) = \begin{cases} 2, & x = -4, \\ 0, & -4 < x < 0, \\ x, & 0 \le x < 4, \\ 2, & x = 4 \end{cases}$$

16 $F(x) =$ \dst $\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi x + \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\pi x$;

$$F(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ x^2, & -1 < x < 0, \\ \frac{1}{2}, & x = 0, \\ 1 - x^2, & 0 < x < 1, \\ \frac{1}{2}, & x = 1 \end{cases}$$

17 $F(x) = \backslash\mathrm{dst}\frac{3}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}$

18 $F(x) = \backslash\mathrm{dst}\frac{5}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi}{3}\cos\frac{n\pi x}{3} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos n\pi - \cos\frac{2n\pi}{3}\right)\sin\frac{n\pi x}{3}$

20 $F(x) = \backslash\mathrm{dst}\frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}\sin nx\right)$

21 $F(x) = \backslash\mathrm{dst} - \pi\cos x - \frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}(-1)^n\frac{n}{n^2-1}\sin nx$

22 $F(x) = \backslash\mathrm{dst}1 - \frac{1}{2}\cos x - \pi\sin x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$

23 $F(x) = -\backslash\mathrm{dst}\frac{2\sin k\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2-k^2}\sin nx$

24 $F(x) = \backslash\mathrm{dst}\frac{\sin k\pi}{\pi}\left[\frac{1}{k} - 2k\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-k^2}\cos nx\right]$

11.3 Fourier Expansions II

1 $C(x) = \backslash\mathrm{dst}\frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}$

2 $C(x) = \backslash\mathrm{dst}\frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi x$

3 $C(x) = \backslash\mathrm{dst} - \frac{2L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{L}$

4 $C(x) = \backslash\mathrm{dst}\frac{1-\cos k\pi}{k\pi} - \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n\cos k\pi]}{n^2-k^2}\cos nx.$

5 $C(x) = \backslash\mathrm{dst}\frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\frac{(2n-1)\pi x}{L}$

6 $C(x) = \backslash\mathrm{dst} - \frac{2L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}$

7 $C(x) = \backslash\mathrm{dst}\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$

8 $C(x) = \backslash\mathrm{dst}\frac{e^{\pi}-1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n e^{\pi}-1]}{(n^2+1)}\cos nx$

9 $\backslash\mathrm{dst}C(x) = \frac{L^2}{6} - \frac{L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{L}$

10 $C(x) = \backslash\mathrm{dst} - \frac{2L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{L}$

11 $S(x) = \backslash\mathrm{dst}\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{L}$

12 $S(x) = \backslash\mathrm{dst}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x$

13 $S(x) = \backslash\mathrm{dst}\frac{2}{\pi}\sum_{n=1}^{\infty}[1-(-1)^n\cos k\pi]\frac{n}{n^2-k^2}\sin nx$

14 $S(x) = \backslash\mathrm{dst}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-\cos\frac{n\pi}{2}\right]\sin\frac{n\pi x}{L}$

15 $S(x) = \backslash\mathrm{dst}\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{L}$

16 $S(x) = \backslash\mathrm{dst}\frac{\pi}{2}\sin x - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)^2}\sin 2nx$

17 $S(x) = \backslash\mathrm{dst} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n[(-1)^n e^{\pi}-1]}{(n^2+1)}\sin nx$

18 $C_M(x) = \backslash\mathrm{dst} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\frac{(2n-1)\pi x}{2L}$

19 $C_M(x) = \backslash\mathrm{dst} - \frac{4L^2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left[1-\frac{8}{(2n-1)^2\pi^2}\right]\cos\frac{(2n-1)\pi x}{2L}$

20 $C_M(x) = -\backslash\mathrm{dst}\frac{4}{\pi}\sum_{n=1}^{\infty}\left[(-1)^n+\frac{2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}.$

21 $C_M(x) =$ \dst $-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos\frac{(2n+1)\pi}{4}\cos\frac{(2n-1)\pi x}{2L}$

22 $C_M(x) =$ \dst $\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{2n-1}{(2n-3)(2n+1)}\cos\frac{(2n-1)x}{2}$

23 $C_M(x) = -$ \dst $\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-3)(2n+1)}\cos\frac{(2n-1)x}{2}$

24 $C_M(x) = -$ \dst $\frac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[1+\frac{4(-1)^n}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$

25 $S_M(x) =$ \dst $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{2L}$

26 $S_M(x) =$ \dst $-\frac{16L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[(-1)^n+\frac{2}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$

27 $S_M(x) =$ \dst $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[1-\cos\frac{(2n-1)\pi)}{4}\right]\sin\frac{(2n-1)\pi x}{2L}$

28 $S_M(x) =$ \dst $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n-1}{(2n-3)(2n+1)}\sin\frac{(2n-1)x}{2}$

29 $S_M(x) =$ \dst $\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-3)(2n+1)}\sin\frac{(2n-1)x}{2}$

30 $S_M(x) =$ \dst $\frac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[(-1)^n+\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$

31 $C(x) =$ \dst $-\frac{7L^4}{5}-\frac{144L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos\frac{n\pi x}{L}$

32 $C(x) =$ \dst $-\frac{2L^4}{5}-\frac{48L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1+(-1)^n2}{n^4}\cos\frac{n\pi x}{L}$

33 $C(x) =$ \dst $\frac{3L^4}{5}-\frac{48L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{2+(-1)^n}{n^4}\cos\frac{n\pi x}{L}$

34 \dst $C(x) = \frac{L^4}{30}-\frac{3L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\cos\frac{2n\pi x}{L}$

36 $S(x) =$ \dst $\frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}$

37 $S(x) =$ \dst $-\frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(1+(-1)^n2)}{n^3}\sin\frac{n\pi x}{L}$

38 $S(x) =$ \dst $-\frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{L}$

39 $S(x) =$ \dst $\frac{96L^4}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin\frac{(2n-1)\pi x}{L}$

40 $S(x) =$ \dst $-\frac{720L^5}{\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\sin\frac{n\pi x}{L}$

41 $S(x) =$ \dst $-\frac{240L^5}{\pi^5}\sum_{n=1}^{\infty}\frac{1+(-1)^n2}{n^5}\sin\frac{n\pi x}{L}$

43 $C_M(x) = -$ \dst $\frac{64L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n+\frac{3}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$

44 $C_M(x) = -\backslash\mathrm{dst}\frac{32L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{2L}$

45 $C_M(x) = -\backslash\mathrm{dst}\frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n + \frac{2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$

46 $C_M(x) = \backslash\mathrm{dst}\frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$

47 $C_M(x) = \backslash\mathrm{dst}\frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{8}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$

48 $C_M(x) = -\backslash\mathrm{dst}\frac{384L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$

49 $C_M(x) = -\backslash\mathrm{dst}\frac{768L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$

51 $S_M(x) = \backslash\mathrm{dst}\frac{32L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2L}$

52 $S_M(x) = \backslash\mathrm{dst} - \frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$

53 $S_M(x) = \backslash\mathrm{dst}\frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{2}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$

54 $S_M(x) = \backslash\mathrm{dst}\frac{192L^3}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2L}$

55 $S_M(x) = \backslash\mathrm{dst}\frac{1536L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[(-1)^n + \frac{3}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$

56 $S_M(x) = \backslash\mathrm{dst}\frac{384L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$