10 Linear Systems of Differential Equations

10 Linear Systems of Differential Equations

IN THIS CHAPTER we consider systems of differential equations involving more than one unknown function. Such systems arise in many physical applications.

SECTION 10.1 presents examples of physical situations that lead to systems of differential equations.

SECTION 10.2 discusses linear systems of differential equations.

SECTION 10.3 deals with the basic theory of homogeneous linear systems.

SECTIONS 10.4, 10.5, AND 10.6 present the theory of constant coefficient homogeneous systems.

SECTION 10.7 presents the method of variation of parameters for nonhomogeneous linear systems.

507

Answers to selected exercises in this chapter

10.1 Introduction to Systems of Differential Equations

Many physical situations are modelled by systems of n differential equations in n unknown functions, where $n \geq 2$. The next three examples illustrate physical problems that lead to systems of differential equations. In these examples and throughout this chapter we'll denote the independent variable by t.

Example 10.1.1

Tanks T_1 and T_2 contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 10.1.1). A solution with 1 pound of salt per gallon is pumped into T_1 from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into T_2 from an external source at 4 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 3 gal/min. T_1 is drained at 6 gal/min and T_2 is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

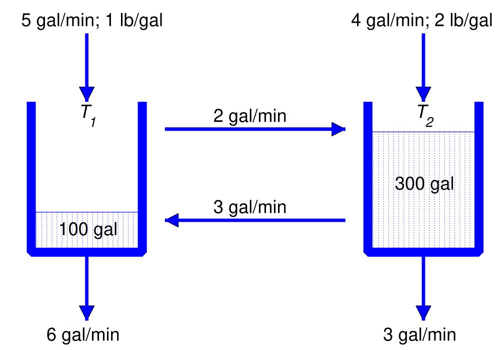

Figure 10.1.1.

Solution As in Section 4.2, let rate in and rate out denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$\begin{aligned} Q_1' &= (\text{rate in})_1 - (\text{rate out})_1, \\ Q_2' &= (\text{rate in})_2 - (\text{rate out})_2. \end{aligned}$$

Note that the volumes of the solutions in T_1 and T_2 remain constant at 100 gallons and 300 gallons, respectively.

T_1 receives salt from the external source at the rate of

$$(1 \text{ lb/gal}) \ \times \ (5 \text{ gal/min}) = \ 5 \text{ lb/min},$$

and from T_2 at the rate of

$$\begin{aligned} (\text{lb/gal in } T_2) \times \ (3 \text{ gal/min}) \ &= \frac{1}{300} Q_2 \times 3 \\ &= \frac{1}{100} Q_2 \text{ lb/min}. \end{aligned}$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100} Q_2. \tag{10.1.1}$$

Solution leaves T_1 at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to T_2; hence,

$$(\text{rate out})_1 = (\text{ lb/gal in T}_1) \times (8 \text{ gal/min}) \ = \frac{1}{100} Q_1 \times 8 = \frac{2}{25} Q_1. \tag{10.1.2}$$

Eqns. (10.1.1) and (10.1.2) imply that

$$Q_1' = 5 + \frac{1}{100} Q_2 - \frac{2}{25} Q_1. \tag{10.1.3}$$

T_2 receives salt from the external source at the rate of

$$(2 \text{ lb/gal}) \times (4 \text{ gal/min}) = 8 \text{ lb/min},$$

and from T_1 at the rate of

$$\begin{aligned}(\text{lb/gal in } T_1) \times (2 \text{ gal/min}) &= \frac{1}{100} Q_1 \times 2 \\ &= \frac{1}{50} Q_1 \text{ lb/min.}\end{aligned}$$

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50} Q_1. \tag{10.1.4}$$

Solution leaves T_2 at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to T_1; hence,

$$(\text{rate out})_2 = (\text{ lb/gal in T}_2) \times (6 \text{ gal/min}) = \frac{1}{300} Q_2 \times 6 = \frac{1}{50} Q_2. \tag{10.1.5}$$

Eqns. (10.1.4) and (10.1.5) imply that

$$Q_2' = 8 + \frac{1}{50} Q_1 - \frac{1}{50} Q_2. \tag{10.1.6}$$

We say that (10.1.3) and (10.1.6) form a system of two first order equations in two unknowns, and write them together as

Extension "bbox" failed to load

Example 10.1.2

A mass m_1 is suspended from a rigid support on a spring S_1 and a second mass m_2 is suspended from the first on a spring S_2 (Figure 10.1.2). The springs obey Hooke's law, with spring constants k_1 and k_2. Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants c_1 and c_2. Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1 and y_2, assuming that the masses of the springs are negligible and that vertical external forces F_1 and F_2 also act on the objects.

Solution In equilibrium, S_1 supports both m_1 and m_2 and S_2 supports only m_2. Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1 \Delta\ell_1 \quad \text{and} \quad m_2 g = k_2 \Delta\ell_2. \tag{10.1.7}$$

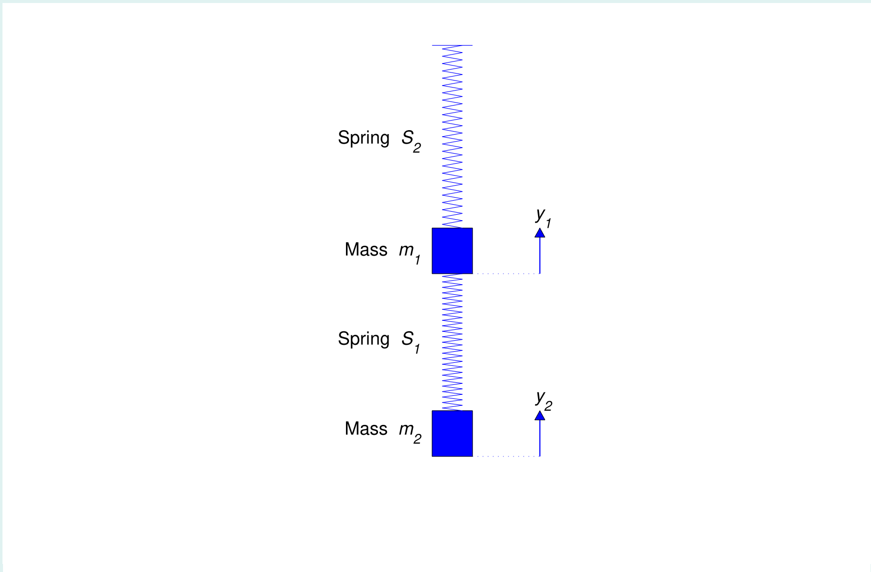

Figure 10.1.2.

Let H_1 be the Hooke's law force acting on m_1, and let D_1 be the damping force on m_1. Similarly, let H_2 and D_2 be the Hooke's law and damping forces acting on m_2. According to Newton's second law of motion,

$$\begin{aligned} m_1y_1'' &= -m_1g + H_1 + D_1 + F_1, \\ m_2y_2'' &= -m_2g + H_2 + D_2 + F_2. \end{aligned} \tag{10.1.8}$$

When the displacements are y_1 and y_2, the change in length of S_1 is $-y_1 + \Delta\ell_1$ and the change in length of S_2 is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on m_1, while only S_2 exerts a Hooke's law force on m_2. These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \tag{10.1.9}$$

When the velocities are y_1' and y_2', S_1 and S_2 are changing length at the rates $-y_1'$ and $-y_2' + y_1'$, respectively. Both springs exert damping forces on m_1, while only S_2 exerts a damping force on m_2. Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1y_1' + c_2(y_2' - y_1') \quad \text{and} \quad D_2 = -c_2(y_2' - y_1'). \tag{10.1.10}$$

From (10.1.8), (10.1.9), and (10.1.10),

$$\begin{aligned} m_1y_1'' &= -m_1g + k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \\ &\quad -c_1y_1' + c_2(y_2' - y_1') + F_1 \\ &= -(m_1g - k_1\Delta\ell_1 + k_2\Delta\ell_2) - k_1y_1 + k_2(y_2 - y_1) \\ &\quad -c_1y_1' + c_2(y_2' - y_1') + F_1 \end{aligned} \tag{10.1.11}$$

and

$$\begin{aligned} m_2y_2'' &= -m_2g + k_2(-y_2 + y_1 + \Delta\ell_2) - c_2(y_2' - y_1') + F_2 \\ &= -(m_2g - k_2\Delta\ell_2) - k_2(y_2 - y_1) - c_2(y_2' - y_1') + F_2. \end{aligned} \tag{10.1.12}$$

From (10.1.7),

$$m_1g-k_1\Delta\ell_1+k_2\Delta\ell_2=-m_2g+k_2\Delta\ell_2=0.$$

Therefore we can rewrite (10.1.11) and (10.1.12) as

$$\begin{aligned} m_1y_1'' &= -(c_1+c_2)y_1'+c_2y_2'-(k_1+k_2)y_1+k_2y_2+F_1\\ m_2y_2'' &= c_2y_1'-c_2y_2'+k_2y_1-k_2y_2+F_2. \quad \text{\bbox} \end{aligned}$$

Example 10.1.3

Let $\mathbf{X}=\mathbf{X}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$ be the position vector at time t of an object with mass m, relative to a rectangular coordinate system with origin at Earth's center (Figure 10.1.3). According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F}=\mathbf{F}(x,y,z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F}=\frac{K}{\|\mathbf{X}\|^2}\left(-\frac{\mathbf{X}}{\|\mathbf{X}\|}\right)=-K\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}, \tag{10.1.13}$$

where K is a constant. To determine K, we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\|=K\frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3}=\frac{K}{\|\mathbf{X}\|^2}=\frac{K}{(x^2+y^2+z^2)}.$$

Let R be Earth's radius. Since $\|\mathbf{F}\|=mg$ when the object is at Earth's surface,

$$mg=\frac{K}{R^2},\quad \text{so}\quad K=mgR^2.$$

Therefore we can rewrite (10.1.13) as

$$\mathbf{F}=-mgR^2\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F}=m\mathbf{X}''$; that is,

$$m(x''\,\mathbf{i}+y''\,\mathbf{j}+z''\,\mathbf{k})=-mgR^2\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Cancelling the common factor m and equating components on the two sides of this equation yields the system

$$\begin{aligned} x'' &= -\text{\dst}\frac{gR^2x}{(x^2+y^2+z^2)^{3/2}}\\ y'' &= -\text{\dst}\frac{gR^2y}{(x^2+y^2+z^2)^{3/2}}\\ z'' &= -\text{\dst}\frac{gR^2z}{(x^2+y^2+z^2)^{3/2}}. \end{aligned} \tag{10.1.14}$$

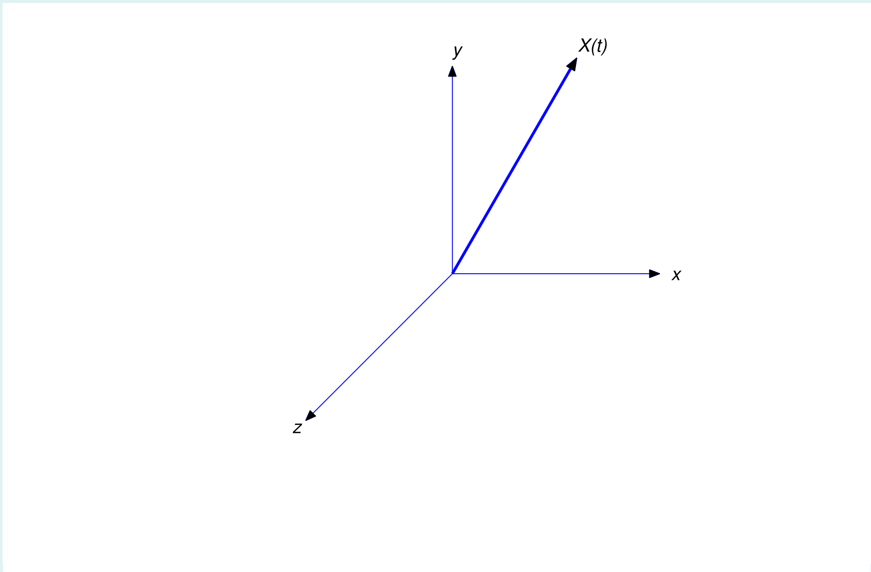

Figure 10.1.3.

Rewriting Higher Order Systems as First Order Systems

A system of the form

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \dots, y_n) \\ y_2' &= g_2(t, y_1, y_2, \dots, y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \dots, y_n) \end{aligned} \tag{10.1.15}$$

is called a first order system, since the only derivatives occurring in it are first derivatives. The derivative of each of the unknowns may depend upon the independent variable and all the unknowns, but not on the derivatives of other unknowns. When we wish to emphasize the number of unknown functions in (10.1.15) we will say that (10.1.15) is an $n \times n$ system.

Systems involving higher order derivatives can often be reformulated as first order systems by introducing additional unknowns. The next two examples illustrate this.

Example 10.1.4

Rewrite the system

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2) y_1' + c_2 y_2' - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned} \tag{10.1.16}$$

derived in Example 10.1.2 as a system of first order equations.

Solution If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so (10.1.16) becomes

$$\begin{aligned} m_1 v_1' &= -(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 v_2' &= c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2. \end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the 4×4 first order system

$$\begin{aligned}
y_1' &= v_1 \\
y_2' &= v_2 \\
v_1' &= \backslash\mathrm{dst}\frac{1}{m_1}\left[-(c_1+c_2)v_1+c_2v_2-(k_1+k_2)y_1+k_2y_2+F_1\right] \\
v_2' &= \backslash\mathrm{dst}\frac{1}{m_2}\left[c_2v_1-c_2v_2+k_2y_1-k_2y_2+F_2\right].
\end{aligned} \tag{10.1.17}$$

Remark

The difference in form between (10.1.15) and (10.1.17), due to the way in which the unknowns are denoted in the two systems, isn't important; (10.1.17) is a first order system, in that each equation in (10.1.17) expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

Example 10.1.5

Rewrite the system

$$\begin{aligned}
x'' &= f(t,x,x',y,y',y'') \\
y''' &= g(t,x,x',y,y'y'')
\end{aligned}$$

as a first order system.

Solution We regard x, x', y, y', and y'' as unknown functions, and rename them

$$x=x_1,\ x'=x_2,\quad y=y_1,\quad y'=y_2,\quad y''=y_3.$$

These unknowns satisfy the system

$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= f(t,x_1,x_2,y_1,y_2,y_3) \\
y_1' &= y_2 \\
y_2' &= y_3 \\
y_3' &= g(t,x_1,x_2,y_1,y_2,y_3).
\end{aligned}$$

Rewriting Scalar Differential Equations as Systems

In this chapter we'll refer to differential equations involving only one unknown function as scalar differential equations. Scalar differential equations can be rewritten as systems of first order equations by the method illustrated in the next two examples.

Example 10.1.6

Rewrite the equation

$$y^{(4)}+4y'''+6y''+4y'+y=0 \tag{10.1.18}$$

as a 4×4 first order system.

Solution We regard y, y', y'', and y''' as unknowns and rename them

$$y = y_1, \quad y' = y_2, \quad y'' = y_3, \quad \text{and} \quad y''' = y_4.$$

Then $y^{(4)} = y_4'$, so (10.1.18) can be written as

$$y_4' + 4y_4 + 6y_3 + 4y_2 + y_1 = 0.$$

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -4y_4 - 6y_3 - 4y_2 - y_1.\text{\bbox} \end{aligned}$$

Example 10.1.7

Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

Solution We regard x, x', and x'' as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \quad \text{and} \quad y_3' = x'''.$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t, y_1, y_2, y_3). \end{aligned}$$

Since systems of differential equations involving higher derivatives can be rewritten as first order systems by the method used in Examples 10.1.5 –10.1.7 , we'll consider only first order systems.

Numerical Solution of Systems

The numerical methods that we studied in Chapter 3 can be extended to systems, and most differential equation software packages include programs to solve systems of equations. We won't go into detail on numerical methods for systems; however, for illustrative purposes we'll describe the Runge-Kutta method for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20} \end{aligned}$$

at equally spaced points t_0, t_1, …, $t_n = b$ in an interval $[t_0, b]$. Thus,

$$t_i = t_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$h = \frac{b - t_0}{n}.$$

We'll denote the approximate values of y_1 and y_2 at these points by $y_{10}, y_{11}, \dots, y_{1n}$ and $y_{20}, y_{21}, \dots, y_{2n}$. The Runge-Kutta method computes these approximate values as follows: given y_{1i} and y_{2i}, compute

$$\begin{aligned}
I_{1i} &= g_1(t_i, y_{1i}, y_{2i}),\\
J_{1i} &= g_2(t_i, y_{1i}, y_{2i}),\\
I_{2i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right),\\
J_{2i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right),\\
I_{3i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right),\\
J_{3i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right),\\
I_{4i} &= g_1(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}),\\
J_{4i} &= g_2(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}),
\end{aligned}$$

and

$$\begin{aligned}
y_{1,i+1} &= y_{1i} + \frac{h}{6}(I_{1i} + 2I_{2i} + 2I_{3i} + I_{4i}),\\
y_{2,i+1} &= y_{2i} + \frac{h}{6}(J_{1i} + 2J_{2i} + 2J_{3i} + J_{4i})
\end{aligned}$$

for $i = 0$, …, $n - 1$. Under appropriate conditions on g_1 and g_2, it can be shown that the global truncation error for the Runge-Kutta method is $O(h^4)$, as in the scalar case considered in Section 3.3.

10.1 Exercises

1. Tanks T_1 and T_2 contain 50 gallons and 100 gallons of salt solutions, respectively. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 1 gal/min, and a solution with 3 pounds of salt per gallon is pumped into T_2 from an external source at 2 gal/min. The solution from T_1 is pumped into T_2 at 3 gal/min, and the solution from T_2 is pumped into T_1 at 4 gal/min. T_1 is drained at 2 gal/min and T_2 is drained at 1 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

Show answer

2. Two 500 gallon tanks T_1 and T_2 initially contain 100 gallons each of salt solution. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 6 gal/min, and a solution with 1 pound of salt per gallon is pumped into T_2 from an external source at 5 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 1 gal/min. Both tanks are drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2 that's valid until a tank is about to overflow. Assume that both mixtures are well stirred.

Show answer

3. A mass m_1 is suspended from a rigid support on a spring S_1 with spring constant k_1 and damping constant c_1. A second mass m_2 is suspended from the first on a spring S_2 with spring constant k_2 and damping constant c_2, and a third mass m_3 is suspended from the second on a spring S_3 with spring constant k_3 and damping constant c_3. Let $y_1 = y_1(t)$, $y_2 = y_2(t)$, and $y_3 = y_3(t)$ be the displacements of the three masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1, y_2 and y_3, assuming that the masses of the springs are negligible and that vertical external forces F_1, F_2, and F_3 also act on the masses.

Show answer

4. Let $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of an object with mass m, expressed in terms of a rectangular coordinate system with origin at Earth's center (Figure 10.1.3). Derive a system of differential equations for x, y, and z, assuming that the object moves under Earth's gravitational force (given by Newton's law of gravitation, as in Example 10.1.3) and a resistive force proportional to the speed of the object. Let α be the constant of proportionality.

Show answer

5. Rewrite the given system as a first order system.

(a) $\begin{aligned} x''' &= f(t, x, y, y') \\ y'' &= g(t, y, y') \end{aligned}$	**(b)** $\begin{aligned} u' &= f(t, u, v, v', w') \\ v'' &= g(t, u, v, v', w) \\ w'' &= h(t, u, v, v', w, w') \end{aligned}$
(c) $y''' = f(t, y, y', y'')$	**(d)** $y^{(4)} = f(t, y)$
(e) $\begin{aligned} x'' &= f(t, x, y) \\ y'' &= g(t, x, y) \end{aligned}$	

Show answer

6. Rewrite the system (10.1.14) of differential equations derived in Example 10.1.3 as a first order system.

Show answer

7. Formulate a version of Euler's method (Section 3.1) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned}$$

on an interval $[t_0, b]$.

8. Formulate a version of the improved Euler method (Section 3.2) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned}$$

on an interval $[t_0, b]$.

10.2 Linear Systems of Differential Equations

A first order system of differential equations that can be written in the form

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t) \end{aligned} \tag{10.2.1}$$

is called a linear system.

The linear system (10.2.1) can be written in matrix form as

$$\backslash\mathrm{col}y'n = \backslash\mathrm{matfunc}ann\backslash\mathrm{col}yn + \backslash\mathrm{colfunc}fn,$$

or more briefly as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \tag{10.2.2}$$

where

$$\mathbf{y} = \backslash\mathrm{col}yn, \quad A(t) = \backslash\mathrm{matfunc}ann, \quad \text{and} \quad \mathbf{f}(t) = \backslash\mathrm{colfunc}fn.$$

We call A the coefficient matrix of (10.2.2) and $\mathbf{f}$ the forcing function. We'll say that A and $\mathbf{f}$ are continuous if their entries are continuous. If $\mathbf{f} = \mathbf{0}$, then (10.2.2) is homogeneous; otherwise, (10.2.2) is nonhomogeneous.

An initial value problem for (10.2.2) consists of finding a solution of (10.2.2) that equals a given constant vector

$$\mathbf{k} = \backslash\mathrm{col}kn.$$

at some initial point t_0. We write this initial value problem as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}.$$

The next theorem gives sufficient conditions for the existence of solutions of initial value problems for (10.2.2). We omit the proof.

Theorem 10.2.1

Suppose the coefficient matrix A and the forcing function $\mathbf{f}$ are continuous on (a,b), let t_0 be in (a,b), and let $\mathbf{k}$ be an arbitrary constant n-vector. Then the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

has a unique solution on (a,b).

Example 10.2.1

a. Write the system

$$\begin{aligned} y_1' &= \ \ y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + \ \ y_2 + \ \ e^{4t} \end{aligned} \tag{10.2.3}$$

in matrix form and conclude from Theorem 10.2.1 that every initial value problem for (10.2.3) has a unique solution on $(-\infty,\infty)$.

b. Verify that

$$\mathbf{y} = \frac{1}{5}\backslash\text{twocol}87e^{4t} + c_1\backslash\text{twocol}11e^{3t} + c_2\backslash\text{twocol}1{-1}e^{-t} \tag{10.2.4}$$

is a solution of (10.2.3) for all values of the constants c_1 and c_2.

c. Find the solution of the initial value problem

$$\mathbf{y}' = \backslash\text{twobytwo}1221\mathbf{y} + \backslash\text{twocol}21e^{4t}, \quad \mathbf{y}(0) = \frac{1}{5}\backslash\text{twocol}322. \tag{10.2.5}$$

Solution (a) The system (10.2.3) can be written in matrix form as

$$\mathbf{y}' = \backslash\text{twobytwo}1221\mathbf{y} + \backslash\text{twocol}21e^{4t}.$$

An initial value problem for (10.2.3) can be written as

$$\mathbf{y}' = \backslash\text{twobytwo}1221\mathbf{y} + \backslash\text{twocol}21e^{4t}, \quad y(t_0) = \backslash\text{twocol}k_1k_2.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty,\infty)$, Theorem 10.2.1 implies that this problem has a unique solution on $(-\infty,\infty)$.

Solution (b) If $\mathbf{y}$ is given by (10.2.4), then

$$\begin{aligned} A\mathbf{y}+\mathbf{f} &= \frac{1}{5}\backslash\text{twobytwo}1221\backslash\text{twocol}87e^{4t} + c_1\backslash\text{twobytwo}1221\backslash\text{twocol}11e^{3t} \\ &\quad + c_2\backslash\text{twobytwo}1221\backslash\text{twocol}1{-}1e^{-t} + \backslash\text{twocol}21e^{4t} \\ &= \frac{1}{5}\backslash\text{twocol}2223e^{4t} + c_1\backslash\text{twocol}33e^{3t} + c_2\backslash\text{twocol}{-}11e^{-t} + \backslash\text{twocol}21e^{4t} \\ &= \frac{1}{5}\backslash\text{twocol}3228e^{4t} + 3c_1\backslash\text{twocol}11e^{3t} - c_2\backslash\text{twocol}1{-}1e^{-t} = \mathbf{y}'. \end{aligned}$$

Solution (c) We must choose c_1 and c_2 in (10.2.4) so that

$$\frac{1}{5}\backslash\text{twocol}87 + c_1\backslash\text{twocol}11 + c_2\backslash\text{twocol}1{-}1 = \frac{1}{5}\backslash\text{twocol}322,$$

which is equivalent to

$$\backslash\text{twobytwo}111{-}1\backslash\text{twocol}c_1c_2 = \backslash\text{twocol}{-}13.$$

Solving this system yields $c_1 = 1$, $c_2 = -2$, so

$$\mathbf{y} = \frac{1}{5}\backslash\text{twocol}87e^{4t} + \backslash\text{twocol}11e^{3t} - 2\backslash\text{twocol}1{-}1e^{-t}$$

is the solution of (10.2.5).

Remark

The theory of $n \times n$ linear systems of differential equations is analogous to the theory of the scalar n-th order equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = F(t), \tag{10.2.6}$$

as developed in Sections 9.1. For example, by rewriting (10.2.6) as an equivalent linear system it can be shown that Theorem 10.2.1 implies Theorem 9.1.1 (Exercise 12).

10.2 Exercises

1. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

a. $\begin{array}{rcl} y_1' &=& 2y_1+4y_2 \\ y_2' &=& 4y_1+2y_2; \end{array}$ $\quad \mathbf{y}=c_1\backslash\text{twocol}11e^{6t}+c_2\backslash\text{twocol}1{-1}e^{-2t}$

b. $\begin{array}{rcl} y_1' &=& -2y_1-2y_2 \\ y_2' &=& -5y_1+\ y_2; \end{array}$ $\quad \mathbf{y}=c_1\backslash\text{twocol}11e^{-4t}+c_2\backslash\text{twocol}{-2}5e^{3t}$

c. $\begin{array}{rcl} y_1' &=& -4y_1-10y_2 \\ y_2' &=& 3y_1+\ 7y_2; \end{array}$ $\quad \mathbf{y}=c_1\backslash\text{twocol}{-5}3e^{2t}+c_2\backslash\text{twocol}2{-1}e^{t}$

d. $\begin{array}{rcl} y_1' &=& 2y_1+\ y_2 \\ y_2' &=& y_1+2y_2; \end{array}$ $\quad \mathbf{y}=c_1\backslash\text{twocol}11e^{3t}+c_2\backslash\text{twocol}1{-1}e^{t}$

Show answer

2. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1, c_2, and c_3.

a. $\begin{array}{rcl} y_1' &=& -y_1+2y_2+3y_3 \\ y_2' &=& y_2+6y_3 \\ y_3' &=& -2y_3; \end{array}$

$\mathbf{y}=c_1\backslash\text{threecol}110e^{t}+c_2\backslash\text{threecol}100e^{-t}+c_3\backslash\text{threecol}1{-2}1e^{-2t}$

b. $\begin{array}{rcl} y_1' &=& 2y_2+2y_3 \\ y_2' &=& 2y_1\qquad+2y_3 \\ y_3' &=& 2y_1+2y_2; \end{array}$

$\mathbf{y}=c_1\backslash\text{threecol}{-1}01e^{-2t}+c_2\backslash\text{threecol}0{-1}1e^{-2t}+c_3\backslash\text{threecol}111e^{4t}$

c. $\begin{array}{rcl} y_1' &=& -y_1+2y_2+2y_3 \\ y_2' &=& 2y_1-\ y_2+2y_3 \\ y_3' &=& 2y_1+2y_2-\ y_3; \end{array}$

$\mathbf{y}=c_1\backslash\text{threecol}{-1}01e^{-3t}+c_2\backslash\text{threecol}0{-1}1e^{-3t}+c_3\backslash\text{threecol}111e^{3t}$

d. $\begin{array}{rcl} y_1' &=& 3y_1-\ y_2-\ y_3 \\ y_2' &=& -2y_1+3y_2+2y_3 \\ y_3' &=& 4y_1-\ y_2-2y_3; \end{array}$

$\mathbf{y}=c_1\backslash\text{threecol}101e^{2t}+c_2\backslash\text{threecol}1{-1}1e^{3t}+c_3\backslash\text{threecol}1{-3}7e^{-t}$

Show answer

3. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

a. $\begin{array}{rclrcl} y_1' &=& y_1+\ y_2 & y_1(0) &=& 1 \\ y_2' &=& -2y_1+4y_2, & y_2(0) &=& 0; \end{array}$ $\quad \mathbf{y}=2\backslash\text{twocol}11e^{2t}-\backslash\text{twocol}12e^{3t}$

b. $\begin{array}{rclrcl} y_1' &=& 5y_1+3y_2 & y_1(0) &=& 12 \\ y_2' &=& -y_1+y_2, & y_2(0) &=& -6; \end{array}$ $\quad \mathbf{y}=3\backslash\text{twocol}1{-1}e^{2t}+3\backslash\text{twocol}3{-1}e^{4t}$

Show answer

4. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

a. $$\begin{array}{rcl} y_1' & = & 6y_1 + 4y_2 + 4y_3 \\ y_2' & = & -7y_1 - 2y_2 - y_3, \\ y_3' & = & 7y_1 + 4y_2 + 3y_3 \end{array} , \quad \begin{array}{rcl} y_1(0) & = & 3 \\ y_2(0) & = & -6 \\ y_3(0) & = & 4 \end{array}$$

$$\mathbf{y} = \backslash\text{threecol}1{-}11e^{6t} + 2\backslash\text{threecol}1{-}21e^{2t} + \backslash\text{threecol}0{-}11e^{-t}$$

b. $$\begin{array}{rcl} y_1' & = & 8y_1 + 7y_2 + \ \ 7y_3 \\ y_2' & = & -5y_1 - 6y_2 - \ \ 9y_3, \\ y_3' & = & 5y_1 + 7y_2 + 10y_3, \end{array} \quad \begin{array}{rcl} y_1(0) & = & 2 \\ y_2(0) & = & -4 \\ y_3(0) & = & 3 \end{array}$$

$$\mathbf{y} = \backslash\text{threecol}1{-}11e^{8t} + \backslash\text{threecol}0{-}11e^{3t} + \backslash\text{threecol}1{-}21e^{t}$$

Show answer

5. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

a. $$\begin{array}{rcl} y_1' & = & -3y_1 + 2y_2 + 3 - 2t \\ y_2' & = & -5y_1 + 3y_2 + 6 - 3t \end{array}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 2\cos t \\ 3\cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} 2\sin t \\ 3\sin t + \cos t \end{bmatrix} + \backslash\text{twocol}1t$$

b. $$\begin{array}{rcl} y_1' & = & 3y_1 + y_2 - 5e^t \\ y_2' & = & -y_1 + y_2 + e^t \end{array}$$

$$\mathbf{y} = c_1\backslash\text{twocol}{-}11e^{2t} + c_2 \begin{bmatrix} 1+t \\ -t \end{bmatrix} e^{2t} + \backslash\text{twocol}13e^{t}$$

c. $$\begin{array}{rcl} y_1' & = & -y_1 - 4y_2 + 4e^t + 8te^t \\ y_2' & = & -y_1 - \ \ y_2 + e^{3t} + (4t+2)e^t \end{array}$$

$$\mathbf{y} = c_1\backslash\text{twocol}21e^{-3t} + c_2\backslash\text{twocol}{-}21e^{t} + \begin{bmatrix} e^{3t} \\ 2te^{t} \end{bmatrix}$$

d. $$\begin{array}{rcl} y_1' & = & -6y_1 - 3y_2 + 14e^{2t} + 12e^t \\ y_2' & = & y_1 - 2y_2 + 7e^{2t} - 12e^t \end{array}$$

$$\mathbf{y} = c_1\backslash\text{twocol}{-}31e^{-5t} + c_2\backslash\text{twocol}{-}11e^{-3t} + \begin{bmatrix} e^{2t} + 3e^{t} \\ 2e^{2t} - 3e^{t} \end{bmatrix}$$

Show answer

6. Convert the linear scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y(t) = F(t) \qquad \text{(A)}$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

and show that A and $\mathbf{f}$ are continuous on an interval (a, b) if and only if (A) is normal on (a, b).

7. A matrix function

$$Q(t) = \backslash\text{matfunc}qrs$$

is said to be differentiable if its entries $\{q_{ij}\}$ are differentiable. Then the derivative Q' is defined by

$$Q'(t) = \backslash\text{matfunc}q'rs.$$

a. Prove: If P and Q are differentiable matrices such that $P+Q$ is defined and if c_1 and c_2 are constants, then

$$(c_1P + c_2Q)' = c_1P' + c_2Q'.$$

b. Prove: If P and Q are differentiable matrices such that PQ is defined, then

$$(PQ)' = P'Q + PQ'.$$

8. Verify that $Y' = AY$.

a. $\backslash\text{dst}Y = \backslash\text{twobytwo}e^{6t}e^{-2t}e^{6t}-e^{-2t}, \quad A = \backslash\text{twobytwo}2442$

b. $\backslash\text{dst}Y = \backslash\text{twobytwo}e^{-4t}-2e^{3t}e^{-4t}5e^{3t}, \quad A = \backslash\text{twobytwo}-2-2-51$

c. $\backslash\text{dst}Y = \backslash\text{twobytwo}-5e^{2t}2e^{t}3e^{2t}-e^{t}, \quad A = \backslash\text{twobytwo}-4-1037$

d. $\backslash\text{dst}Y = \backslash\text{twobytwo}e^{3t}e^{t}e^{3t}-e^{t}, \quad A = \backslash\text{twobytwo}2112$

e. $Y = \begin{bmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}, \quad A = \backslash\text{threebythree}-12301600-2$

f. $\backslash\text{dst}Y = \backslash\text{cthreebythree}-e^{-2t}-e^{-2t}e^{4t}0\quad e^{-2t}e^{4t}e^{-2t}0e^{4t}, \quad A = \backslash\text{threebythree}022202220$

g. $\backslash\text{dst}Y = \backslash\text{cthreebythree}e^{3t}e^{-3t}0e^{3t}0-e^{-3t}e^{3t}e^{-3t}\quad e^{-3t}, \quad A = \backslash\text{threebythree}-966-636-663$

h. $Y = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix}, \quad A = \backslash\text{threebythree}3-1-1-2324-1-2$

9. Suppose

$$\mathbf{y}_1 = \backslash\text{twocol}y_{11}y_{21} \quad \text{and} \quad \mathbf{y}_2 = \backslash\text{twocol}y_{12}y_{22}$$

are solutions of the homogeneous system

$$\mathbf{y}' = A(t)\mathbf{y}, \tag{A}$$

and define

$$Y = \backslash\text{twobytwo}y_{11}y_{12}y_{21}y_{22}.$$

a. Show that $Y' = AY$.

b. Show that if $\mathbf{c}$ is a constant vector then $\mathbf{y} = Y\mathbf{c}$ is a solution of (A).

c. State generalizations of **(a)** and **(b)** for $n \times n$ systems.

10. Suppose Y is a differentiable square matrix.

a. Find a formula for the derivative of Y^2.

b. Find a formula for the derivative of Y^n, where n is any positive integer.

c. State how the results obtained in **(a)** and **(b)** are analogous to results from calculus concerning scalar functions.

Show answer

11. It can be shown that if Y is a differentiable and invertible square matrix function, then Y^{-1} is differentiable.

a. Show that $(Y^{-1})' = -Y^{-1}Y'Y^{-1}$. (Hint: Differentiate the identity $Y^{-1}Y = I$.)

b. Find the derivative of $Y^{-n} = (Y^{-1})^n$, where n is a positive integer.

c. State how the results obtained in **(a)** and **(b)** are analogous to results from calculus concerning scalar functions.

12. Show that Theorem 10.2.1 implies Theorem 9.1.1.

Hint

Write the scalar equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x)$$

as an $n \times n$ system of linear equations.

13. Suppose $\mathbf{y}$ is a solution of the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), and that the $n \times n$ matrix P is invertible and differentiable on (a, b). Find a matrix B such that the function $\mathbf{x} = P\mathbf{y}$ is a solution of $\mathbf{x}' = B\mathbf{x}$ on (a, b).

Show answer

10.3 Basic Theory of Homogeneous Linear System

In this section we consider homogeneous linear systems $\mathbf{y}' = A(t)\mathbf{y}$, where $A = A(t)$ is a continuous $n \times n$ matrix function on an interval (a, b). The theory of linear homogeneous systems has much in common with the theory of linear homogeneous scalar equations, which we considered in Sections 2.1, 5.1, and 9.1.

Whenever we refer to solutions of $\mathbf{y}' = A(t)\mathbf{y}$ we'll mean solutions on (a, b). Since $\mathbf{y} \equiv \mathbf{0}$ is obviously a solution of $\mathbf{y}' = A(t)\mathbf{y}$, we call it the trivial solution. Any other solution is nontrivial.

If $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ are vector functions defined on an interval (a, b) and c_1, c_2, …, c_n are constants, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n \tag{10.3.1}$$

is a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …,$\mathbf{y}_n$. It's easy show that if $\mathbf{y}_1$, $\mathbf{y}_2$, …,$\mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then so is any linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ (Exercise 1). We say that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) on if every solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) can be written as a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$, as in (10.3.1). In this case we say that (10.3.1) is the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

It can be shown that if A is continuous on (a, b) then $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b) (Exercises 15 and 16). The next definition will help to characterize fundamental sets of solutions of $\mathbf{y}' = A(t)\mathbf{y}$.

We say that a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n-vector functions is linearly independent on (a, b) if the only constants c_1, c_2, …, c_n such that

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) = 0, \quad a < t < b, \tag{10.3.2}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If (10.3.2) holds for some set of constants c_1, c_2, …, c_n that are not all zero, then $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is linearly dependent on (a, b)

The next theorem is analogous to Theorems 5.1.3 and 9.1.2.

Theorem 10.3.1

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b). Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of n solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is a fundamental set if and only if it's linearly independent on (a, b).

Example 10.3.1

Show that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are linearly independent on every interval (a, b).

Solution Suppose

$$c_1 \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

We must show that $c_1 = c_2 = c_3 = 0$. Rewriting this equation in matrix form yields

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{aligned} \begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} &= e^t \begin{vmatrix} e^{3t} & e^{3t} \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 0 \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix} \\ &= e^t(-e^{3t}) + e^{2t}(-e^{2t}) = -2e^{4t}. \end{aligned}$$

Since this determinant is never zero, $c_1 = c_2 = c_3 = 0$. ∎

We can use the method in Example 10.3.1 to test n solutions $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of any $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ for linear independence on an interval (a, b) on which A is continuous. To explain this (and for other purposes later), it's useful to write a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ in a different way. We first write the vector functions in terms of their components as

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \ldots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}.$$

If

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$$

then

$$\begin{aligned}\mathbf{y} &= c_1\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix} + c_2\begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix} \\ &= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \backslash\mathrm{col}cn.\end{aligned}$$

This shows that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = Y\mathbf{c}, \tag{10.3.3}$$

where

$$\mathbf{c} = \backslash\mathrm{col}cn$$

and

$$Y = [\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n] = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}; \tag{10.3.4}$$

that is, the columns of Y are the vector functions $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$.

For reference below, note that

$$\begin{aligned} Y' &= [\mathbf{y}_1'\ \mathbf{y}_2'\ \cdots\ \mathbf{y}_n'] \\ &= [A\mathbf{y}_1\ A\mathbf{y}_2\ \cdots\ A\mathbf{y}_n] \\ &= A[\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n] = AY; \end{aligned}$$

that is, Y satisfies the matrix differential equation

$$Y' = AY.$$

The determinant of Y,

$$W = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \tag{10.3.5}$$

is called the Wronskian (http://www-history.mcs.st-and.ac.uk/Mathematicians/Wronski.html) of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$. It can be shown (Exercises 2 and 3) that this definition is analogous to definitions of the Wronskian of scalar functions given in Sections 5.1 and 9.1. The next theorem is analogous to Theorems 5.1.4 and 9.1.3. The proof is sketched in Exercise 4 for $n = 2$ and in Exercise 5 for general n.

Theorem 10.3.2

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b), let $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), and let t_0 be in (a, b). Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is given by

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \left[a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)\right] ds\right), \quad a < t < b. \tag{10.3.6}$$

Therefore, either W has no zeros in (a, b) or $W \equiv 0$ on (a, b).

Remark

The sum of the diagonal entries of a square matrix A is called the trace of A, denoted by tr(A). Thus, for an $n \times n$ matrix A,

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

and (10.3.6) can be written as

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\, ds\right), \quad a < t < b.$$

The next theorem is analogous to Theorems 5.1.6 and 9.1.4.

Theorem 10.3.3

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and let $\mathbf{y}_1$, $\mathbf{y}_2$, …,$\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$, where c_1, c_2, …, c_n are arbitrary constants.

b. $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

c. $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is linearly independent on (a, b).

d. The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at some point in (a, b).

e. The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at all points in (a, b).

We say that Y in (10.3.4) is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if any (and therefore all) of the statements (**a**)-(**e**) of Theorem 10.3.2 are true for the columns of Y. In this case, (10.3.3) implies that the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ can be written as $\mathbf{y} = Y\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant n-vector.

Example 10.3.2

The vector functions

$$\mathbf{y}_1 = \text{\twocol}{-e^{2t}}{2e^{2t}} \quad \text{and} \quad \mathbf{y}_2 = \text{\twocol}{-e^{-t}}{e^{-t}}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \backslash\mathrm{twobytwo}{-4}{-3}65\mathbf{y} \tag{10.3.7}$$

on $(-\infty, \infty)$. (Verify.)

a. Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition (10.3.5)

b. Verify Abel's formula (10.3.6) for the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

c. Find the general solution of (10.3.7).

d. Solve the initial value problem

$$\mathbf{y}' = \backslash\mathrm{twobytwo}{-4}{-3}65\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \tag{10.3.8}$$

Solution (a) From (10.3.5)

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{2t}e^{-t}\backslash\mathrm{twobytwo}{-1}{-1}21 = e^t. \tag{10.3.9}$$

Solution (b) Here

$$A = \backslash\mathrm{twobytwo}{-4}{-3}65,$$

so tr$(A) = -4 + 5 = 1$. If t_0 is an arbitrary real number then (10.3.6) implies that

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} 1\,ds\right) = \begin{vmatrix} -e^{2t_0} & -e^{-t_0} \\ 2e^{2t_0} & e^{-t_0} \end{vmatrix} e^{(t-t_0)} = e^{t_0}e^{t-t_0} = e^t,$$

which is consistent with (10.3.9).

Solution (c) Since $W(t) \neq 0$, Theorem 10.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of (10.3.7) and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for (10.3.7). Therefore the general solution of (10.3.7) is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1\backslash\mathrm{twocol}{-e^{2t}}2e^{2t} + c_2\backslash\mathrm{twocol}{-e^{-t}}e^{-t} = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{10.3.10}$$

Solution (d) Setting $t = 0$ in (10.3.10) and imposing the initial condition in (10.3.8) yields

$$c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} -c_1 - c_2 &= 4 \\ 2c_1 + c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = -1$, $c_2 = -3$. Substituting these values into (10.3.10) yields

$$\mathbf{y} = -\begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} - 3\begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{2t} + 3e^{-t} \\ -2e^{2t} - 3e^{-t} \end{bmatrix}$$

as the solution of (10.3.8).

10.3 Exercises

1. Prove: If $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then any linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ is also a solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

2. In Section 5.1 the Wronskian of two solutions y_1 and y_2 of the scalar second order equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{A}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

a. Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of two solutions of this system.

b. Apply Eqn. (10.3.6) to the system derived in **(a)**, and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

Show answer

3. In Section 9.1 the Wronskian of n solutions y_1, y_2, ..., y_n of the $n-$th order equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \tag{A}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

a. Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of n solutions of this system.

b. Apply Eqn. (10.3.6) to the system derived in **(a)**, and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

Show answer

4. Suppose

$$\mathbf{y}_1 = \backslash\mathrm{twocol} y_{11} y_{21} \quad \text{and} \quad \mathbf{y}_2 = \backslash\mathrm{twocol} y_{12} y_{22}$$

are solutions of the 2×2 system $\mathbf{y}' = A\mathbf{y}$ on (a, b), and let

$$Y = \backslash\mathrm{twobytwo} y_{11} y_{12} y_{21} y_{22} \quad \text{and} \quad W = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix};$$

thus, W is the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

a. Deduce from the definition of determinant that

$$W' = \begin{vmatrix} y'_{11} & y'_{12} \\ y_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} \\ y'_{21} & y'_{22} \end{vmatrix}.$$

b. Use the equation $Y' = A(t)Y$ and the definition of matrix multiplication to show that

$$[y'_{11} \quad y'_{12}] = a_{11}[y_{11} \quad y_{12}] + a_{12}[y_{21} \quad y_{22}]$$

and

$$[y'_{21} \quad y'_{22}] = a_{21}[y_{11} \quad y_{12}] + a_{22}[y_{21} \quad y_{22}].$$

c. Use properties of determinants to deduce from **(a)** and **(a)** that

$$\begin{vmatrix} y'_{11} & y'_{12} \\ y_{21} & y_{22} \end{vmatrix} = a_{11}W \quad \text{and} \quad \begin{vmatrix} y_{11} & y_{12} \\ y'_{21} & y'_{22} \end{vmatrix} = a_{22}W.$$

d. Conclude from **(c)** that

$$W' = (a_{11} + a_{22})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s)] \, ds \right) \quad a < t < b.$$

5. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b). Let

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where the columns of Y are solutions of $\mathbf{y}' = A(t)\mathbf{y}$. Let

$$r_i = [y_{i1}\, y_{i2}\, \dots\, y_{in}]$$

be the ith row of Y, and let W be the determinant of Y.

a. Deduce from the definition of determinant that

$$W' = W_1 + W_2 + \cdots + W_n,$$

where, for $1 \le m \le n$, the ith row of W_m is r_i if $i \ne m$, and r_m' if $i = m$.

b. Use the equation $Y' = AY$ and the definition of matrix multiplication to show that

$$r_m' = a_{m1}r_1 + a_{m2}r_2 + \cdots + a_{mn}r_n.$$

c. Use properties of determinants to deduce from **(b)** that

$$\det(W_m) = a_{mm}W.$$

d. Conclude from **(a)** and **(c)** that

$$W' = (a_{11} + a_{22} + \cdots + a_{nn})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)]\, ds\right), \quad a < t < b.$$

6. Suppose the $n \times n$ matrix A is continuous on (a, b) and t_0 is a point in (a, b). Let Y be a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

a. Show that $Y(t_0)$ is invertible.

b. Show that if $\mathbf{k}$ is an arbitrary n-vector then the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y} = Y(t)Y^{-1}(t_0)\mathbf{k}.$$

7. Let

$$A=\twobytwo2442,\quad \mathbf{y}_1=\begin{bmatrix} e^{6t}\\ e^{6t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} e^{-2t}\\ -e^{-2t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix} -3\\ 9\end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1,\mathbf{y}_2\}$ is a fundamental set of solutions for $\mathbf{y}'=A\mathbf{y}$.

b. Solve the initial value problem

$$\mathbf{y}'=A\mathbf{y},\quad \mathbf{y}(0)=\mathbf{k}. \tag{A}$$

c. Use the result of Exercise 6**(b)** to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

Show answer

8. Repeat Exercise 7 with

$$A=\twobytwo{-2}{-2}{-5}1,\quad \mathbf{y}_1=\begin{bmatrix} e^{-4t}\\ e^{-4t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} -2e^{3t}\\ 5e^{3t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix} 10\\ -4\end{bmatrix}.$$

Show answer

9. Repeat Exercise 7 with

$$A=\twobytwo{-4}{-10}37,\quad \mathbf{y}_1=\begin{bmatrix} -5e^{2t}\\ 3e^{2t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} 2e^{t}\\ -e^{t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix} -19\\ 11\end{bmatrix}.$$

Show answer

10. Repeat Exercise 7 with

$$A=\twobytwo2112,\quad \mathbf{y}_1=\begin{bmatrix} e^{3t}\\ e^{3t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} e^{t}\\ -e^{t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix} 2\\ 8\end{bmatrix}.$$

Show answer

11. Let

$$A=\threebythree3{-1}{-1}{-2}324{-1}{-2},$$
$$\mathbf{y}_1=\begin{bmatrix} e^{2t}\\ 0\\ e^{2t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} e^{3t}\\ -e^{3t}\\ e^{3t}\end{bmatrix},\quad \mathbf{y}_3=\begin{bmatrix} e^{-t}\\ -3e^{-t}\\ 7e^{-t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix} 2\\ -7\\ 20\end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ is a fundamental set of solutions for $\mathbf{y}'=A\mathbf{y}$.

b. Solve the initial value problem

$$\mathbf{y}'=A\mathbf{y},\quad \mathbf{y}(0)=\mathbf{k}. \tag{A}$$

c. Use the result of Exercise 6**(b)** to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

Show answer

12. Repeat Exercise 11 with

$A =$ \threebythree022202220,

$$\mathbf{y}_1 = \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ -9 \\ 12 \end{bmatrix}.$$

Show answer

13. Repeat Exercise 11 with

$A =$ \threebythree−12301600−2,

$$\mathbf{y}_1 = \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix}.$$

Show answer

14. Suppose Y and Z are fundamental matrices for the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$. Then some of the four matrices YZ^{-1}, $Y^{-1}Z$, $Z^{-1}Y$, ZY^{-1} are necessarily constant. Identify them and prove that they are constant.

Show answer

15. Suppose the columns of an $n \times n$ matrix Y are solutions of the $n \times n$ system $\mathbf{y}' = A\mathbf{y}$ and C is an $n \times n$ constant matrix.

a. Show that the matrix $Z = YC$ satisfies the differential equation $Z' = AZ$.

b. Show that Z is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if and only if C is invertible and Y is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$.

16. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and t_0 is in (a, b). For $i = 1$, 2, …, n, let $\mathbf{y}_i$ be the solution of the initial value problem $\mathbf{y}_i' = A(t)\mathbf{y}_i, \; \mathbf{y}_i(t_0) = \mathbf{e}_i$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

that is, the jth component of $\mathbf{e}_i$ is 1 if $j = i$, or 0 if $j \neq i$.

a. Show that$\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

b. Conclude from **(a)** and Exercise 15 that $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b).

17. Show that Y is a fundamental matrix for the system $\mathbf{y}' = A(t)\mathbf{y}$ if and only if Y^{-1} is a fundamental matrix for $\mathbf{y}' = -A^T(t)\mathbf{y}$, where A^T denotes the transpose of A. *Hint: See Exercise 11.*

18. Let Z be the fundamental matrix for the constant coefficient system $\mathbf{y}' = A\mathbf{y}$ such that $Z(0) = I$.

a. Show that $Z(t)Z(s) = Z(t + s)$ for all s and t. *Hint: For fixed s let $\Gamma_1(t) = Z(t)Z(s)$ and $\Gamma_2(t) = Z(t + s)$. Show that Γ_1 and Γ_2 are both solutions of the matrix initial value problem $\Gamma' = A\Gamma, \quad \Gamma(0) = Z(s)$. Then conclude from Theorem 10.2.1 that $\Gamma_1 = \Gamma_2$.*

b. Show that $(Z(t))^{-1} = Z(-t)$.

c. The matrix Z defined above is sometimes denoted by e^{tA}. Discuss the motivation for this notation.

10.4 Constant Coefficient Homogeneous Systems I

We'll now begin our study of the homogeneous system

$$\mathbf{y}' = A\mathbf{y}, \tag{10.4.1}$$

where A is an $n \times n$ constant matrix. Since A is continuous on $(-\infty, \infty)$, Theorem 10.2.1 implies that all solutions of (10.4.1) are defined on $(-\infty, \infty)$. Therefore, when we speak of solutions of $\mathbf{y}' = A\mathbf{y}$, we'll mean solutions on $(-\infty, \infty)$.

In this section we assume that all the eigenvalues of A are real and that A has a set of n linearly independent eigenvectors. In the next two sections we consider the cases where some of the eigenvalues of A are complex, or where A does not have n linearly independent eigenvectors.

In Example 10.3.2 we showed that the vector functions

$$\mathbf{y}_1 = \twocol{-e^{2t}}{2e^{2t}} \quad \text{and} \quad \mathbf{y}_2 = \twocol{-e^{-t}}{e^{-t}}$$

form a fundamental set of solutions of the system

$$\mathbf{y}' = \twobytwo{-4}{-3}{6}{5}\mathbf{y}, \tag{10.4.2}$$

but we did not show how we obtained $\mathbf{y}_1$ and $\mathbf{y}_2$ in the first place. To see how these solutions can be obtained we write (10.4.2) as

$$\begin{array}{rcl} y_1' &=& -4y_1 - 3y_2 \\ y_2' &=& 6y_1 + 5y_2 \end{array} \tag{10.4.3}$$

and look for solutions of the form

$$y_1 = x_1 e^{\lambda t} \quad \text{and} \quad y_2 = x_2 e^{\lambda t}, \tag{10.4.4}$$

where x_1, x_2, and λ are constants to be determined. Differentiating (10.4.4) yields

$$y_1' = \lambda x_1 e^{\lambda t} \quad \text{and} \quad y_2' = \lambda x_2 e^{\lambda t}.$$

Substituting this and (10.4.4) into (10.4.3) and canceling the common factor $e^{\lambda t}$ yields

$$\begin{array}{rcl} -4x_1 - 3x_2 &=& \lambda x_1 \\ 6x_1 + 5x_2 &=& \lambda x_2. \end{array}$$

For a given λ, this is a homogeneous algebraic system, since it can be rewritten as

$$\begin{array}{rcl} (-4-\lambda)x_1 - 3x_2 &=& 0 \\ 6x_1 + (5-\lambda)x_2 &=& 0. \end{array} \tag{10.4.5}$$

The trivial solution $x_1 = x_2 = 0$ of this system isn't useful, since it corresponds to the trivial solution $y_1 \equiv y_2 \equiv 0$ of (10.4.3), which can't be part of a fundamental set of solutions of (10.4.2). Therefore we consider only those values of λ for which (10.4.5) has nontrivial solutions. These are the values of λ for which the determinant of (10.4.5) is zero; that is,

$$\begin{aligned}\begin{vmatrix}-4-\lambda & -3\\ 6 & 5-\lambda\end{vmatrix} &= (-4-\lambda)(5-\lambda)+18\\ &= \lambda^2-\lambda-2\\ &= (\lambda-2)(\lambda+1)=0,\end{aligned}$$

which has the solutions $\lambda_1=2$ and $\lambda_2=-1$.

Taking $\lambda=2$ in (10.4.5) yields

$$\begin{aligned}-6x_1-3x_2&=0\\ 6x_1+3x_2&=0,\end{aligned}$$

which implies that $x_1=-x_2/2$, where x_2 can be chosen arbitrarily. Choosing $x_2=2$ yields the solution $y_1=-e^{2t}$, $y_2=2e^{2t}$ of (10.4.3). We can write this solution in vector form as

$$\mathbf{y}_1=\backslash\mathrm{twocol}{-1}\quad 2e^{2t}. \tag{10.4.6}$$

Taking $\lambda=-1$ in (10.4.5) yields the system

$$\begin{aligned}-3x_1-3x_2&=0\\ 6x_1+6x_2&=0,\end{aligned}$$

so $x_1=-x_2$. Taking $x_2=1$ here yields the solution $y_1=-e^{-t}$, $y_2=e^{-t}$ of (10.4.3). We can write this solution in vector form as

$$\mathbf{y}_2=\backslash\mathrm{twocol}{-1}\quad 1e^{-t}. \tag{10.4.7}$$

In (10.4.6) and (10.4.7) the constant coefficients in the arguments of the exponential functions are the eigenvalues of the coefficient matrix in (10.4.2), and the vector coefficients of the exponential functions are associated eigenvectors. This illustrates the next theorem.

Theorem 10.4.1

Suppose the $n\times n$ constant matrix A has n real eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_n$. Then the functions

$$\mathbf{y}_1=\mathbf{x}_1e^{\lambda_1t},\ \mathbf{y}_2=\mathbf{x}_2e^{\lambda_2t},\ \dots,\ \mathbf{y}_n=\mathbf{x}_ne^{\lambda_nt}$$

form a fundamental set of solutions of $\mathbf{y}'=A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y}=c_1\mathbf{x}_1e^{\lambda_1t}+c_2\mathbf{x}_2e^{\lambda_2t}+\cdots+c_n\mathbf{x}_ne^{\lambda_nt}.$$

Proof Differentiating $\mathbf{y}_i=\mathbf{x}_ie^{\lambda_it}$ and recalling that $A\mathbf{x}_i=\lambda_i\mathbf{x}_i$ yields

$$\mathbf{y}_i'=\lambda_i\mathbf{x}_ie^{\lambda_it}=A\mathbf{x}_ie^{\lambda_it}=A\mathbf{y}_i.$$

This shows that $\mathbf{y}_i$ is a solution of $\mathbf{y}'=A\mathbf{y}$.

The Wronskian of $\{\mathbf{y}_1,\mathbf{y}_2,\dots,\mathbf{y}_n\}$ is

$$\left|\begin{array}{cccc} x_{11}e^{\lambda_1 t} & x_{12}e^{\lambda_2 t} & \cdots & x_{1n}e^{\lambda_n t} \\ x_{21}e^{\lambda_1 t} & x_{22}e^{\lambda_2 t} & \cdots & x_{2n}e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}e^{\lambda_1 t} & x_{n2}e^{\lambda_2 t} & \cdots & x_{nn}e^{\lambda x_n t} \end{array}\right| = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t}\left|\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ \backslash\text{vspace}23pt x_{21} & x_{22} & \cdots & x_{2n} \\ \backslash\text{vspace}23pt\vdots & \vdots & \ddots & \vdots \\ \backslash\text{vspace}23pt x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right|.$$

Since the columns of the determinant on the right are $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, which are assumed to be linearly independent, the determinant is nonzero. Therefore Theorem 10.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 10.4.1

a. Find the general solution of

$$\mathbf{y}' = \backslash\text{twobytwo}2442\mathbf{y}. \tag{10.4.8}$$

b. Solve the initial value problem

$$\mathbf{y}' = \backslash\text{twobytwo}2442\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \tag{10.4.9}$$

Solution (a) The characteristic polynomial of the coefficient matrix A in (10.4.8) is

$$\begin{aligned} \left|\begin{array}{cc} 2-\lambda & 4 \\ 4 & 2-\lambda \end{array}\right| &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of A. To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10.4.10}$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in (10.4.10) yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of (10.4.8). Setting $\lambda = -2$ in (10.4.10) yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of (10.4.8). From Theorem 10.4.1, the general solution of (10.4.8) is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \tag{10.4.11}$$

Solution (b) To satisfy the initial condition in (10.4.9), we must choose c_1 and c_2 in (10.4.11) so that

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned} c_1 - c_2 &= 5 \\ c_1 + c_2 &= -1, \end{aligned}$$

so $c_1 = 2, c_2 = -3$. Therefore the solution of (10.4.9) is

$$\mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

or, in terms of components,

$$y_1 = 2e^{6t} + 3e^{-2t}, \quad y_2 = 2e^{6t} - 3e^{-2t}.$$

Example 10.4.2

a. Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}. \tag{10.4.12}$$

b. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}. \tag{10.4.13}$$

Solution (a) The characteristic polynomial of the coefficient matrix A in (10.4.12) is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-3)(\lambda+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10.4.14}$$

with $\lambda = 2, 3, -1$. With $\lambda = 2$, the augmented matrix of (10.4.14) is

$$\left[\begin{array}{rrrcr} 1 & -1 & -1 & \vdots & 0 \\ -2 & 1 & 2 & \vdots & 0 \\ 4 & -1 & -4 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of (10.4.12). With $\lambda = 3$, the augmented matrix of (10.4.14) is

$$\left[\begin{array}{rrrcr} 0 & -1 & -1 & \vdots & 0 \\ -2 & 0 & 2 & \vdots & 0 \\ 4 & -1 & -5 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right] e^{3t}$$

as a solution of (10.4.12). With $\lambda = -1$, the augmented matrix of (10.4.14) is

$$\left[\begin{array}{rrrcr} 4 & -1 & -1 & \vdots & 0 \\ -2 & 4 & 2 & \vdots & 0 \\ 4 & -1 & -1 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -\frac{1}{7} & \vdots & 0 \\ 0 & 1 & \frac{3}{7} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \left[\begin{array}{r} 1 \\ -3 \\ 7 \end{array}\right] e^{-t}$$

as a solution of (10.4.12). By Theorem 10.4.1, the general solution of (10.4.12) is

$$\mathbf{y} = c_1 \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right] e^{2t} + c_2 \left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right] e^{3t} + c_3 \left[\begin{array}{r} 1 \\ -3 \\ 7 \end{array}\right] e^{-t},$$

which can also be written as

$$\mathbf{y} = \left[\begin{array}{ccc} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right]. \tag{10.4.15}$$

Solution (b) To satisfy the initial condition in (10.4.13) we must choose c_1, c_2, c_3 in (10.4.15) so that

$$\left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & 1 & 7 \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] = \left[\begin{array}{r} 2 \\ -1 \\ 8 \end{array}\right].$$

Solving this system yields $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Hence, the solution of (10.4.13) is

$$\begin{aligned}\mathbf{y} &= \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}.\end{aligned}$$

Example 10.4.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y}. \tag{10.4.16}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.4.16) is

$$\begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+5)^2.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 1, while $\lambda_2 = -5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ are solutions of the system with augmented matrix

$$\left[\begin{array}{rrrcr} -4 & 2 & 2 & \vdots & 0 \\ 2 & -4 & 2 & \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \tag{10.4.17}$$

of (10.4.16). Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\left[\begin{array}{rrrcr} 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \end{array}\right].$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose x_2 and x_3 arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of (10.4.16) are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and (10.4.17), Theorem 10.4.1 implies that the general solution of (10.4.16) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\text{\twocol} y_1' y_2' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{\twocol} y_1 y_2. \tag{10.4.18}$$

It is convenient to think of a "y_1-y_2 plane," where a point is identified by rectangular coordinates (y_1, y_2). If $\mathbf{y} = \text{\dst\twocol} y_1 y_2$ is a non-constant solution of (10.4.18), then the point $(y_1(t), y_2(t))$ moves along a curve C in the y_1-y_2 plane as t varies from $-\infty$ to ∞. We call C the trajectory of $\mathbf{y}$. (We also say that C is a trajectory of the system (10.4.18).) I's important to note that C is the trajectory of infinitely many solutions of (10.4.18), since if τ is any real number, then $\mathbf{y}(t - \tau)$ is a solution of (10.4.18) (Exercise 28(b)), and $(y_1(t-\tau), y_2(t-\tau))$ also moves along C as t varies from $-\infty$ to ∞. Moreover, Exercise 28**(c)** implies that distinct trajectories of (10.4.18) can't intersect, and that two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of (10.4.18) have the same trajectory if and only if $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$ for some τ.

From Exercise 28**(a)**, a trajectory of a nontrivial solution of (10.4.18) can't contain $(0, 0)$, which we define to be the trajectory of the trivial solution $\mathbf{y} \equiv 0$. More generally, if $\mathbf{y} = \text{\dst\twocol} k_1 k_2 \neq \mathbf{0}$ is a constant solution of (10.4.18) (which could occur if zero is an eigenvalue of the matrix of (10.4.18)), we define the trajectory of $\mathbf{y}$ to be the single point (k_1, k_2).

To be specific, this is the question: What do the trajectories look like, and how are they traversed? In this section we'll answer this question, assuming that the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

of (10.4.18) has real eigenvalues λ_1 and λ_2 with associated linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then the general solution of (10.4.18) is

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}. \tag{10.4.19}$$

We'll consider other situations in the next two sections.

We leave it to you (Exercise 35) to classify the trajectories of (10.4.18) if zero is an eigenvalue of A. We'll confine our attention here to the case where both eigenvalues are nonzero. In this case the simplest situation is where $\lambda_1 = \lambda_2 \neq 0$, so (10.4.19) becomes

$$\mathbf{y} = (c_1\mathbf{x}_1 + c_2\mathbf{x}_2)e^{\lambda_1 t}.$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, an arbitrary vector $\mathbf{x}$ can be written as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Therefore the general solution of (10.4.18) can be written as $\mathbf{y} = \mathbf{x}e^{\lambda_1 t}$ where $\mathbf{x}$ is an arbitrary 2-vector, and the trajectories of nontrivial solutions of (10.4.18) are half-lines through (but not including) the origin. The direction of motion is away from the origin if $\lambda_1 > 0$ (Figure 10.4.1), toward it if $\lambda_1 < 0$ (Figure 10.4.2). (In these and the next figures an arrow through a point indicates the direction of motion along the trajectory through the point.)

Figure 10.4.1. *Trajectories of a 2×2 system with a repeated positive eigenvalue*

Figure 10.4.2. *Trajectories of a 2×2 system with a repeated negative eigenvalue*

Now suppose $\lambda_2 > \lambda_1$, and let L_1 and L_2 denote lines through the origin parallel to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. By a half-line of L_1 (or L_2), we mean either of the rays obtained by removing the origin from L_1 (or L_2).

Letting $c_2 = 0$ in (10.4.19) yields $\mathbf{y} = c_1\mathbf{x}_1e^{\lambda_1 t}$. If $c_1 \neq 0$, the trajectory defined by this solution is a half-line of L_1. The direction of motion is away from the origin if $\lambda_1 > 0$, toward the origin if $\lambda_1 < 0$. Similarly, the trajectory of $\mathbf{y} = c_2\mathbf{x}_2e^{\lambda_2 t}$ with $c_2 \neq 0$ is a half-line of L_2.

Henceforth, we assume that c_1 and c_2 in (10.4.19) are both nonzero. In this case, the trajectory of (10.4.19) can't intersect L_1 or L_2, since every point on these lines is on the trajectory of a solution for which either $c_1 = 0$ or $c_2 = 0$. (Remember: distinct trajectories can't intersect!). Therefore the trajectory of (10.4.19) must lie entirely in one of the four open sectors bounded by L_1 and L_2, but do not any point on L_1 or L_2. Since the initial point $(y_1(0), y_2(0))$ defined by

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is on the trajectory, we can determine which sector contains the trajectory from the signs of c_1 and c_2, as shown in Figure 10.4.3.

The direction of $\mathbf{y}(t)$ in (10.4.19) is the same as that of

$$e^{-\lambda_2 t}\mathbf{y}(t) = c_1\mathbf{x}_1e^{-(\lambda_2-\lambda_1)t} + c_2\mathbf{x}_2 \tag{10.4.20}$$

and of

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2e^{(\lambda_2-\lambda_1)t}. \tag{10.4.21}$$

Since the right side of (10.4.20) approaches $c_2\mathbf{x}_2$ as $t \to \infty$, the trajectory is asymptotically parallel to L_2 as $t \to \infty$. Since the right side of (10.4.21) approaches $c_1\mathbf{x}_1$ as $t \to -\infty$, the trajectory is asymptotically parallel to L_1 as $t \to -\infty$.

The shape and direction of traversal of the trajectory of (10.4.19) depend upon whether λ_1 and λ_2 are both positive, both negative, or of opposite signs. We'll now analyze these three cases.

Henceforth $\|\mathbf{u}\|$ denote the length of the vector $\mathbf{u}$.

Figure 10.4.3. *Four open sectors bounded by L_1 and L_2*

Figure 10.4.4. *Two positive eigenvalues; motion away from origin*

Case 1: $\lambda_2 > \lambda_1 > 0$

Figure 10.4.4 shows some typical trajectories. In this case, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is not only asymptotically parallel to L_1 as $t \to -\infty$, but is actually asymptotically tangent to L_1 at the origin. On the other hand, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to\infty} \left\|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\right\| = \lim_{t\to\infty} \|c_1\mathbf{x}_1 e^{\lambda_1 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_2 as $t \to \infty$, it's not asymptotically tangent to L_2. The direction of motion along each trajectory is away from the origin.

Case 2: $0 > \lambda_2 > \lambda_1$

Figure 10.4.5 shows some typical trajectories. In this case, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is asymptotically tangent to L_2 at the origin as $t \to \infty$. On the other hand, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to-\infty} \left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to-\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_1 as $t \to -\infty$, it's not asymptotically tangent to it. The direction of motion along each trajectory is toward the origin.

Figure 10.4.5. *Two negative eigenvalues; motion toward the origin*

Figure 10.4.6. *Eigenvalues of different signs*

Case 3: $\lambda_2 > 0 > \lambda_1$

Figure 10.4.6 shows some typical trajectories. In this case,

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \left\|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\right\| = \lim_{t\to\infty} \|c_1\mathbf{x}_1 e^{\lambda_1 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_2 as $t \to \infty$. Similarly,

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to-\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_1 as $t \to -\infty$. The direction of motion is toward the origin on L_1 and away from the origin on L_2. The direction of motion along any other trajectory is away from L_1, toward L_2.

10.4 Exercises

In Exercises 1–15 find the general solution.

1. \dst$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}$

Show answer

2. \dst$\mathbf{y}' = \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \mathbf{y}$

Show answer

3. \dst$\mathbf{y}' = \frac{1}{5} \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$

Show answer

4. \dst$\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$

Show answer

5. \dst$\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$

Show answer

6. \dst$\mathbf{y}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \mathbf{y}$

Show answer

7. \dst$\mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y}$

Show answer

8. \dst$\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \mathbf{y}$

Show answer

9. \dst$\mathbf{y}' = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \mathbf{y}$

Show answer

10. \dst$\mathbf{y}' = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}$

Show answer

11. \dst$\mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \mathbf{y}$

Show answer

12. \dst$\mathbf{y}' = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{y}$

Show answer

13. $\dst\mathbf{y}' = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}$

Show answer

14. $\dst\mathbf{y}' = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \mathbf{y}$

Show answer

15. $\dst\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y}$

Show answer

In Exercises 16–27 solve the initial value problem.

16. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}-74-67\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}2-4$

Show answer

17. $\backslash\text{dst}\mathbf{y}' = \frac{1}{6}\backslash\text{twobytwo}72-22\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}0-3$

Show answer

18. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}21-1224-15\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}53$

Show answer

19. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}-74-67\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}-17$

Show answer

20. $\backslash\text{dst}\mathbf{y}' = \frac{1}{6}\backslash\text{threebythree}1204-10003\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}471$

Show answer

21. $\backslash\text{dst}\mathbf{y}' = \frac{1}{3}\backslash\text{threebythree}2-23-443210\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}115$

Show answer

22. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}6-3-821-23-3-5\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}0-1-1$

Show answer

23. $\backslash\text{dst}\mathbf{y}' = \frac{1}{3}\backslash\text{threebythree}24-715-5-44-1\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}413$

Show answer

24. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}30111-27103\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}276$

Show answer

25. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}-2-5-1-4-11453\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}8-10-4$

Show answer

26. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}3-104-204-42\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}7102$

Show answer

27. $\backslash\text{dst}\mathbf{y}' = \begin{bmatrix} -2 & 2 & 6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}6-107$

Show answer

28. Let A be an $n \times n$ constant matrix. Then Theorem 10.2.1 implies that the solutions of

$$\mathbf{y}' = A\mathbf{y} \tag{A}$$

are all defined on $(-\infty, \infty)$.

a. Use Theorem 10.2.1 to show that the only solution of (A) that can ever equal the zero vector is $\mathbf{y} \equiv \mathbf{0}$.

b. Suppose $\mathbf{y}_1$ is a solution of (A) and $\mathbf{y}_2$ is defined by $\mathbf{y}_2(t) = \mathbf{y}_1(t-\tau)$, where τ is an arbitrary real number. Show that $\mathbf{y}_2$ is also a solution of (A).

c. Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of (A) and there are real numbers t_1 and t_2 such that $\mathbf{y}_1(t_1) = \mathbf{y}_2(t_2)$. Show that $\mathbf{y}_2(t) = \mathbf{y}_1(t-\tau)$ for all t, where $\tau = t_2 - t_1$. *Hint: Show that $\mathbf{y}_1(t-\tau)$ and $\mathbf{y}_2(t)$ are solutions of the same initial value problem for (A), and apply the uniqueness assertion of Theorem 10.2.1.*

In Exercises 29- 34 describe and graph trajectories of the given system.

29. C/G $\mathbf{y}' = \dst\twobytwo111{-1}\mathbf{y}$

Show answer

30. C/G $\mathbf{y}' = \dst\twobytwo{-4}3{-2}{-11}\mathbf{y}$

Show answer

31. C/G $\mathbf{y}' = \dst\twobytwo9{-3}{-1}11\mathbf{y}$

Show answer

32. C/G $\mathbf{y}' = \dst\twobytwo{-1}{-10}{-5}4\mathbf{y}$

Show answer

33. C/G $\mathbf{y}' = \dst\twobytwo5{-4}110\mathbf{y}$

Show answer

34. C/G $\mathbf{y}' = \dst\twobytwo{-7}13{-5}\mathbf{y}$

Show answer

35. Suppose the eigenvalues of the 2×2 matrix A are $\lambda = 0$ and $\mu \neq 0$, with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Let L_1 be the line through the origin parallel to $\mathbf{x}_1$.

a. Show that every point on L_1 is the trajectory of a constant solution of $\mathbf{y}' = A\mathbf{y}$.

b. Show that the trajectories of nonconstant solutions of $\mathbf{y}' = A\mathbf{y}$ are half-lines parallel to $\mathbf{x}_2$ and on either side of L_1, and that the direction of motion along these trajectories is away from L_1 if $\mu > 0$, or toward L_1 if $\mu < 0$.

The matrices of the systems in Exercises 36-41 are singular. Describe and graph the trajectories of nonconstant solutions of the given systems.

36. C/G $\mathbf{y}' = \dst\twobytwo{-1}{1}{1}{-1}\mathbf{y}$

Show answer

37. C/G $\mathbf{y}' = \dst\twobytwo{-1}{-3}{2}{6}\mathbf{y}$

Show answer

38. C/G $\mathbf{y}' = \dst\twobytwo{1}{-3}{-1}{3}\mathbf{y}$

Show answer

39. C/G $\mathbf{y}' = \dst\twobytwo{1}{-2}{-1}{2}\mathbf{y}$

Show answer

40. C/G $\mathbf{y}' = \dst\twobytwo{-4}{-4}{1}{1}\mathbf{y}$

Show answer

41. C/G $\mathbf{y}' = \dst\twobytwo{3}{-1}{-3}{1}\mathbf{y}$

Show answer

42. L Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t, and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition,

$$P' = aP \quad \text{and} \quad Q' = bQ, \tag{A}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (A) is replaced by

$$\begin{aligned} P' &= aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. (Since negative population doesn't make sense, this system holds only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$.

a. For several choices of a, b, α, and β, verify experimentally (by graphing trajectories of (A) in the P-Q plane) that there's a constant $\rho > 0$ (depending upon a, b, α, and β) with the following properties:

   i. If $Q_0 > \rho P_0$, then P decreases monotonically to zero in finite time, during which Q remains positive.

   ii. If $Q_0 < \rho P_0$, then Q decreases monotonically to zero in finite time, during which P remains positive.

b. Conclude from **(a)** that exactly one of the species becomes extinct in finite time if $Q_0 \neq \rho P_0$. Determine experimentally what happens if $Q_0 = \rho P_0$.

c. Confirm your experimental results and determine γ by expressing the eigenvalues and associated eigenvectors of

$$A = \twobytwo{a}{-\alpha}{-\beta}{b}$$

in terms of a, b, α, and β, and applying the geometric arguments developed at the end of this section.

10.5 Constant Coefficient Homogeneous Systems II

We saw in Section 10.4 that if an $n \times n$ constant matrix A has n real eigenvalues λ_1, λ_2, …, λ_n (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, then the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{x}_1e^{\lambda_1 t} + c_2\mathbf{x}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_ne^{\lambda_n t}.$$

In this section we consider the case where A has n real eigenvalues, but does not have n linearly independent eigenvectors. It is shown in linear algebra that this occurs if and only if A has at least one eigenvalue of multiplicity $r > 1$ such that the associated eigenspace has dimension less than r. In this case A is said to be defective. Since it's beyond the scope of this book to give a complete analysis of systems with defective coefficient matrices, we will restrict our attention to some commonly occurring special cases.

Example 10.5.1

Show that the system

$$\mathbf{y}' = \backslash\mathrm{twobytwo}11{-}254{-}9\mathbf{y} \tag{10.5.1}$$

does not have a fundamental set of solutions of the form $\{\mathbf{x}_1e^{\lambda_1 t}, \mathbf{x}_2e^{\lambda_2 t}\}$, where λ_1 and λ_2 are eigenvalues of the coefficient matrix A of (10.5.1) and $\mathbf{x}_1$, and $\mathbf{x}_2$ are associated linearly independent eigenvectors.

Solution The characteristic polynomial of A is

$$\begin{aligned} \backslash\mathrm{twochar}11{-}254{-}9 &= (\lambda - 11)(\lambda + 9) + 100 \\ &= \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2. \end{aligned}$$

Hence, $\lambda = 1$ is the only eigenvalue of A. The augmented matrix of the system $(A - I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{rrcr} 10 & -25 & \vdots & 0 \\ 4 & -10 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrcr} 1 & -\backslash\mathrm{dst}\frac{5}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = 5x_2/2$ where x_2 is arbitrary. Therefore all eigenvectors of A are scalar multiples of $\mathbf{x}_1 = \backslash\mathrm{dst}\backslash\mathrm{twocol}52$, so A does not have a set of two linearly independent eigenvectors. ∎

From Example 10.5.1, we know that all scalar multiples of $\mathbf{y}_1 = \backslash\mathrm{dst}\backslash\mathrm{twocol}52e^t$ are solutions of (10.5.1); however, to find the general solution we must find a second solution $\mathbf{y}_2$ such that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is linearly independent. Based on your recollection of the procedure for solving a constant coefficient scalar equation

$$ay'' + by' + cy = 0$$

in the case where the characteristic polynomial has a repeated root, you might expect to obtain a second solution of (10.5.1) by multiplying the first solution by t. However, this yields $\mathbf{y}_2 = \backslash\mathrm{dst}\backslash\mathrm{twocol}52te^t$, which doesn't work, since

$$\mathbf{y}_2' = \backslash\mathrm{twocol}52(te^t + e^t), \quad \text{while} \quad \backslash\mathrm{twobytwo}11{-}254{-}9\mathbf{y}_2 = \backslash\mathrm{twocol}52te^t.$$

The next theorem shows what to do in this situation.

Theorem 10.5.1

Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 2 and the associated eigenspace has dimension 1; that is, all λ_1-eigenvectors of A are scalar multiples of an eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{10.5.2}$$

Moreover, if $\mathbf{u}$ is any such vector then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t} \quad \text{and} \quad \mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \tag{10.5.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

A complete proof of this theorem is beyond the scope of this book. The difficulty is in proving that there's a vector $\mathbf{u}$ satisfying (10.5.2), since $\det(A - \lambda_1 I) = 0$. We'll take this without proof and verify the other assertions of the theorem.

We already know that $\mathbf{y}_1$ in (10.5.3) is a solution of $\mathbf{y}' = A\mathbf{y}$. To see that $\mathbf{y}_2$ is also a solution, we compute

$$\begin{aligned} \mathbf{y}_2' - A\mathbf{y}_2 &= \lambda_1\mathbf{u}e^{\lambda_1 t} + \mathbf{x}e^{\lambda_1 t} + \lambda_1\mathbf{x}te^{\lambda_1 t} - A\mathbf{u}e^{\lambda_1 t} - A\mathbf{x}te^{\lambda_1 t} \\ &= (\lambda_1\mathbf{u} + \mathbf{x} - A\mathbf{u})e^{\lambda_1 t} + (\lambda_1\mathbf{x} - A\mathbf{x})te^{\lambda_1 t}. \end{aligned}$$

Since $A\mathbf{x} = \lambda_1\mathbf{x}$, this can be written as

$$\mathbf{y}_2' - A\mathbf{y}_2 = -\left((A - \lambda_1 I)\mathbf{u} - \mathbf{x}\right)e^{\lambda_1 t},$$

and now (10.5.2) implies that $\mathbf{y}_2' = A\mathbf{y}_2$.

To see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, suppose c_1 and c_2 are constants such that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1\mathbf{x}e^{\lambda_1 t} + c_2(\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}) = \mathbf{0}. \tag{10.5.4}$$

We must show that $c_1 = c_2 = 0$. Multiplying (10.5.4) by $e^{-\lambda_1 t}$ shows that

$$c_1\mathbf{x} + c_2(\mathbf{u} + \mathbf{x}t) = \mathbf{0}. \tag{10.5.5}$$

By differentiating this with respect to t, we see that $c_2\mathbf{x} = \mathbf{0}$, which implies $c_2 = 0$, because $\mathbf{x} \neq \mathbf{0}$. Substituting $c_2 = 0$ into (10.5.5) yields $c_1\mathbf{x} = \mathbf{0}$, which implies that $c_1 = 0$, again because $\mathbf{x} \neq \mathbf{0}$

Example 10.5.2

Use Theorem 10.5.1 to find the general solution of the system

$$\mathbf{y}' = \twobytwo{11}{-25}{4}{-9}\mathbf{y} \tag{10.5.6}$$

considered in Example 10.5.1.

Solution In Example 10.5.1 we saw that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2 of the coefficient matrix A in (10.5.6), and that all of the eigenvectors of A are multiples of

$$\mathbf{x} = \twocol52.$$

Therefore

$$\mathbf{y}_1 = \twocol52e^t$$

is a solution of (10.5.6). From Theorem 10.5.1, a second solution is given by $\mathbf{y}_2 = \mathbf{u}e^t + \mathbf{x}te^t$, where $(A - I)\mathbf{u} = \mathbf{x}$. The augmented matrix of this system is

$$\left[\begin{array}{rrcr} 10 & -25 & \vdots & 5 \\ 4 & -10 & \vdots & 2 \end{array}\right],$$

which is row equivalent to

$$\dst\left[\begin{array}{rrcr} 1 & -\frac{5}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore the components of $\mathbf{u}$ must satisfy

$$u_1 - \frac{5}{2}u_2 = \frac{1}{2},$$

where u_2 is arbitrary. We choose $u_2 = 0$, so that $u_1 = 1/2$ and

$$\mathbf{u} = \twocol\frac{1}{2}0.$$

Thus,

$$\mathbf{y}_2 = \twocol10\frac{e^t}{2} + \twocol52te^t.$$

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by Theorem 10.5.1, they form a fundamental set of solutions of (10.5.6). Therefore the general solution of (10.5.6) is

$$\mathbf{y} = c_1\twocol52e^t + c_2\left(\twocol10\frac{e^t}{2} + \twocol52te^t\right).\bbox$$

Note that choosing the arbitrary constant u_2 to be nonzero is equivalent to adding a scalar multiple of $\mathbf{y}_1$ to the second solution $\mathbf{y}_2$ (Exercise 33).

Example 10.5.3

Find the general solution of

$$\mathbf{y}' = \threebythree34{-1}021{-2}22{-5}\mathbf{y}. \tag{10.5.7}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.5.7) is

$$\begin{vmatrix} 3-\lambda & 4 & -10 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -5-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)^2.$$

Hence, the eigenvalues are $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = -1$ with multiplicity 2.

Eigenvectors associated with $\lambda_1 = 1$ must satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} 2 & 4 & -10 & \vdots & 0 \\ 2 & 0 & -2 & \vdots & 0 \\ 2 & 2 & -6 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 2x_3$, where x_3 is arbitrary. Choosing $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \backslash\text{threecol}121.$$

Therefore

$$\mathbf{y}_1 = \backslash\text{threecol}121e^t$$

is a solution of (10.5.7).

Eigenvectors associated with $\lambda_2 = -1$ satisfy $(A + I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} 4 & 4 & -10 & \vdots & 0 \\ 2 & 2 & -2 & \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_3 = 0$ and $x_1 = -x_2$, where x_2 is arbitrary. Choosing $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \backslash\text{threecol}-110,$$

so

$$\mathbf{y}_2 = \backslash\text{threecol}-110e^{-t}$$

is a solution of (10.5.7).

Since all the eigenvectors of A associated with $\lambda_2=-1$ are multiples of $\mathbf{x}_2$, we must now use Theorem 10.5.1 to find a third solution of (10.5.7) in the form

$$\mathbf{y}_3=\mathbf{u}e^{-t}+\backslash\mathrm{threecol}{-1}10te^{-t}, \tag{10.5.8}$$

where $\mathbf{u}$ is a solution of $(A+I)\mathbf{u}=\mathbf{x}_2$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} 4 & 4 & -10 & \vdots & -1 \\ 2 & 2 & -2 & \vdots & 1 \\ 2 & 2 & -4 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $u_3=1/2$ and $u_1=1-u_2$, where u_2 is arbitrary. Choosing $u_2=0$ yields

$$\mathbf{u}=\backslash\mathrm{threecol}10\frac{1}{2},$$

and substituting this into (10.5.8) yields the solution

$$\mathbf{y}_3=\backslash\mathrm{threecol}201\frac{e^{-t}}{2}+\backslash\mathrm{threecol}{-1}10te^{-t}$$

of (10.5.7).

Since the Wronskian of $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ at $t=0$ is

$$\left|\begin{array}{rrr} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{array}\right|=\frac{1}{2},$$

$\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ is a fundamental set of solutions of (10.5.7). Therefore the general solution of (10.5.7) is

$$\mathbf{y}=c_1\backslash\mathrm{threecol}121e^{t}+c_2\backslash\mathrm{threecol}{-1}10e^{-t}+c_3\left(\backslash\mathrm{threecol}201\frac{e^{-t}}{2}+\backslash\mathrm{threecol}{-1}10te^{-t}\right).$$

THEOREM 10.5.2

Suppose the $n\times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is one–dimensional; that is, all eigenvectors associated with λ_1 are scalar multiples of the eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A-\lambda_1 I)\mathbf{u}=\mathbf{x}, \tag{10.5.9}$$

and, if $\mathbf{u}$ is any such vector, there are infinitely many vectors $\mathbf{v}$ such that

$$(A-\lambda_1 I)\mathbf{v}=\mathbf{u}. \tag{10.5.10}$$

If $\mathbf{u}$ satisfies (10.5.9) and $\mathbf{v}$ satisfies (10.5.10), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2} \end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Again, it's beyond the scope of this book to prove that there are vectors $\mathbf{u}$ and $\mathbf{v}$ that satisfy (10.5.9) and (10.5.10). Theorem 10.5.1 implies that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave the rest of the proof to you (Exercise 34).

Example 10.5.4

Use Theorem 10.5.2 to find the general solution of

$$\mathbf{y}' = \threebythree11113{-1}022\mathbf{y}. \tag{10.5.11}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.5.11) is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -(\lambda-2)^3.$$

Hence, $\lambda_1 = 2$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A-2I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$, so the eigenvectors are all scalar multiples of

$$\mathbf{x}_1 = \threecol101.$$

Therefore

$$\mathbf{y}_1 = \threecol101 e^{2t}$$

is a solution of (10.5.11).

We now find a second solution of (10.5.11) in the form

$$\mathbf{y}_2 = \mathbf{u}e^{2t} + \threecol101te^{2t},$$

where $\mathbf{u}$ satisfies $(A-2I)\mathbf{u} = \mathbf{x}_1$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 1 & 1 & \vdots & 1 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 1 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & -\frac{1}{2} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Letting $u_3 = 0$ yields $u_1 = -1/2$ and $u_2 = 1/2$; hence,

$$\mathbf{u} = \frac{1}{2}\threecol-110$$

and

$$\mathbf{y}_2 = \threecol-110\frac{e^{2t}}{2} + \threecol101te^{2t}$$

is a solution of (10.5.11).

We now find a third solution of (10.5.11) in the form

$$\mathbf{y}_3 = \mathbf{v}e^{2t} + \threecol-110\frac{te^{2t}}{2} + \threecol101\frac{t^2e^{2t}}{2}$$

where $\mathbf{v}$ satisfies $(A-2I)\mathbf{v} = \mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 1 & 1 & \vdots & -\frac{1}{2} \\ 1 & 1 & -1 & \vdots & \frac{1}{2} \\ 0 & 2 & 0 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Letting $v_3 = 0$ yields $v_1 = 1/2$ and $v_2 = 0$; hence,

$$\mathbf{v} = \frac{1}{2}\threecol100.$$

Therefore

$$\mathbf{y}_3 = \backslash\mathrm{threecol}100\frac{e^{2t}}{2} + \backslash\mathrm{threecol}{-110}\frac{te^{2t}}{2} + \backslash\mathrm{threecol}101\frac{t^2e^{2t}}{2}$$

is a solution of (10.5.11). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.5.2, they form a fundamental set of solutions of (10.5.11). Therefore the general solution of (10.5.11) is

$$\begin{aligned}\mathbf{y} = {} & \backslash\mathrm{dst}c_1\backslash\mathrm{threecol}101e^{2t} + c_2\left(\backslash\mathrm{threecol}{-110}\frac{e^{2t}}{2} + \backslash\mathrm{threecol}101te^{2t}\right) \\ & + c_3\backslash\mathrm{dst}\left(\backslash\mathrm{threecol}100\frac{e^{2t}}{2} + \backslash\mathrm{threecol}{-110}\frac{te^{2t}}{2} + \backslash\mathrm{threecol}101\frac{t^2e^{2t}}{2}\right).\end{aligned}$$

Theorem 10.5.3

Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is two–dimensional; that is, all eigenvectors of A associated with λ_1 are linear combinations of two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there are constants α and β $($not both zero$)$ such that if

$$\mathbf{x}_3 = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \tag{10.5.12}$$

then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_3. \tag{10.5.13}$$

If $\mathbf{u}$ satisfies (10.5.13), then

$$\begin{aligned}\mathbf{y}_1 &= \mathbf{x}_1e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{x}_2e^{\lambda_1 t}, \text{and} \\ \mathbf{y}_3 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}_3te^{\lambda_1 t},\end{aligned} \tag{10.5.14}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

We omit the proof of this theorem.

Example 10.5.5

Use Theorem 10.5.3 to find the general solution of

$$\mathbf{y}' = \backslash\mathrm{threebythree}001{-111}{-102}\mathbf{y}. \tag{10.5.15}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.5.15) is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda - 1)^3.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 0 & 1 & \vdots & 0 \\ -1 & 0 & 1 & \vdots & 0 \\ -1 & 0 & 1 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and x_2 is arbitrary, so the eigenvectors are of the form

$$\mathbf{x}_1 = \backslash\mathrm{threecol} x_3 x_2 x_3 = x_3 \backslash\mathrm{threecol} 101 + x_2 \backslash\mathrm{threecol} 010.$$

Therefore the vectors

$$\mathbf{x}_1 = \backslash\mathrm{threecol} 101 \quad \text{and} \quad \mathbf{x}_2 = \backslash\mathrm{threecol} 010 \tag{10.5.16}$$

form a basis for the eigenspace, and

$$\mathbf{y}_1 = \backslash\mathrm{threecol} 101 e^t \quad \text{and} \quad \mathbf{y}_2 = \backslash\mathrm{threecol} 010 e^t$$

are linearly independent solutions of (10.5.15).

To find a third linearly independent solution of (10.5.15), we must find constants α and β (not both zero) such that the system

$$(A - I)\mathbf{u} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \tag{10.5.17}$$

has a solution $\mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 0 & 1 & \vdots & \alpha \\ -1 & 0 & 1 & \vdots & \beta \\ -1 & 0 & 1 & \vdots & \alpha \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & -\alpha \\ 0 & 0 & 0 & \vdots & \beta-\alpha \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]. \tag{10.5.18}$$

Therefore (10.5.17) has a solution if and only if $\beta = \alpha$, where α is arbitrary. If $\alpha = \beta = 1$ then (10.5.12) and (10.5.16) yield

$$\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2 = \backslash\mathrm{threecol} 101 + \backslash\mathrm{threecol} 010 = \backslash\mathrm{threecol} 111,$$

and the augmented matrix (10.5.18) becomes

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

This implies that $u_1 = -1 + u_3$, while u_2 and u_3 are arbitrary. Choosing $u_2 = u_3 = 0$ yields

$$\mathbf{u} = \threecol{-1}00.$$

Therefore (10.5.14) implies that

$$\mathbf{y}_3 = \mathbf{u}e^t + \mathbf{x}_3 te^t = \threecol{-1}00 e^t + \threecol111 te^t$$

is a solution of (10.5.15). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.5.3, they form a fundamental set of solutions for (10.5.15). Therefore the general solution of (10.5.15) is

$$\mathbf{y} = c_1\threecol101 e^t + c_2\threecol010 e^t + c_3\left(\threecol{-1}00 e^t + \threecol111 te^t\right).\bbox$$

Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\twocol{y_1'}{y_2'} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right] \twocol{y_1}{y_2} \tag{10.5.19}$$

under the assumptions of this section; that is, when the matrix

$$A = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right]$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. In this case we know from Theorem 10.5.1 that the general solution of (10.5.19) is

$$\mathbf{y} = c_1\mathbf{x}e^{\lambda_1 t} + c_2(\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}), \tag{10.5.20}$$

where $\mathbf{x}$ is an eigenvector of A and $\mathbf{u}$ is any one of the infinitely many solutions of

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{10.5.21}$$

We assume that $\lambda_1 \neq 0$.

Figure 10.5.1. *Positive and negative half-planes*

Let L denote the line through the origin parallel to $\mathbf{x}$. By a half-line of L we mean either of the rays obtained by removing the origin from L. Eqn. (10.5.20) is a parametric equation of the half-line of L in the direction of $\mathbf{x}$ if $c_1 > 0$, or of the half-line of L in the direction of $-\mathbf{x}$ if $c_1 < 0$. The origin is the trajectory of the trivial solution $\mathbf{y} \equiv \mathbf{0}$.

Henceforth, we assume that $c_2 \neq 0$. In this case, the trajectory of (10.5.20) can't intersect L, since every point of L is on a trajectory obtained by setting $c_2 = 0$. Therefore the trajectory of (10.5.20) must lie entirely in one of the open half-planes bounded by L, but does not contain any point on L. Since the initial point $(y_1(0), y_2(0))$ defined by $\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{u}$ is on the trajectory, we can determine which half-plane contains the trajectory from the sign of c_2, as shown in Figure 10.5.1. For convenience we'll call the half-plane where $c_2 > 0$ the positive half-plane. Similarly, the-half plane where $c_2 < 0$ is the negative half-plane. You should convince yourself (Exercise 35) that even though there are infinitely many vectors $\mathbf{u}$ that satisfy (10.5.21), they all define the same positive and negative half-planes. In the figures simply regard $\mathbf{u}$ as an arrow pointing to the positive half-plane, since wen't attempted to give $\mathbf{u}$

its proper length or direction in comparison with $\mathbf{x}$. For our purposes here, only the relative orientation of $\mathbf{x}$ and $\mathbf{u}$ is important; that is, whether the positive half-plane is to the right of an observer facing the direction of $\mathbf{x}$ (as in Figures 10.5.2 and 10.5.5), or to the left of the observer (as in Figures 10.5.3 and 10.5.4).

Multiplying (10.5.20) by $e^{-\lambda_1 t}$ yields

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x} + c_2\mathbf{u} + c_2 t\mathbf{x}.$$

Since the last term on the right is dominant when $|t|$ is large, this provides the following information on the direction of $\mathbf{y}(t)$:

a. Along trajectories in the positive half-plane ($c_2 > 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $\mathbf{x}$ as $t \to \infty$ and the direction of $-\mathbf{x}$ as $t \to -\infty$.
b. Along trajectories in the negative half-plane ($c_2 < 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $-\mathbf{x}$ as $t \to \infty$ and the direction of $\mathbf{x}$ as $t \to -\infty$.

Since

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 > 0,$$

or

$$\lim_{t-\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 < 0,$$

there are four possible patterns for the trajectories of (10.5.19), depending upon the signs of c_2 and λ_1. Figures 10.5.2-10.5.5 illustrate these patterns, and reveal the following principle:

If λ_1 and c_2 have the same sign then the direction of the traectory approaches the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$. If λ_1 and c_2 have opposite signs then the direction of the trajectory approaches the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$.

Figure 10.5.2. *Positive eigenvalue; motion away from the origin*

Figure 10.5.3. *Positive eigenvalue; motion away from the origin*

Figure 10.5.4. *Negative eigenvalue; motion toward the origin*

Figure 10.5.5. *Negative eigenvalue; motion toward the origin*

10.5 Exercises

In Exercises 1–12 find the general solution.

1. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}34{-}17\mathbf{y}$

Show answer

2. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}0{-}11{-}2\mathbf{y}$

Show answer

3. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}{-}74{-}1{-}11\mathbf{y}$

Show answer

4. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}31{-}11\mathbf{y}$

Show answer

5. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}412{-}3{-}8\mathbf{y}$

Show answer

6. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}{-}109{-}42\mathbf{y}$

Show answer

7. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}{-}1316{-}911\mathbf{y}$

Show answer

8. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}021{-}461042\mathbf{y}$

Show answer

9. $\backslash\text{dst}\mathbf{y}' = \frac{1}{3}\backslash\text{threebythree}11{-}3{-}4{-}43{-}210\mathbf{y}$

Show answer

10. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}{-}11{-}1{-}202{-}13{-}1\mathbf{y}$

Show answer

11. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}4{-}2{-}2{-}23{-}12{-}13\mathbf{y}$

Show answer

12. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}6{-}532{-}13211\mathbf{y}$

Show answer

In Exercises 13–23 solve the initial value problem.

13. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}-118-2-3\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}62$

Show answer

14. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}15-916-9\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}58$

Show answer

15. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}-3-41-7\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}23$

Show answer

16. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}-724-617\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}31$

Show answer

17. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}-73-3-1\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}02$

Show answer

18. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}-1101-1-2-1-1-1\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}65-7$

Show answer

19. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}-221-221-332\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}-6-20$

Show answer

20. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}-7-44-101-9-56\mathbf{y}, \quad \mathbf{y(0)} = \backslash\text{threecol}\mathbf{-69-1}$

Show answer

21. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}-1-4-1361-3-23\mathbf{y,} \quad \mathbf{y(0)} = \backslash\text{threecol}\mathbf{-213}$

Show answer

22. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}4-8-4-3-1-31-19\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}-41-3$

Show answer

23. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}-5-111-7113-408\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}022$

Show answer

The coefficient matrices in Exercises 24–32 have eigenvalues of multiplicity 3. Find the general solution.

24. $\dst\mathbf{y}' = \threebythree5{-1}1{-1}9{-3}{-2}24\mathbf{y}$

Show answer

25. $\dst\mathbf{y}' = \threebythree110{-1}22232{-1}6\mathbf{y}$

Show answer

26. $\dst\mathbf{y}' = \threebythree{-6}{-4}{-4}2{-1}1231\mathbf{y}$

Show answer

27. $\dst\mathbf{y}' = \threebythree02{-2}{-1}5{-3}111\mathbf{y}$

Show answer

28. $\dst\mathbf{y}' = \threebythree{-2}{-12}10{2}{-24}11{2}{-24}8\mathbf{y}$

Show answer

29. $\dst\mathbf{y}' = \threebythree{-1}{-12}8{1}{-9}41{-6}1\mathbf{y}$

Show answer

30. $\dst\mathbf{y}' = \threebythree{-4}0{-1}{-1}{-3}{-1}10{-2}\mathbf{y}$

Show answer

31. $\dst\mathbf{y}' = \threebythree{-3}{-3}44{5}{-8}23{-5}\mathbf{y}$

Show answer

32. $\mathbf{y}' = \threebythree{-3}{-1}0{1}{-1}0{-1}{-1}{-2}\mathbf{y}$

Show answer

33. Under the assumptions of Theorem 10.5.1, suppose $\mathbf{u}$ and $\hat{\mathbf{u}}$ are vectors such that

$$(A-\lambda_1 I)\mathbf{u}=\mathbf{x}\quad\text{and}\quad(A-\lambda_1 I)\hat{\mathbf{u}}=\mathbf{x},$$

and let

$$\mathbf{y}_2=\mathbf{u}e^{\lambda_1 t}+\mathbf{x}te^{\lambda_1 t}\quad\text{and}\quad\hat{\mathbf{y}}_2=\hat{\mathbf{u}}e^{\lambda_1 t}+\mathbf{x}te^{\lambda_1 t}.$$

Show that $\mathbf{y}_2-\hat{\mathbf{y}}_2$ is a scalar multiple of $\mathbf{y}_1=\mathbf{x}e^{\lambda_1 t}$.

34. Under the assumptions of Theorem 10.5.2, let

$$\begin{aligned}\mathbf{y}_1&=\mathbf{x}e^{\lambda_1 t},\\ \mathbf{y}_2&=\mathbf{u}e^{\lambda_1 t}+\mathbf{x}te^{\lambda_1 t},\ \text{and}\\ \mathbf{y}_3&=\mathbf{v}e^{\lambda_1 t}+\mathbf{u}te^{\lambda_1 t}+\mathbf{x}\frac{t^2e^{\lambda_1 t}}{2}.\end{aligned}$$

Complete the proof of Theorem 10.5.2 by showing that $\mathbf{y}_3$ is a solution of $\mathbf{y}'=A\mathbf{y}$ and that $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ is linearly independent.

35. Suppose the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. Let $\mathbf{x}$ be a λ_1-eigenvector of A. Show that if $(A - \lambda_1 I)\mathbf{u}_1 = \mathbf{x}$ and $(A - \lambda_1 I)\mathbf{u}_2 = \mathbf{x}$, then $\mathbf{u}_2 - \mathbf{u}_1$ is parallel to $\mathbf{x}$. Conclude from this that all vectors $\mathbf{u}$ such that $(A - \lambda_1 I)\mathbf{u} = \mathbf{x}$ define the same positive and negative half-planes with respect to the line L through the origin parallel to $\mathbf{x}$.

In Exercises 36- 45 plot trajectories of the given system.

36. C/G $\mathbf{y}' =$ \dst\twobytwo$-3-141\mathbf{y}$

37. C/G $\mathbf{y}' =$ \dst\twobytwo$2-110\mathbf{y}$

38. C/G $\mathbf{y}' =$ \dst\twobytwo$-1-335\mathbf{y}$

39. C/G $\mathbf{y}' =$ \dst\twobytwo$-53-31\mathbf{y}$

40. C/G $\mathbf{y}' =$ \dst\twobytwo$-2-334\mathbf{y}$

41. C/G $\mathbf{y}' =$ \dst\twobytwo$-4-332\mathbf{y}$

42. C/G $\mathbf{y}' =$ \dst\twobytwo$0-11-2\mathbf{y}$

43. C/G $\mathbf{y}' =$ \dst\twobytwo$01-12\mathbf{y}$

44. C/G $\mathbf{y}' =$ \dst\twobytwo$-21-10\mathbf{y}$

45. C/G $\mathbf{y}' =$ \dst\twobytwo$0-41-4\mathbf{y}$

10.6 Constant Coefficient Homogeneous Systems III

We now consider the system $\mathbf{y}' = A\mathbf{y}$, where A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$. We continue to assume that A has real entries, so the characteristic polynomial of A has real coefficients. This implies that $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of A.

An eigenvector $\mathbf{x}$ of A associated with $\lambda = \alpha + i\beta$ will have complex entries, so we'll write

$$\mathbf{x} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ have real entries; that is, $\mathbf{u}$ and $\mathbf{v}$ are the real and imaginary parts of $\mathbf{x}$. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}). \tag{10.6.1}$$

Taking complex conjugates here and recalling that A has real entries yields

$$A(\mathbf{u} - i\mathbf{v}) = (\alpha - i\beta)(\mathbf{u} - i\mathbf{v}),$$

which shows that $\mathbf{x} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated with $\overline{\lambda} = \alpha - i\beta$. The complex conjugate eigenvalues λ and $\overline{\lambda}$ can be separately associated with linearly independent solutions $\mathbf{y}' = A\mathbf{y}$; however, we won't pursue this approach, since solutions obtained in this way turn out to be complex–valued. Instead, we'll obtain solutions of $\mathbf{y}' = A\mathbf{y}$ in the form

$$\mathbf{y} = f_1\mathbf{u} + f_2\mathbf{v} \tag{10.6.2}$$

where f_1 and f_2 are real–valued scalar functions. The next theorem shows how to do this.

Theorem 10.6.1

Let A be an $n \times n$ matrix with real entries. Let $\lambda = \alpha + i\beta$ ($\beta \neq 0$) be a complex eigenvalue of A and let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ be an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Then $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

which are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}), \tag{10.6.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof A function of the form (10.6.2) is a solution of $\mathbf{y}' = A\mathbf{y}$ if and only if

$$f_1'\mathbf{u} + f_2'\mathbf{v} = f_1 A\mathbf{u} + f_2 A\mathbf{v}. \tag{10.6.4}$$

Carrying out the multiplication indicated on the right side of (10.6.1) and collecting the real and imaginary parts of the result yields

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha\mathbf{u} - \beta\mathbf{v}) + i(\alpha\mathbf{v} + \beta\mathbf{u}).$$

Equating real and imaginary parts on the two sides of this equation yields

$$\begin{array}{rcl} A\mathbf{u} & = & \alpha\mathbf{u} - \beta\mathbf{v} \\ A\mathbf{v} & = & \alpha\mathbf{v} + \beta\mathbf{u}. \end{array}$$

We leave it to you (Exercise 25) to show from this that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero. Substituting from these equations into (10.6.4) yields

$$\begin{aligned} f_1'\mathbf{u} + f_2'\mathbf{v} &= f_1(\alpha\mathbf{u} - \beta\mathbf{v}) + f_2(\alpha\mathbf{v} + \beta\mathbf{u}) \\ &= (\alpha f_1 + \beta f_2)\mathbf{u} + (-\beta f_1 + \alpha f_2)\mathbf{v}. \end{aligned}$$

This is true if

$$\begin{array}{rcr} f_1' & = & \alpha f_1 + \beta f_2 \\ f_2' & = & -\beta f_1 + \alpha f_2, \end{array} \quad \text{or, equivalently,} \quad \begin{array}{rcr} f_1' - \alpha f_1 & = & \beta f_2 \\ f_2' - \alpha f_2 & = & -\beta f_1. \end{array}$$

If we let $f_1 = g_1 e^{\alpha t}$ and $f_2 = g_2 e^{\alpha t}$, where g_1 and g_2 are to be determined, then the last two equations become

$$\begin{array}{rcr} g_1' & = & \beta g_2 \\ g_2' & = & -\beta g_1, \end{array}$$

which implies that

$$g_1'' = \beta g_2' = -\beta^2 g_1,$$

so

$$g_1'' + \beta^2 g_1 = 0.$$

The general solution of this equation is

$$g_1 = c_1 \cos \beta t + c_2 \sin \beta t.$$

Moreover, since $g_2 = g_1'/\beta$,

$$g_2 = -c_1 \sin \beta t + c_2 \cos \beta t.$$

Multiplying g_1 and g_2 by $e^{\alpha t}$ shows that

$$\begin{aligned} f_1 &= e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t), \\ f_2 &= e^{\alpha t}(-c_1 \sin \beta t + c_2 \cos \beta t). \end{aligned}$$

Substituting these into (10.6.2) shows that

$$\begin{aligned} \mathbf{y} &= e^{\alpha t}\left[(c_1 \cos \beta t + c_2 \sin \beta t)\mathbf{u} + (-c_1 \sin \beta t + c_2 \cos \beta t)\mathbf{v}\right] \\ &= c_1 e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) + c_2 e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t) \end{aligned} \tag{10.6.5}$$

is a solution of $\mathbf{y}' = A\mathbf{y}$ for any choice of the constants c_1 and c_2. In particular, by first taking $c_1 = 1$ and $c_2 = 0$ and then taking $c_1 = 0$ and $c_2 = 1$, we see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave it to you to verify that they are, respectively, the real and imaginary parts of (10.6.3) (Exercise 26), and that they are linearly independent (Exercise 27).

Example 10.6.1

Find the general solution of

$$\mathbf{y}' = \backslash\text{twobytwo}4{-5}5{-2}\mathbf{y}. \tag{10.6.6}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.6) is

$$\left| \begin{array}{cc} 4-\lambda & -5 \\ 5 & -2-\lambda \end{array} \right| = (\lambda - 1)^2 + 16.$$

Hence, $\lambda = 1 + 4i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A - (1+4i)\,I)\,\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccccc} 3-4i & -5 & \vdots & 0 \\ 5 & -3-4i & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cccc} 1 & -\frac{3+4i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array} \right].$$

Therefore $x_1 = (3+4i)x_2/5$. Taking $x_2 = 5$ yields $x_1 = 3+4i$, so

$$\mathbf{x} = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 4t + i \sin 4t) \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix},$$

which are linearly independent solutions of (10.6.6). The general solution of (10.6.6) is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix}.$$

Example 10.6.2

Find the general solution of

$$\mathbf{y}' = \backslash\text{twobytwo}-1439-616\mathbf{y}. \tag{10.6.7}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.7) is

$$\begin{vmatrix} -14-\lambda & 39 \\ -6 & 16-\lambda \end{vmatrix} = (\lambda-1)^2 + 9.$$

Hence, $\lambda = 1+3i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A-(1+3i)I)\,\mathbf{x} = \mathbf{0}$. The augmented augmented matrix of this system is

$$\left[\begin{array}{ccccc} -15-3i & 39 & \vdots & 0 \\ -6 & 15-3i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cccc} 1 & \frac{-5+i}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = (5-i)/2$. Taking $x_2 = 2$ yields $x_1 = 5-i$, so

$$\mathbf{x} = \begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^{t}(\cos 3t + i\sin 3t)\begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t\begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t\begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix},$$

which are linearly independent solutions of (10.6.7). The general solution of (10.6.7) is

$$\mathbf{y} = c_1e^t\begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} + c_2e^t\begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix}.$$

Example 10.6.3

Find the general solution of

$$\mathbf{y}' = \backslash\text{threebythree}{-554{-876100}}\mathbf{y}. \tag{10.6.8}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.8) is

$$\begin{vmatrix} -5-\lambda & 5 & 4 \\ -8 & 7-\lambda & 6 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. The augmented matrix of $(A-2I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{rrrcr} -7 & 5 & 4 & \vdots & 0 \\ -8 & 5 & 6 & \vdots & 0 \\ 1 & 0 & -2 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -2 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \backslash\text{threecol}221,$$

so

$$\mathbf{y}_1 = \backslash\text{threecol}221e^{2t}$$

is a solution of (10.6.8).

The augmented matrix of $(A-iI)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{cccc} -5-i & 5 & 4 & \vdots & 0 \\ -8 & 7-i & 6 & \vdots & 0 \\ 1 & 0 & -i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cccc} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & 1-i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -(1-i)x_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}.$$

The real and imaginary parts of

$$(\cos t + i \sin t) \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix},$$

which are solutions of (10.6.8). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (10.6.8). The general solution of (10.6.8) is

$$\mathbf{y} = c_1 \text{\threecol}221 e^{2t} + c_2 \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix}.$$

Example 10.6.4

Find the general solution of

$$\mathbf{y}' = \text{\threebythree}1{-1}{-2}1321{-1}2\mathbf{y}. \tag{10.6.9}$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.9) is

$$\left|\begin{array}{ccc} 1-\lambda & -1 & -2 \\ 1 & 3-\lambda & 2 \\ 1 & -1 & 2-\lambda \end{array}\right| = -(\lambda-2)\left((\lambda-2)^2+4\right).$$

Hence, the eigenvalues of A are $\lambda_1=2$, $\lambda_2=2+2i$, and $\lambda_3=2-2i$. The augmented matrix of $(A-2I)\mathbf{x}=\mathbf{0}$ is

$$\left[\begin{array}{rrrcr} -1 & -1 & -2 & \vdots & 0 \\ 1 & 1 & 2 & \vdots & 0 \\ 1 & -1 & 0 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1=x_2=-x_3$. Taking $x_3=1$ yields

$$\mathbf{x}_1 = \threecol{-1}{-1}{1},$$

so

$$\mathbf{y}_1 = \threecol{-1}{-1}{1} e^{2t}$$

is a solution of (10.6.9).

The augmented matrix of $(A-(2+2i)I)\,\mathbf{x}=\mathbf{0}$ is

$$\left[\begin{array}{cccc c} -1-2i & -1 & -2 & \vdots & 0 \\ 1 & 1-2i & 2 & \vdots & 0 \\ 1 & -1 & -2i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1=ix_3$ and $x_2=-ix_3$. Taking $x_3=1$ yields the eigenvector

$$\mathbf{x}_2 = \threecol{i}{-i}{1}$$

The real and imaginary parts of

$$e^{2t}(\cos 2t + i\sin 2t)\threecol{i}{-i}{1}$$

are

$$\mathbf{y}_2 = e^{2t}\begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^{2t}\begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix},$$

which are solutions of (10.6.9). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -2,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (10.6.9). The general solution of (10.6.9) is

$$\mathbf{y} = c_1\text{\threecol}{-1}{-1}{1}e^{2t} + c_2e^{2t}\begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} + c_3e^{2t}\begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix}.$$

Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\text{\twocol}{y_1'}{y_2'} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\text{\twocol}{y_1}{y_2} \tag{10.6.10}$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a complex eigenvalue $\lambda = \alpha + i\beta$ ($\beta \neq 0$) and $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. To describe the trajectories accurately it's necessary to introduce a new rectangular coordinate system in the y_1-y_2 plane. This raises a point that hasn't come up before: It is always possible to choose $\mathbf{x}$ so that $(\mathbf{u}, \mathbf{v}) = 0$. A special effort is required to do this, since not every eigenvector has this property. However, if we know an eigenvector that doesn't, we can multiply it by a suitable complex constant to obtain one that does. To see this, note that if $\mathbf{x}$ is a λ-eigenvector of A and k is an arbitrary real number, then

$$\mathbf{x}_1 = (1 + ik)\mathbf{x} = (1 + ik)(\mathbf{u} + i\mathbf{v}) = (\mathbf{u} - k\mathbf{v}) + i(\mathbf{v} + k\mathbf{u})$$

is also a λ-eigenvector of A, since

$$A\mathbf{x}_1 = A((1 + ik)\mathbf{x}) = (1 + ik)A\mathbf{x} = (1 + ik)\lambda\mathbf{x} = \lambda((1 + ik)\mathbf{x}) = \lambda\mathbf{x}_1.$$

The real and imaginary parts of $\mathbf{x}_1$ are

$$\mathbf{u}_1 = \mathbf{u} - k\mathbf{v} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + k\mathbf{u}, \tag{10.6.11}$$

so

$$(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{u} - k\mathbf{v}, \mathbf{v} + k\mathbf{u}) = -\left[(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v})\right].$$

Therefore $(\mathbf{u}_1, \mathbf{v}_1) = 0$ if

$$(\mathbf{u},\mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u},\mathbf{v}) = 0. \tag{10.6.12}$$

If $(\mathbf{u},\mathbf{v}) \neq 0$ we can use the quadratic formula to find two real values of k such that $(\mathbf{u}_1,\mathbf{v}_1) = 0$ (Exercise 28).

Example 10.6.5

In Example 10.6.1 we found the eigenvector

$$\mathbf{x} = \backslash\text{ctwocol}3 + 4i5 = \backslash\text{twocol}35 + i\backslash\text{twocol}40$$

for the matrix of the system (10.6.6). Here $\mathbf{u} = \backslash\text{dst}\backslash\text{twocol}35$ and $\mathbf{v} = \backslash\text{twocol}40$ are not orthogonal, since $(\mathbf{u},\mathbf{v}) = 12$. Since $\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 = -18$, (10.6.12) is equivalent to

$$2k^2 - 3k - 2 = 0.$$

The zeros of this equation are $k_1 = 2$ and $k_2 = -1/2$. Letting $k = 2$ in (10.6.11) yields

$$\mathbf{u}_1 = \mathbf{u} - 2\mathbf{v} = \backslash\text{twocol}{-5}\quad 5 \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + 2\mathbf{u} = \backslash\text{twocol}1010,$$

and $(\mathbf{u}_1,\mathbf{v}_1) = 0$. Letting $k = -1/2$ in (10.6.11) yields

$$\mathbf{u}_1 = \mathbf{u} + \frac{\mathbf{v}}{2} = \backslash\text{twocol}55 \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} - \frac{\mathbf{u}}{2} = \frac{1}{2}\backslash\text{twocol}{-5}\quad 5,$$

and again $(\mathbf{u}_1,\mathbf{v}_1) = 0$. ∎

(The numbers don't always work out as nicely as in this example. You'll need a calculator or computer to do Exercises 29-40.)

Henceforth, we'll assume that $(\mathbf{u},\mathbf{v}) = 0$. Let $\mathbf{U}$ and $\mathbf{V}$ be unit vectors in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively; that is, $\mathbf{U} = \mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{V} = \mathbf{v}/\|\mathbf{v}\|$. The new rectangular coordinate system will have the same origin as the y_1-y_2 system. The coordinates of a point in this system will be denoted by (z_1, z_2), where z_1 and z_2 are the displacements in the directions of $\mathbf{U}$ and $\mathbf{V}$, respectively.

From (10.6.5), the solutions of (10.6.10) are given by

$$\mathbf{y} = e^{\alpha t}\left[(c_1\cos\beta t + c_2\sin\beta t)\mathbf{u} + (-c_1\sin\beta t + c_2\cos\beta t)\mathbf{v}\right]. \tag{10.6.13}$$

For convenience, let's call the curve traversed by $e^{-\alpha t}\mathbf{y}(t)$ a shadow trajectory of (10.6.10). Multiplying (10.6.13) by $e^{-\alpha t}$ yields

$$e^{-\alpha t}\mathbf{y}(t) = z_1(t)\mathbf{U} + z_2(t)\mathbf{V},$$

where

$$\begin{aligned} z_1(t) &= \|\mathbf{u}\|(c_1\cos\beta t + c_2\sin\beta t) \\ z_2(t) &= \|\mathbf{v}\|(-c_1\sin\beta t + c_2\cos\beta t). \end{aligned}$$

Therefore

$$\frac{(z_1(t))^2}{\|\mathbf{u}\|^2} + \frac{(z_2(t))^2}{\|\mathbf{v}\|^2} = c_1^2 + c_2^2$$

(verify!), which means that the shadow trajectories of (10.6.10) are ellipses centered at the origin, with axes of symmetry parallel to $\mathbf{U}$ and $\mathbf{V}$. Since

$$z_1' = \frac{\beta\|\mathbf{u}\|}{\|\mathbf{v}\|} z_2 \quad \text{and} \quad z_2' = -\frac{\beta\|\mathbf{v}\|}{\|\mathbf{u}\|} z_1,$$

the vector from the origin to a point on the shadow ellipse rotates in the same direction that $\mathbf{V}$ would have to be rotated by $\pi/2$ radians to bring it into coincidence with $\mathbf{U}$ (Figures 10.6.1 and 10.6.2).

Figure 10.6.1. *Shadow trajectories traversed clockwise*

Figure 10.6.2. *Shadow trajectories traversed counterclockwise*

If $\alpha = 0$, then any trajectory of (10.6.10) is a shadow trajectory of (10.6.10); therefore, if λ is purely imaginary, then the trajectories of (10.6.10) are ellipses traversed periodically as indicated in Figures 10.6.1 and 10.6.2.

If $\alpha > 0$, then

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals away from the origin as t varies from $-\infty$ to ∞. The direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 10.6.3 and 10.6.4.

If $\alpha < 0$, then

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals toward the origin as t varies from $-\infty$ to ∞. Again, the direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 10.6.5 and 10.6.6.

Figure 10.6.3. $\alpha > 0$*; shadow trajectory spiraling outward*

Figure 10.6.4. $\alpha > 0$*; shadow trajectory spiraling outward*

Figure 10.6.5. $\alpha < 0$*; shadow trajectory spiraling inward*

Figure 10.6.6. $\alpha < 0$*; shadow trajectory spiraling inward*

10.6 Exercises

In Exercises 1–16 find the general solution.

1. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{twobytwo}-12-55\mathbf{y}$

Show answer

2. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{twobytwo}-114-269\mathbf{y}$

Show answer

3. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{twobytwo}12-45\mathbf{y}$

Show answer

4. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{twobytwo}5-63-1\mathbf{y}$

Show answer

5. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}3-31022511\mathbf{y}$

Show answer

6. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}-3311-5-3-373\mathbf{y}$

Show answer

7. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}21-1011101\mathbf{y}$

Show answer

8. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}-31-34-124-23\mathbf{y}$

Show answer

9. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{twobytwo}5-4101\mathbf{y}$

Show answer

10. $\backslash\mathrm{dst}\mathbf{y}' = \frac{1}{3}\backslash\mathrm{twobytwo}7-525\mathbf{y}$

Show answer

11. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{twobytwo}32-51\mathbf{y}$

Show answer

12. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{twobytwo}3452-20-30\mathbf{y}$

Show answer

13. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}11210-1-1-2-1\mathbf{y}$

Show answer

14. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}3-4-2-57-8-1013-8\mathbf{y}$

Show answer

15. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}60-3-3331-26\mathbf{y}'$

Show answer

16. $\backslash\mathrm{dst}\mathbf{y}' = \backslash\mathrm{threebythree}12-202-1100\mathbf{y}'$

Show answer

In Exercises 17–24 solve the initial value problem.

17. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}4-63-2\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}52$

Show answer

18. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}715-31\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}51$

Show answer

19. $\backslash\text{dst}\mathbf{y}' = \backslash\text{twobytwo}7-153-5\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}177$

Show answer

20. $\backslash\text{dst}\mathbf{y}' = \frac{1}{6}\backslash\text{twobytwo}4-252\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{twocol}1-1$

Show answer

21. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}52-1-322132\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}406$

Show answer

22. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}440810-2023-2\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}865$

Show answer

23. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}115-15-618-22-311-15\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}151710$

Show answer

24. $\backslash\text{dst}\mathbf{y}' = \backslash\text{threebythree}4-44-103152-31\mathbf{y}, \quad \mathbf{y}(0) = \backslash\text{threecol}16146$

Show answer

25. Suppose an $n \times n$ matrix A with real entries has a complex eigenvalue $\lambda = \alpha + i\beta$ ($\beta \neq 0$) with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Show that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero.

26. Verify that

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}).$$

27. Show that if the vectors $\mathbf{u}$ and $\mathbf{v}$ are not both $\mathbf{0}$ and $\beta \neq 0$ then the vector functions

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)$$

are linearly independent on every interval. *Hint: There are two cases to consider: (i)* $\{\mathbf{u}, \mathbf{v}\}$ *linearly independent, and (ii)* $\{\mathbf{u}, \mathbf{v}\}$ *linearly dependent. In either case, exploit the the linear independence of* $\{\cos\beta t, \sin\beta t\}$ *on every interval.*

28. Suppose $\mathbf{u} =$ \dst\twocol$u_1 u_2$ and $\mathbf{v} =$ \dst\twocol$v_1 v_2$ are not orthogonal; that is, $(\mathbf{u}, \mathbf{v}) \neq 0$.

a. Show that the quadratic equation

$$(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v}) = 0$$

has a positive root k_1 and a negative root $k_2 = -1/k_1$.

b. Let $\mathbf{u}_1^{(1)} = \mathbf{u} - k_1\mathbf{v}$, $\mathbf{v}_1^{(1)} = \mathbf{v} + k_1\mathbf{u}$, $\mathbf{u}_1^{(2)} = \mathbf{u} - k_2\mathbf{v}$, and $\mathbf{v}_1^{(2)} = \mathbf{v} + k_2\mathbf{u}$, so that $(\mathbf{u}_1^{(1)}, \mathbf{v}_1^{(1)}) = (\mathbf{u}_1^{(2)}, \mathbf{v}_1^{(2)}) = 0$, from the discussion given above. Show that

$$\mathbf{u}_1^{(2)} = \frac{\mathbf{v}_1^{(1)}}{k_1} \quad \text{and} \quad \mathbf{v}_1^{(2)} = -\frac{\mathbf{u}_1^{(1)}}{k_1}.$$

c. Let $\mathbf{U}_1$, $\mathbf{V}_1$, $\mathbf{U}_2$, and $\mathbf{V}_2$ be unit vectors in the directions of $\mathbf{u}_1^{(1)}$, $\mathbf{v}_1^{(1)}$, $\mathbf{u}_1^{(2)}$, and $\mathbf{v}_1^{(2)}$, respectively. Conclude from **(a)** that $\mathbf{U}_2 = \mathbf{V}_1$ and $\mathbf{V}_2 = -\mathbf{U}_1$, and that therefore the counterclockwise angles from $\mathbf{U}_1$ to $\mathbf{V}_1$ and from $\mathbf{U}_2$ to $\mathbf{V}_2$ are both $\pi/2$ or both $-\pi/2$.

In Exercises 29-32 find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the trajectories, and plot some typical trajectories.

29. C/G \dst$\mathbf{y}' =$ \twobytwo$3{-}55{-}3\mathbf{y}$

Show answer

30. C/G \dst$\mathbf{y}' =$ \twobytwo$-1510{-}2515\mathbf{y}$

Show answer

31. C/G \dst$\mathbf{y}' =$ \twobytwo$-48{-}44\mathbf{y}$

Show answer

32. C/G \dst$\mathbf{y}' =$ \twobytwo$-3{-}1533\mathbf{y}$

Show answer

In Exercises 33-40 find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the shadow trajectories, and plot a typical trajectory.

33. C/G \dst$\mathbf{y}' =$ \twobytwo$-56-127\mathbf{y}$

Show answer

34. C/G \dst$\mathbf{y}' =$ \twobytwo$5-126-7\mathbf{y}$

Show answer

35. C/G \dst$\mathbf{y}' =$ \twobytwo$4-59-2\mathbf{y}$

Show answer

36. C/G \dst$\mathbf{y}' =$ \twobytwo$-49-52\mathbf{y}$

Show answer

37. C/G \dst$\mathbf{y}' =$ \twobytwo$-110-10-1\mathbf{y}$

Show answer

38. C/G \dst$\mathbf{y}' =$ \twobytwo$-1-520-1\mathbf{y}$

Show answer

39. C/G \dst$\mathbf{y}' =$ \twobytwo$-710-109\mathbf{y}$

Show answer

40. C/G \dst$\mathbf{y}' =$ \twobytwo$-76-125\mathbf{y}$

Show answer

10.7 Variation of Parameters for Nonhomogeneous Linear Systems

We now consider the nonhomogeneous linear system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

where A is an $n \times n$ matrix function and $\mathbf{f}$ is an n-vector forcing function. Associated with this system is the complementary system $\mathbf{y}' = A(t)\mathbf{y}$.

The next theorem is analogous to Theorems 5.3.2 and 9.1.5. It shows how to find the general solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ if we know a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ and a fundamental set of solutions of the complementary system. We leave the proof as an exercise (Exercise 21).

Theorem 10.7.1

Suppose the $n \times n$ matrix function A and the n-vector function $\mathbf{f}$ are continuous on (a, b). Let $\mathbf{y}_p$ be a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b), and let $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ be a fundamental set of solutions of the complementary equation $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then $\mathbf{y}$ is a solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b) if and only if

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n,$$

where c_1, c_2, …, c_n are constants.

Finding a Particular Solution of a Nonhomogeneous System

We now discuss an extension of the method of variation of parameters to linear nonhomogeneous systems. This method will produce a particular solution of a nonhomogenous system $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ provided that we know a fundamental matrix for the complementary system. To derive the method, suppose Y is a fundamental matrix for the complementary system; that is,

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \quad \cdots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}$$

is a fundamental set of solutions of the complementary system. In Section 10.3 we saw that $Y' = A(t)Y$. We seek a particular solution of

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t) \tag{10.7.1}$$

of the form

$$\mathbf{y}_p = Y\mathbf{u}, \tag{10.7.2}$$

where $\mathbf{u}$ is to be determined. Differentiating (10.7.2) yields

$$\begin{aligned} \mathbf{y}_p' &= Y'\mathbf{u} + Y\mathbf{u}' \\ &= AY\mathbf{u} + Y\mathbf{u}' \text{ (since } Y' = AY) \\ &= A\mathbf{y}_p + Y\mathbf{u}' \text{ (since } Y\mathbf{u} = \mathbf{y}_p). \end{aligned}$$

Comparing this with (10.7.1) shows that $\mathbf{y}_p = Y\mathbf{u}$ is a solution of (10.7.1) if and only if

$$Y\mathbf{u}' = \mathbf{f}.$$

Thus, we can find a particular solution $\mathbf{y}_p$ by solving this equation for $\mathbf{u}'$, integrating to obtain $\mathbf{u}$, and computing $Y\mathbf{u}$. We can take all constants of integration to be zero, since any particular solution will suffice.

Exercise 22 sketches a proof that this method is analogous to the method of variation of parameters discussed in Sections 5.7 and 9.4 for scalar linear equations.

Example 10.7.1

a. Find a particular solution of the system

$$\mathbf{y}' = \backslash\text{twobytwo}1221\mathbf{y} + \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}, \tag{10.7.3}$$

which we considered in Example 10.2.1.

b. Find the general solution of (10.7.3).

Solution (a) The complementary system is

$$\mathbf{y}' = \backslash\text{twobytwo}1221\mathbf{y}. \tag{10.7.4}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3).$$

Using the method of Section 10.4, we find that

$$\mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

are linearly independent solutions of (10.7.4). Therefore

$$Y = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for (10.7.4). We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.3), where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \backslash\text{twocol}u_1'u_2' = \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{vmatrix} = -2e^{2t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= -\backslash\text{dst}\frac{1}{2e^{2t}}\begin{vmatrix} 2e^{4t} & e^{-t} \\ e^{4t} & -e^{-t} \end{vmatrix} &= \backslash\text{dst}\frac{3e^{3t}}{2e^{2t}} &= \backslash\text{dst}\frac{3}{2}e^{t}, \\ u_2' &= -\backslash\text{dst}\frac{1}{2e^{2t}}\begin{vmatrix} e^{3t} & 2e^{4t} \\ e^{3t} & e^{4t} \end{vmatrix} &= \backslash\text{dst}\frac{e^{7t}}{2e^{2t}} &= \backslash\text{dst}\frac{1}{2}e^{5t}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2}\begin{bmatrix} 3e^{t} \\ e^{5t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{10}\begin{bmatrix} 15e^{t} \\ e^{5t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \dst\frac{1}{10}\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 15e^{t} \\ e^{5t} \end{bmatrix} = \dst\frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix}$$

is a particular solution of (10.7.3).

Solution (b) From Theorem 10.7.1, the general solution of (10.7.3) is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + c_1\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \tag{10.7.5}$$

which can also be written as

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

Writing (10.7.5) in terms of coordinates yields

$$\begin{aligned} y_1 &= \frac{8}{5}e^{4t} + c_1e^{3t} + c_2e^{-t} \\ y_2 &= \frac{7}{5}e^{4t} + c_1e^{3t} - c_2e^{-t}, \end{aligned}$$

so our result is consistent with Example 10.2.1. . ∎

If A isn't a constant matrix, it's usually difficult to find a fundamental set of solutions for the system $\mathbf{y}' = A(t)\mathbf{y}$. It is beyond the scope of this text to discuss methods for doing this. Therefore, in the following examples and in the exercises involving systems with variable coefficient matrices we'll provide fundamental matrices for the complementary systems without explaining how they were obtained.

Example 10.7.2

Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 2e^{-2t} \\ 2e^{2t} & 4 \end{bmatrix}\mathbf{y} + \twocol11, \tag{10.7.6}$$

given that

$$Y = \begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix for the complementary system.

Solution We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.6) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix} \twocol{u_1'}{u_2'} = \twocol11.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{vmatrix} = 2e^{6t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= \dst\frac{1}{2e^{6t}} \begin{vmatrix} 1 & -1 \\ 1 & e^{2t} \end{vmatrix} &=& \dst\frac{e^{2t}+1}{2e^{6t}} &=& \dst\frac{e^{-4t}+e^{-6t}}{2} \\ u_2' &= \dst\frac{1}{2e^{6t}} \begin{vmatrix} e^{4t} & 1 \\ e^{6t} & 1 \end{vmatrix} &=& \dst\frac{e^{4t}-e^{6t}}{2e^{6t}} &=& \dst\frac{e^{-2t}-1}{2}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2} \begin{bmatrix} e^{-4t} + e^{-6t} \\ e^{-2t} - 1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = -\frac{1}{24} \begin{bmatrix} 3e^{-4t} + 2e^{-6t} \\ 6e^{-2t} + 12t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = -\frac{1}{24} \begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix} \begin{bmatrix} 3e^{-4t} + 2e^{-6t} \\ 6e^{-2t} + 12t \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 4e^{-2t} + 12t - 3 \\ -3e^{2t}(4t+1) - 8 \end{bmatrix}$$

is a particular solution of (10.7.6).

Example 10.7.3

Find a particular solution of

$$\mathbf{y}' = -\frac{2}{t^2} \begin{bmatrix} t & -3t^2 \\ 1 & -2t \end{bmatrix} \mathbf{y} + t^2\twocol11, \tag{10.7.7}$$

given that

$$Y = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix}$$

is a fundamental matrix for the complementary system on $(-\infty, 0)$ and $(0, \infty)$.

Solution We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.7) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix} \backslash\mathrm{twocol} u_1' u_2' = \backslash\mathrm{twocol} t^2 t^2.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} 2t & 3t^2 \\ 1 & 2t \end{vmatrix} = t^2.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= \backslash\mathrm{dst}\frac{1}{t^2}\begin{vmatrix} t^2 & 3t^2 \\ t^2 & 2t \end{vmatrix} &= \backslash\mathrm{dst}\frac{2t^3-3t^4}{t^2} &= 2t-3t^2, \\ u_2' &= \backslash\mathrm{dst}\frac{1}{t^2}\begin{vmatrix} 2t & t^2 \\ 1 & t^2 \end{vmatrix} &= \backslash\mathrm{dst}\frac{2t^3-t^2}{t^2} &= 2t-1. \end{aligned}$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} 2t-3t^2 \\ 2t-1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} t^2-t^3 \\ t^2-t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix}\begin{bmatrix} t^2-t^3 \\ t^2-t \end{bmatrix} = \begin{bmatrix} t^3(t-1) \\ t^2(t-1) \end{bmatrix}$$

is a particular solution of (10.7.7).

Example 10.7.4

a. Find a particular solution of

$$\mathbf{y}' = \backslash\text{threebythree}2-1-110-11-10\mathbf{y} + \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}. \tag{10.7.8}$$

b. Find the general solution of (10.7.8).

Solution (a) The complementary system for (10.7.8) is

$$\mathbf{y}' = \backslash\text{threebythree}2-1-110-11-10\mathbf{y}. \tag{10.7.9}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda(\lambda-1)^2.$$

Using the method of Section 10.4, we find that

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

are linearly independent solutions of (10.7.9). Therefore

$$Y = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix}$$

is a fundamental matrix for (10.7.9). We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.8), where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \backslash\text{threecol}u_1'u_2'u_3' = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{vmatrix} = -e^{2t}.$$

Thus, by Cramer's rule,

$$u_1' = -\backslash\text{dst}\frac{1}{e^{2t}}\begin{vmatrix} e^t & e^t & e^t \\ 0 & e^t & 0 \\ e^{-t} & 0 & e^t \end{vmatrix} = -\backslash\text{dst}\frac{e^{3t}-e^t}{e^{2t}} = e^{-t}-e^t$$

$$u_2' = -\backslash\text{dst}\frac{1}{e^{2t}}\begin{vmatrix} 1 & e^t & e^t \\ 1 & 0 & 0 \\ 1 & e^{-t} & e^t \end{vmatrix} = -\backslash\text{dst}\frac{1-e^{2t}}{e^{2t}} = 1-e^{-2t}$$

$$u_3' = -\backslash\text{dst}\frac{1}{e^{2t}}\begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^{-t} \end{vmatrix} = \backslash\text{dst}\frac{e^{2t}}{e^{2t}} = 1.$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} e^{-t}-e^t \\ 1-e^{-2t} \\ 1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} -e^t-e^{-t} \\ \backslash\text{dst}\frac{e^{-2t}}{2}+t \\ t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix}\begin{bmatrix} -e^t-e^{-t} \\ \backslash\text{dst}\frac{e^{-2t}}{2}+t \\ t \end{bmatrix} = \begin{bmatrix} e^t(2t-1)-\backslash\text{dst}\frac{e^{-t}}{2} \\ e^t(t-1)-\backslash\text{dst}\frac{e^{-t}}{2} \\ e^t(t-1)-\backslash\text{dst}e^{-t} \end{bmatrix}$$

is a particular solution of (10.7.8).

Solution (a) From Theorem 10.7.1 the general solution of (10.7.8) is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \begin{bmatrix} e^t(2t-1)-\backslash\text{dst}\frac{e^{-t}}{2} \\ e^t(t-1)-\backslash\text{dst}\frac{e^{-t}}{2} \\ e^t(t-1)-\backslash\text{dst}e^{-t} \end{bmatrix} + c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix} + c_3\begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix},$$

which can be written as

$$\mathbf{y} = \mathbf{y}_p + Y\mathbf{c} = \begin{bmatrix} e^t(2t-1)-\backslash\text{dst}\frac{e^{-t}}{2} \\ e^t(t-1)-\backslash\text{dst}\frac{e^{-t}}{2} \\ e^t(t-1)-\backslash\text{dst}e^{-t} \end{bmatrix} + \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix}\mathbf{c}$$

where $\mathbf{c}$ is an arbitrary constant vector.

Example 10.7.5

Find a particular solution of

$$\mathbf{y}' = \frac{1}{2}\begin{bmatrix} 3 & e^{-t} & -e^{2t} \\ 0 & 6 & 0 \\ -e^{-2t} & e^{-3t} & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ e^{t} \\ e^{-t} \end{bmatrix}, \tag{10.7.10}$$

given that

$$Y = \begin{bmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}$$

is a fundamental matrix for the complementary system.

Solution We seek a particular solution of (10.7.10) in the form $\mathbf{y}_p = Y\mathbf{u}$, where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \backslash\mathrm{threecol} u_1' u_2' u_3' = \begin{bmatrix} 1 \\ e^{t} \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = -2e^{4t}.$$

By Cramer's rule,

$$\begin{aligned}
u_1' &= -\backslash\mathrm{dst}\frac{1}{2e^{4t}}\begin{vmatrix} 1 & 0 & e^{2t} \\ e^{t} & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} &= \backslash\mathrm{dst}\frac{e^{4t}}{2e^{4t}} &= \backslash\mathrm{dst}\frac{1}{2} \\
u_2' &= -\backslash\mathrm{dst}\frac{1}{2e^{4t}}\begin{vmatrix} e^{t} & 1 & e^{2t} \\ 0 & e^{t} & e^{3t} \\ e^{-t} & e^{-t} & 0 \end{vmatrix} &= \backslash\mathrm{dst}\frac{e^{3t}}{2e^{4t}} &= \backslash\mathrm{dst}\frac{1}{2}e^{-t} \\
u_3' &= -\backslash\mathrm{dst}\frac{1}{2e^{4t}}\begin{vmatrix} e^{t} & 0 & 1 \\ 0 & e^{3t} & e^{t} \\ e^{-t} & 1 & e^{-t} \end{vmatrix} &= -\backslash\mathrm{dst}\frac{e^{3t}-2e^{2t}}{2e^{4t}} &= \backslash\mathrm{dst}\frac{2e^{-2t}-e^{-t}}{2}.
\end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2}\begin{bmatrix} 1 \\ e^{-t} \\ 2e^{-2t} - e^{-t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} t \\ -e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{2} \begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ -e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix} = \backslash\mathrm{dst}\frac{1}{2} \begin{bmatrix} e^t(t+1) - 1 \\ -e^t \\ e^{-t}(t-1) \end{bmatrix}$$

is a particular solution of (10.7.10).

10.7 Exercises

In Exercises 1–10 find a particular solution.

1. \dsty $\mathbf{y}' =$ \twobytwo $-1-4-1-1\mathbf{y} + \begin{bmatrix} 21e^{4t} \\ 8e^{-3t} \end{bmatrix}$

Show answer

2. \dsty $\mathbf{y}' = \frac{1}{5}$\twobytwo $-43-2-11\mathbf{y} + \begin{bmatrix} 50e^{3t} \\ 10e^{-3t} \end{bmatrix}$

Show answer

3. \dsty $\mathbf{y}' =$ \twobytwo $1221\mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}$

Show answer

4. \dsty $\mathbf{y}' =$ \twobytwo $-4-365\mathbf{y} + \begin{bmatrix} 2 \\ -2e^{t} \end{bmatrix}$

Show answer

5. \dsty $\mathbf{y}' =$ \twobytwo $-6-31-2\mathbf{y} + \begin{bmatrix} 4e^{-3t} \\ 4e^{-5t} \end{bmatrix}$

Show answer

6. \dsty $\mathbf{y}' =$ \twobytwo $01-10\mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}$

Show answer

7. \dsty $\mathbf{y}' =$ \threebythree $31-1351-624\mathbf{y} + \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

Show answer

8. \dsty $\mathbf{y}' =$ \threebythree $3-1-1-2324-1-2\mathbf{y} + \begin{bmatrix} 1 \\ e^{t} \\ e^{t} \end{bmatrix}$

Show answer

9. \dsty $\mathbf{y}' =$ \threebythree $-3222-3222-3\mathbf{y} + \begin{bmatrix} e^{t} \\ e^{-5t} \\ e^{t} \end{bmatrix}$

Show answer

10. \dsty $\mathbf{y}' = \frac{1}{3}$\threebythree $11-3-4-43-210\mathbf{y} + \begin{bmatrix} e^{t} \\ e^{t} \\ e^{t} \end{bmatrix}$

Show answer

In Exercises 11–20 find a particular solution, given that Y is a fundamental matrix for the complementary system.

11. \dst$\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\mathbf{y} + t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}; \quad Y = t\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

Show answer

12. \dst$\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} t \\ t^2 \end{bmatrix}; \quad Y = t\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$

Show answer

13. \dst$\mathbf{y}' = \frac{1}{t^2-1}\begin{bmatrix} t & -1 \\ -1 & t \end{bmatrix}\mathbf{y} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad Y = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix}$

Show answer

14. \dst$\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 1 & -2e^{-t} \\ 2e^t & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}; \quad Y = \begin{bmatrix} 2 & e^{-t} \\ e^t & 2 \end{bmatrix}$

Show answer

15. \dst$\mathbf{y}' = \frac{1}{2t^4}\begin{bmatrix} 3t^3 & t^6 \\ 1 & -3t^3 \end{bmatrix}\mathbf{y} + \frac{1}{t}\begin{bmatrix} t^2 \\ 1 \end{bmatrix}; \quad Y = \frac{1}{t^2}\begin{bmatrix} t^3 & t^4 \\ -1 & t \end{bmatrix}$

Show answer

16. \dst$\mathbf{y}' = \begin{bmatrix} \text{\dst}\frac{1}{t-1} & -\text{\dst}\frac{e^{-t}}{t-1} \\ \text{\dst}\frac{e^t}{t+1} & \text{\dst}\frac{1}{t+1} \end{bmatrix}\mathbf{y} + \begin{bmatrix} t^2-1 \\ t^2-1 \end{bmatrix}; \quad Y = \begin{bmatrix} t & e^{-t} \\ e^t & t \end{bmatrix}$

Show answer

17. \dst$\mathbf{y}' = \frac{1}{t}$\threebythree110021−222$\mathbf{y}$ + \threecol121 $\quad Y = \begin{bmatrix} t^2 & t^3 & 1 \\ t^2 & 2t^3 & -1 \\ 0 & 2t^3 & 2 \end{bmatrix}$

Show answer

18. \dst$\mathbf{y}' = \begin{bmatrix} 3 & e^t & e^{2t} \\ e^{-t} & 2 & e^t \\ e^{-2t} & e^{-t} & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^{3t} \\ 0 \\ 0 \end{bmatrix}; \quad Y = \begin{bmatrix} e^{5t} & e^{2t} & 0 \\ e^{4t} & 0 & e^t \\ e^{3t} & -1 & -1 \end{bmatrix}$

Show answer

19. \dst$\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & t \\ 0 & -t & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} t \\ t \\ t \end{bmatrix}; \quad Y = t\begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix}$

Show answer

20. \dst$\mathbf{y}' = -\frac{1}{t}\begin{bmatrix} e^{-t} & -t & 1-e^{-t} \\ e^{-t} & 1 & -t-e^{-t} \\ e^{-t} & -t & 1-e^{-t} \end{bmatrix}\mathbf{y} + \frac{1}{t}\begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}; \quad Y = \frac{1}{t}\begin{bmatrix} e^t & e^{-t} & t \\ e^t & -e^{-t} & e^{-t} \\ e^t & e^{-t} & 0 \end{bmatrix}$

Show answer

21. Prove Theorem 10.7.1.

22. a. Convert the scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = F(t) \tag{A}$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t). \tag{B}$$

b. Suppose (A) is normal on an interval (a, b) and $\{y_1, y_2, \dots, y_n\}$ is a fundamental set of solutions of

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = 0 \tag{C}$$

on (a, b). Find a corresponding fundamental matrix Y for

$$\mathbf{y}' = A(t)\mathbf{y} \tag{D}$$

on (a, b) such that

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is a solution of (C) if and only if $\mathbf{y} = Y\mathbf{c}$ with

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is a solution of (D).

c. Let $y_p = u_1y_1 + u_1y_2 + \cdots + u_ny_n$ be a particular solution of (A), obtained by the method of variation of parameters for scalar equations as given in Section 9.4, and define

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Show that $\mathbf{y}_p = Y\mathbf{u}$ is a solution of (B).

d. Let $\mathbf{y}_p = Y\mathbf{u}$ be a particular solution of (B), obtained by the method of variation of parameters for systems as given in this section. Show that $y_p = u_1y_1 + u_1y_2 + \cdots + u_ny_n$ is a solution of (A).

Show answer

23. Suppose the $n \times n$ matrix function A and the n–vector function $\mathbf{f}$ are continuous on (a,b). Let t_0 be in (a,b), let $\mathbf{k}$ be an arbitrary constant vector, and let Y be a fundamental matrix for the homogeneous system $\mathbf{y}' = A(t)\mathbf{y}$. Use variation of parameters to show that the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y}(t) = Y(t)\left(Y^{-1}(t_0)\mathbf{k} + \int_{t_0}^{t} Y^{-1}(s)\mathbf{f}(s)\,ds\right).$$

Chapter 10 Linear Systems of Differential Equations

10.1 Introduction to Systems of Differential Equations

1
$$\begin{aligned} Q_1' &= \backslash\mathrm{dst}2 - \tfrac{1}{10}Q_1 + \tfrac{1}{25}Q_2 \\ Q_2' &= \backslash\mathrm{dst}6 + \tfrac{3}{50}Q_1 - \tfrac{1}{20}Q_2. \end{aligned}$$

2
$$\begin{aligned} Q_1' &= \backslash\mathrm{dst}12 - \tfrac{5}{100+2t}Q_1 + \tfrac{1}{100+3t}Q_2 \\ Q_2' &= \backslash\mathrm{dst}5 + \tfrac{1}{50+t}Q_1 - \tfrac{4}{100+3t}Q_2. \end{aligned}$$

3 $m_1y_1'' = -(c_1+c_2)y_1' + c_2y_2' - (k_1+k_2)y_1 + k_2y_2 + F_1$

$$m_2y_2'' = (c_2-c_3)y_1' - (c_2+c_3)y_2' + c_3y_3' + (k_2-k_3)y_1 - (k_2+k_3)y_2 + k_3y_3 + F_2$$

$$m_3y_3'' = c_3y_1' + c_3y_2' - c_3y_3' + k_3y_1 + k_3y_2 - k_3y_3 + F_3$$

4 $x'' = -\backslash\mathrm{dst}\frac{\alpha}{m}x' + \frac{gR^2x}{(x^2+y^2+z^2)^{3/2}}$ $y'' = -\backslash\mathrm{dst}\frac{\alpha}{m}y' + \frac{gR^2y}{(x^2+y^2+z^2)^{3/2}}$

z''
$= -\backslash\mathrm{dst}\frac{\alpha}{m}z' + \frac{gR^2z}{(x^2+y^2+z^2)^{3/2}}$

5

(a) $\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= f(t,x_1,y_1,y_2) \\ y_1' &= y_2 \\ y_2' &= g(t,y_1,y_2) \end{aligned}$	**(b)** $\begin{aligned} u_1' &= f(t,u_1,v_1,v_2,w_2) \\ ;v_1' &= v_2 \\ v_2' &= g(t,u_1,v_1,v_2,w_1) \\ w_1' &= w_2 \\ w_2' &= h(t,u_1,v_1,v_2,w_1,w_2) \end{aligned}$

(c
y_1'
y_2'
y_3'

(e)
$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t,x_1,y_1) \\ y_1' &= y_2 \\ y_2' &= g(t,x_1,y_1) \end{aligned}$$

6

$\begin{aligned} x' &= x_1 \\ y' &= y_1 \\ z' &= z_1 \end{aligned}$	$\backslash\mathrm{dst}$ $\begin{aligned} x_1' &= -\backslash\mathrm{dst}\frac{gR^2x}{(x^2+y^2+z^2)^{3/2}} \\ y_1' &= -\backslash\mathrm{dst}\frac{gR^2y}{(x^2+y^2+z^2)^{3/2}} \\ z_1' &= -\backslash\mathrm{dst}\frac{gR^2z}{(x^2+y^2+z^2)^{3/2}} \end{aligned}$

10.2 Linear Systems of Differential Equations

1 **(a)** $\mathbf{y}' =$ \twobytwo$2442\mathbf{y}$ **(b)** $\mathbf{y}' =$ \twobytwo$-2-2-51\mathbf{y}$

(c)
$\mathbf{y}'$
$=$ \twobytwo$-4-1037\mathbf{y}$
(d) $\mathbf{y}'$
$=$ \twobytwo$2112\mathbf{y}$

2 **(a)** $\mathbf{y}' =$ \threebythree$-12301600-2\mathbf{y}$ **(b)** $\mathbf{y}' =$ \threebythree$022202220\mathbf{y}$

(c)
$\mathbf{y}'$
$=$ \threebythree$-1222-1222-1\mathbf{y}$
(d)
$\mathbf{y}'$
$=$ \threebythree$3-1-1-2324-1-2\mathbf{y}$

3 **(a)** $\mathbf{y}' =$ \twobytwo$11-24\mathbf{y},\ \mathbf{y}(0) =$ \twocol$10$ **(b)** $\mathbf{y}' =$ \twobytwo$53-11\mathbf{y},\ \mathbf{y}(0) =$ \twocol$9-5$

4 **(a)** $\mathbf{y}' =$ \threebythree$644-7-2-1743\mathbf{y},\ \mathbf{y}(0) =$ \threecol$3-64$

(b)
$\mathbf{y}'$
$=$ \threebythree$877-5-6-95710\mathbf{y},$
$\mathbf{y}(0)$
$=$ \threecol$2-43$

5 **(a)** $\mathbf{y}' =$ \twobytwo$-32-53+$ \twocol$3-2t6-3t$ **(b)** $\mathbf{y}' =$ \twobytwo$31-11\mathbf{y}+\begin{bmatrix} -5e^t \\ e^t \end{bmatrix}$

10 **(a)** \dst$\frac{d}{dt}Y^2 = Y'Y + YY'$

(b)
\dst$\frac{d}{dt}Y^n$
$= Y'Y^{n-1}$
$+ YY'Y^{n-2}$
$+ Y^2Y'Y^{n-3}$
$+ \cdots$
$+ Y^{n-1}Y'$
$= \sum_{r=0}^{n-1} Y^rY'Y^{n-r-1}$

13 $B = (P' + PA)P^{-1}.$

10.3 Basic Theory of Homogeneous Linear System

2 \dst$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -$\dst$\frac{P_2(x)}{P_0(x)} & -$\dst$\frac{P_1(x)}{P_0(x)} \end{bmatrix}\mathbf{y}$

3 $\mathbf{y}' = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -$\dst$\frac{P_n(x)}{P_0(x)} & -$\dst$\frac{P_{n-1}(x)}{P_0(x)} & \cdots & -$\dst$\frac{P_1(x)}{P_0(x)} \end{bmatrix}\mathbf{y}$

7 **(b)** $\mathbf{y} =$ \dst $\begin{bmatrix} 3e^{6t} - 6e^{-2t} \\ 3e^{6t} + 6e^{-2t} \end{bmatrix}$ **(c)** $\mathbf{y} =$ \dst $\frac{1}{2}\begin{bmatrix} e^{6t} + e^{-2t} & e^{6t} - e^{-2t} \\ e^{6t} - e^{-2t} & e^{6t} + e^{-2t} \end{bmatrix}\mathbf{k}$

8 **(b)** $\mathbf{y} =$ \dst $\begin{bmatrix} 6e^{-4t} + 4e^{3t} \\ 6e^{-4t} - 10e^{3t} \end{bmatrix}$ **(c)** $\mathbf{y} =$ \dst $\frac{1}{7}\begin{bmatrix} 5e^{-4t} + 2e^{3t} & 2e^{-4t} - 2e^{3t} \\ 5e^{-4t} - 5e^{3t} & 2e^{-4t} + 5e^{3t} \end{bmatrix}\mathbf{k}$

9 **(b)** $\mathbf{y} =$ \dst $\begin{bmatrix} -15e^{2t} - 4e^{t} \\ 9e^{2t} + 2e^{t} \end{bmatrix}$ **(c)** $\mathbf{y} =$ \dst $\begin{bmatrix} -5e^{2t} + 6e^{t} & -10e^{2t} + 10e^{t} \\ 3e^{2t} - 3e^{t} & 6e^{2t} - 5e^{t} \end{bmatrix}\mathbf{k}$

10 **(b)** $\mathbf{y} =$ \dst $\begin{bmatrix} 5e^{3t} - 3e^{t} \\ 5e^{3t} + 3e^{t} \end{bmatrix}$ **(c)** \dst$\mathbf{y} = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{t} & e^{3t} - e^{t} \\ e^{3t} - e^{t} & e^{3t} + e^{t} \end{bmatrix}\mathbf{k}$

11 **(b)** $\mathbf{y} =$ \dst $\begin{bmatrix} e^{2t} - 2e^{3t} + 3e^{-t} \\ 2e^{3t} - 9e^{-t} \\ e^{2t} - 2e^{3t} + 21e^{-t} \end{bmatrix}$ **(c)** $\mathbf{y} =$ \dst $\frac{1}{6}\begin{bmatrix} 4e^{2t} + 3e^{3t} - e^{-t} & 6e^{2t} - 6e^{3t} & 2e^{2t} - 3e^{3t} + e^{-t} \\ -3e^{3t} + 3e^{-t} & 6e^{3t} & 3e^{3t} - 3e^{-t} \\ 4e^{2t} + 3e^{3t} - 7e^{-t} & 6e^{2t} - 6e^{3t} & 2e^{2t} - 3e^{3t} + 7e^{-t} \end{bmatrix}\mathbf{k}$

12 **(b)** $\mathbf{y} =$ \dst $\frac{1}{3}\begin{bmatrix} -e^{-2t} + e^{4t} \\ -10e^{-2t} + e^{4t} \\ 11e^{-2t} + e^{4t} \end{bmatrix}$ **(c)** $\mathbf{y} =$ \dst $\frac{1}{3}\begin{bmatrix} 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t} \end{bmatrix}\mathbf{k}$

13 **(b)** $\mathbf{y} =$ \dst $\begin{bmatrix} 3e^{t} + 3e^{-t} - e^{-2t} \\ 3e^{t} + 2e^{-2t} \\ -e^{-2t} \end{bmatrix}$ **(c)** $\mathbf{y} =$ \dst $\begin{bmatrix} e^{-t} & e^{t} - e^{-t} & 2e^{t} - 3e^{-t} + e^{-2t} \\ 0 & e^{t} & 2e^{t} - 2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}\mathbf{k}$

14 YZ^{-1} and ZY^{-1}

10.4 Constant Coefficient Homogeneous Systems I

1 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

2 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t/2} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$

3 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$

4 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}$

5 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t}$

6 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$

7 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$

8 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$

9 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{-16t} + c_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$

10 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} -7 \\ -5 \\ 4 \end{bmatrix} e^{2t}$

11 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} -2 \\ -6 \\ 3 \end{bmatrix} e^{-5t}$

12 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t}$

13 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{4t}$

14 $\mathbf{y} = \backslash\mathrm{dst} c_1 \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{5t}$

15 $\mathbf{y} = \backslash\text{dst}c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} e^{6t}$

16 $\mathbf{y} = \backslash\text{dst}-\backslash\text{twocol}26e^{5t} + \backslash\text{twocol}42e^{-5t}$

17 $\mathbf{y} = \backslash\text{dst}\backslash\text{twocol}2-4e^{t/2} + \backslash\text{twocol}-21e^{t}$

18 $\mathbf{y} = \backslash\text{dst}\backslash\text{twocol}77e^{9t} - \backslash\text{twocol}24e^{-3t}$

19 $\mathbf{y} = \backslash\text{dst}\backslash\text{twocol}39e^{5t} - \backslash\text{twocol}42e^{-5t}$

20 $\mathbf{y} = \backslash\text{dst}\backslash\text{threecol}550e^{t/2} + \backslash\text{threecol}001e^{t/2} + \backslash\text{threecol}-120e^{-t/2}$

21 $\mathbf{y} = \backslash\text{dst}\backslash\text{threecol}333e^{t} + \backslash\text{threecol}-2-22e^{-t}$

22 $\mathbf{y} = \backslash\text{dst}\backslash\text{threecol}2-22e^{t} - \backslash\text{threecol}303e^{-2t} + \backslash\text{threecol}110e^{3t}$

23 $\mathbf{y} = \backslash\text{dst}-\backslash\text{threecol}121e^{t} + \backslash\text{threecol}424e^{-t} + \backslash\text{threecol}110e^{2t}$

24 $\mathbf{y} = \backslash\text{dst}\backslash\text{threecol}-2-22e^{2t} - \backslash\text{threecol}030e^{-2t} + \backslash\text{threecol}4124e^{4t}$

25 $\mathbf{y} = \backslash\text{dst}\backslash\text{threecol}-1-11e^{-6t} + \backslash\text{threecol}2-22e^{2t} + \backslash\text{threecol}7-7-7e^{4t}$

26 $\mathbf{y} = \backslash\text{dst}\backslash\text{threecol}144e^{-t} + \backslash\text{threecol}66-2e^{2t}$

27 $\mathbf{y} = \backslash\text{dst}\backslash\text{threecol}4-22 + \backslash\text{threecol}3-96e^{4t} + \backslash\text{threecol}-11-1e^{2t}$

29 Half lines of $L_1 : y_2 = y_1$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories

> are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

30 Half lines of $L_1: y_2 = -2y_1$ and $L_2: y_2 = -y_1/3$ are trajectories

> other trajectories are asymptotically parallel to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

31 Half lines of $L_1: y_2 = y_1/3$ and $L_2: y_2 = -y_1$ are trajectories other trajectories

> are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically parallel to L_2 as $t \to \infty$.

32 Half lines of $L_1 : y_2 = y_1/2$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories

> are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

33 Half lines of $L_1 : y_2 = -y_1/4$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories

> are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically parallel to L_2 as $t \to \infty$.

34 Half lines of $L_1: y_2 = -y_1$ and $L_2: y_2 = 3y_1$ are trajectories other trajectories

> are asymptotically parallel to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

36 Points on $L_2: y_2 = y_1$ are trajectories of constant solutions. The trajectories

> of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, traversed toward L_1.

37 Points on $L_1: y_2 = -y_1/3$ are trajectories of constant solutions. The trajectories

> of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, traversed away from L_1.

38 Points on $L_1: y_2 = y_1/3$ are trajectories of constant solutions. The trajectories

> of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, \dst\twocol 1−1, traversed away from L_1.

39 Points on $L_1: y_2 = y_1/2$ are trajectories of constant solutions. The trajectories

> of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, L_1.

40 Points on $L_2: y_2 = -y_1$ are trajectories of constant solutions. The trajectories

> of nonconstant solutions are half-lines on either side of L_2, parallel to $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, traversed toward L_1.

41 Points on $L_1 : y_2 = 3y_1$ are trajectories of constant solutions. The trajectories

> of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, traversed away from L_1.

10.5 Constant Coefficient Homogeneous Systems II

1 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{twocol}21e^{5t} + c_2\left(\backslash\mathrm{twocol}{-1}\ \ 0e^{5t} + \backslash\mathrm{twocol}21te^{5t}\right).$

2 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{twocol}11e^{-t} + c_2\left(\backslash\mathrm{twocol}10e^{-t} + \backslash\mathrm{twocol}11te^{-t}\right)$

3 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{twocol}{-2}\ \ 1e^{-9t} + c_2\left(\backslash\mathrm{twocol}{-1}\ \ 0e^{-9t} + \backslash\mathrm{twocol}{-2}\ \ 1te^{-9t}\right)$

4 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{twocol}{-1}\ \ 1e^{2t} + c_2\left(\backslash\mathrm{twocol}{-1}\ \ 0e^{2t} + \backslash\mathrm{twocol}{-1}\ \ 1te^{2t}\right)$

5 $c_1\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\frac{e^{-2t}}{3} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}te^{-2t}\right)$

6 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{twocol}32e^{-4t} + c_2\left(\backslash\mathrm{twocol}{-1}\ \ 0\frac{e^{-4t}}{2} + \backslash\mathrm{twocol}32te^{-4t}\right)$

7 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{twocol}43e^{-t} + c_2\left(\backslash\mathrm{twocol}{-1}\ \ 0\frac{e^{-t}}{3} + \backslash\mathrm{twocol}43te^{-t}\right)$

8 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{threecol}{-1}{-1}2 + c_2\backslash\mathrm{threecol}112e^{4t} + c_3\left(\backslash\mathrm{threecol}010\frac{e^{4t}}{2} + \backslash\mathrm{threecol}112te^{4t}\right)$

9 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{threecol}{-1}11e^{t} + c_2\backslash\mathrm{threecol}1{-1}1e^{-t} + c_3\left(\backslash\mathrm{threecol}030e^{-t} + \backslash\mathrm{threecol}1{-1}1te^{-t}\right).$

10 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{threecol}011e^{2t} + c_2\backslash\mathrm{threecol}101e^{-2t} + c_3\left(\backslash\mathrm{threecol}110\frac{e^{-2t}}{2} + \backslash\mathrm{threecol}101te^{-2t}\right)$

11 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{threecol}{-2}{-3}1e^{2t} + c_2\backslash\mathrm{threecol}0{-1}1e^{4t} + c_3\left(\backslash\mathrm{threecol}100\frac{e^{4t}}{2} + \backslash\mathrm{threecol}0{-1}1te^{4t}\right)$

12 $\mathbf{y} = \backslash\mathrm{dst}c_1\backslash\mathrm{threecol}{-1}{-1}1e^{-2t} + c_2\backslash\mathrm{threecol}111e^{4t} + c_3\left(\backslash\mathrm{threecol}100\frac{e^{4t}}{2} + \backslash\mathrm{threecol}111te^{4t}\right).$

13 $\mathbf{y} = \backslash\mathrm{dst}\backslash\mathrm{twocol}62e^{-7t} - \backslash\mathrm{twocol}84te^{-7t}$

14 $\mathbf{y} = \backslash\mathrm{dst}\backslash\mathrm{twocol}58e^{3t} - \backslash\mathrm{twocol}1216te^{3t}$

15 $\mathbf{y} = \backslash\mathrm{dst}\backslash\mathrm{twocol}23e^{-5t} - \backslash\mathrm{twocol}84te^{-5t}$

16 $\mathbf{y} = \backslash\mathrm{dst}\backslash\mathrm{twocol}31e^{5t} - \backslash\mathrm{twocol}126te^{5t}$

17 $\mathbf{y} = \backslash\mathrm{dst}\backslash\mathrm{twocol}02e^{-4t} + \backslash\mathrm{twocol}66te^{-4t}$

18 $\mathbf{y} = \backslash\mathrm{dst}\backslash\mathrm{threecol}48{-6}e^{t} + \backslash\mathrm{threecol}2{-3}{-1}e^{-2t} + \backslash\mathrm{threecol}{-1}10te^{-2t}$

19 $\mathbf{y} = \backslash\mathrm{dst}\backslash\mathrm{threecol}336e^{2t} - \backslash\mathrm{threecol}956 + \backslash\mathrm{threecol}220t$

20 $\mathbf{y} = \dst-\threecol202e^{-3t} + \threecol-491e^{t} - \threecol044te^{t}$

21 $\mathbf{y} = \dst\threecol-222e^{4t} + \threecol0-11e^{2t} + \threecol3-33te^{2t}$

22 $\mathbf{y} = \dst-\threecol110e^{-4t} + \threecol-32-3e^{8t} + \threecol80-8te^{8t}$

23 $\mathbf{y} = \dst\threecol363e^{4t} - \threecol341 + \threecol844t$

24 $\mathbf{y} = \dst c_1\threecol011e^{6t} + c_2\left(\threecol-110\frac{e^{6t}}{4} + \threecol011te^{6t}\right)$

$+c_3\left(\backslash t\right.$

25 $\mathbf{y} = \dst c_1\threecol-111e^{3t} + c_2\left(\threecol100\frac{e^{3t}}{2} + \threecol-111te^{3t}\right)$

$+c_3\left(\backslash t\right.$

26 $\mathbf{y} = \dst c_1\threecol0-11e^{-2t} + c_2\left(\threecol-110e^{-2t} + \threecol0-11te^{-2t}\right)$

$+c_3\left(\backslash t\right.$

27 $\mathbf{y} = \dst c_1\threecol011e^{2t} + c_2\left(\threecol110\frac{e^{2t}}{2} + \threecol011te^{2t}\right)$

$+c_3\left(\backslash t\right.$

28 $\mathbf{y} = \dst c_1\threecol-212e^{-6t} + c_2\left(-\threecol610\frac{e^{-6t}}{6} + \threecol-212te^{-6t}\right)$

$+c_3\left(-\right.$

29 $\mathbf{y} = \dst c_1\threecol-401e^{-3t} + c_2\threecol610e^{-3t} + c_3\left(\threecol100e^{-3t} + \threecol211te^{-3t}\right)$

30 $\mathbf{y} = \dst c_1\threecol-101e^{-3t} + c_2\threecol010e^{-3t} + c_3\left(\threecol100e^{-3t} + \threecol-1-11te^{-3t}\right)$

31 $\mathbf{y} = \dst c_1\threecol201e^{-t} + c_2\threecol-320e^{-t} + c_3\left(\threecol100\frac{e^{-t}}{2} + \threecol-121te^{-t}\right)$

32 $\mathbf{y} = \dst c_1\threecol-110e^{-2t} + c_2\threecol001e^{-2t} + c_3\left(\threecol-100e^{-2t} + \threecol1-11te^{-2t}\right)$

10.6 Constant Coefficient Homogeneous Systems III

1 $\mathbf{y} = \backslash\mathrm{dst}c_1e^{2t}\backslash\mathrm{ctwocol}3\cos t+\sin t5\cos t + c_2e^{2t}\backslash\mathrm{ctwocol}3\sin t-\cos t5\sin t.$

2 $\mathbf{y} = \backslash\mathrm{dst}c_1e^{-t}\backslash\mathrm{ctwocol}5\cos 2t+\sin 2t13\cos 2t + c_2e^{-t}\backslash\mathrm{ctwocol}5\sin 2t-\cos 2t13\sin 2t.$

3 $\mathbf{y} = \backslash\mathrm{dst}c_1e^{3t}\backslash\mathrm{ctwocol}\cos 2t+\sin 2t2\cos 2t + c_2e^{3t}\backslash\mathrm{ctwocol}\sin 2t-\cos 2t2\sin 2t.$

4 $\mathbf{y} = \backslash\mathrm{dst}c_1e^{2t}\backslash\mathrm{ctwocol}\cos 3t-\sin 3t\cos 3t + c_2e^{2t}\backslash\mathrm{ctwocol}\sin 3t+\cos 3t\sin 3t.$

5 $\backslash\mathrm{dst}\mathbf{y} = c_1\backslash\mathrm{threecol}{-1}{-1}2e^{-2t} + c_2e^{4t}\backslash\mathrm{cthreecol}\cos 2t-\sin 2t\cos 2t+\sin 2t2\cos 2t + c_3e^{4t}\backslash\mathrm{cthreecol}\sin 2t+\cos 2t\sin 2t-\cos 2t2\sin 2t.$

6 $\backslash\mathrm{dst}\mathbf{y} = c_1\backslash\mathrm{threecol}{-1}{-1}1e^{-t} + c_2e^{-2t}\backslash\mathrm{cthreecol}\cos 2t-\sin 2t-\cos 2t-\sin 2t2\cos 2t + c_3e^{-2t}\backslash\mathrm{cthreecol}\sin 2t+\cos 2t-\sin 2t+\cos 2t2\sin 2t$

7 $\backslash\mathrm{dst}\mathbf{y} = c_1\backslash\mathrm{threecol}111e^{2t} + c_2e^{t}\backslash\mathrm{threecol}-\sin t\sin t\cos t + c_3e^{t}\backslash\mathrm{threecol}\cos t-\cos t\sin t$

8 $\backslash\mathrm{dst}\mathbf{y} = c_1\backslash\mathrm{threecol}{-1}11e^{t} + c_2e^{-t}\backslash\mathrm{cthreecol}-\sin 2t-\cos 2t2\cos 2t2\cos 2t + c_3e^{-t}\backslash\mathrm{cthreecol}\cos 2t-\sin 2t2\sin 2t2\sin 2t$

9 $\backslash\mathrm{dst}\mathbf{y} = c_1e^{3t}\begin{bmatrix}\cos 6t-3\sin 6t\\ 5\cos 6t\end{bmatrix} + c_2e^{3t}\begin{bmatrix}\sin 6t+3\cos 6t\\ 5\sin 6t\end{bmatrix}$

10 $\backslash\mathrm{dst}\mathbf{y} = c_1e^{2t}\begin{bmatrix}\cos t-3\sin t\\ 2\cos t\end{bmatrix} + c_2e^{2t}\begin{bmatrix}\sin t+3\cos t\\ 2\sin t\end{bmatrix}$

11 $\backslash\mathrm{dst}\mathbf{y} = c_1e^{2t}\begin{bmatrix}3\sin 3t-\cos 3t\\ 5\cos 3t\end{bmatrix} + c_2e^{2t}\begin{bmatrix}-3\cos 3t-\sin 3t\\ 5\sin 3t\end{bmatrix}$

12 $\backslash\mathrm{dst}\mathbf{y} = c_1e^{2t}\begin{bmatrix}\sin 4t-8\cos 4t\\ 5\cos 4t\end{bmatrix} + c_2e^{2t}\begin{bmatrix}-\cos 4t-8\sin 4t\\ 5\sin 4t\end{bmatrix}$

13 $\backslash\mathrm{dst}\mathbf{y} = c_1\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}e^{-2t} + c_2e^{t}\begin{bmatrix}\sin t\\ -\cos t\\ \cos t\end{bmatrix} + c_3e^{t}\begin{bmatrix}-\cos t\\ -\sin t\\ \sin t\end{bmatrix}$

14 $\backslash\mathrm{dst}\mathbf{y} = c_1\begin{bmatrix}2\\ 2\\ 1\end{bmatrix}e^{-2t} + c_2e^{2t}\begin{bmatrix}-\cos 3t-\sin 3t\\ -\sin 3t\\ \cos 3t\end{bmatrix} + c_3e^{2t}\begin{bmatrix}-\sin 3t+\cos 3t\\ \cos 3t\\ \sin 3t\end{bmatrix}$

15 $\backslash\mathrm{dst}\mathbf{y} = c_1\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}e^{3t} + c_2e^{6t}\begin{bmatrix}-\sin 3t\\ \sin 3t\\ \cos 3t\end{bmatrix} + c_3e^{6t}\begin{bmatrix}\cos 3t\\ -\cos 3t\\ \sin 3t\end{bmatrix}$

16 \dst$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 e^t \begin{bmatrix} 2\cos t - 2\sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_3 e^t \begin{bmatrix} 2\sin t + 2\cos t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}$

17 \dst$\mathbf{y} = e^t \begin{bmatrix} 5\cos 3t + \sin 3t \\ 2\cos 3t + 3\sin 3t \end{bmatrix}$

18 \dst$\mathbf{y} = e^{4t} \begin{bmatrix} 5\cos 6t + 5\sin 6t \\ \cos 6t - 3\sin 6t \end{bmatrix}$

19 \dst$\mathbf{y} = e^t \begin{bmatrix} 17\cos 3t - \sin 3t \\ 7\cos 3t + 3\sin 3t \end{bmatrix}$

20 $\mathbf{y} =$ \dst$e^{t/2} \begin{bmatrix} \cos(t/2) + \sin(t/2) \\ -\cos(t/2) + 2\sin(t/2) \end{bmatrix}$

21 \dst$\mathbf{y} =$ \threecol$1{-}12e^t + e^{4t} \begin{bmatrix} 3\cos t + \sin t \\ \cos t - 3\sin t \\ 4\cos t - 2\sin t \end{bmatrix}$

22 \dst$\mathbf{y} =$ \threecol$442e^{8t} + e^{2t} \begin{bmatrix} 4\cos 2t + 8\sin 2t \\ -6\sin 2t + 2\cos 2t \\ 3\cos 2t + \sin 2t \end{bmatrix}$

23 \dst$\mathbf{y} =$ \threecol$033e^{-4t} + e^{4t} \begin{bmatrix} 15\cos 6t + 10\sin 6t \\ 14\cos 6t - 8\sin 6t \\ 7\cos 6t - 4\sin 6t \end{bmatrix}$

24 \dst$\mathbf{y} =$ \threecol$6{-}33e^{8t} + \begin{bmatrix} 10\cos 4t - 4\sin 4t \\ 17\cos 4t - \sin 4t \\ 3\cos 4t - 7\sin 4t \end{bmatrix}$

29 $\mathbf{U} =$ \dst$\frac{1}{\sqrt{2}}$\twocol$-11$, $\mathbf{V} =$ \dst$\frac{1}{\sqrt{2}}$\twocol$11$

30 $\mathbf{U} \approx$ \dst\twocol$.5257.8507$, $\mathbf{V} \approx$ \dst\twocol$-.8507\quad .5257$

31 $\mathbf{U} \approx$ \dst\twocol$.8507.5257$,

$\mathbf{V}$

$\approx$ \dst\twocol$-.5257\quad .8507$

32 $\mathbf{U} \approx$ \dst\twocol$-.9732\quad .2298$, $\mathbf{V} \approx$ \dst\twocol$.2298.9732$

33 $\mathbf{U} \approx$ \dst\twocol $\quad .5257.8507$, $\mathbf{V} \approx$ \dst\twocol$-.8507.5257$

34 $\mathbf{U} \approx$ \dst\twocol$-.5257\quad .8507$, $\mathbf{V} \approx$ \dst\twocol$.8507.5257$

35 $\mathbf{U} \approx$ \dst\twocol$-.8817\quad .4719$, $\mathbf{V} \approx$ \dst\twocol$.4719.8817$

36 $\mathbf{U} \approx$ \dst\twocol.8817.4719, $\mathbf{V} \approx$ \dst\twocol$-$.4719 .8817

37 $\mathbf{U} =$ \dst\twocol01, $\mathbf{V} =$ \dst\twocol$-$10

38 $\mathbf{U} =$ \dst\twocol01, $\mathbf{V} =$ \dst\twocol10

39 $\mathbf{U} =$ \dst$\frac{1}{\sqrt{2}}$\twocol11, $\mathbf{V} =$ \dst$\frac{1}{\sqrt{2}}$\twocol$-1$ 1

40 $\mathbf{U} \approx$ \dst\twocol.5257.8507, $\mathbf{V} \approx$ \dst\twocol$-$.8507 .5257

10.7 Variation of Parameters for Nonhomogeneous Linear Systems

1 \dst $\left[\begin{array}{c} 5e^{4t}+e^{-3t}(2+8t) \\ -e^{4t}-e^{-3t}(1-4t) \end{array}\right]$

2 \dst $\left[\begin{array}{c} 13e^{3t}+3e^{-3t} \\ -e^{3t}-11e^{-3t} \end{array}\right]$

3 \dst $\frac{1}{9}\left[\begin{array}{c} 7-6t \\ -11+3t \end{array}\right]$

4 \dst $\left[\begin{array}{c} 5-3e^{t} \\ -6+5e^{t} \end{array}\right]$

5 \dst $\left[\begin{array}{c} e^{-5t}(3+6t)+e^{-3t}(3-2t) \\ -e^{-5t}(3+2t)-e^{-3t}(1-2t) \end{array}\right]$

6 \dst $\left[\begin{array}{c} t \\ 0 \end{array}\right]$

7 \dst $-\frac{1}{6}\left[\begin{array}{c} 2-6t \\ 7+6t \\ 1-12t \end{array}\right]$

8 \dst $-\frac{1}{6}\left[\begin{array}{c} 3e^{t}+4 \\ 6e^{t}-4 \\ 10 \end{array}\right]$

9 \dst $\frac{1}{18}\left[\begin{array}{c} e^{t}(1+12t)-e^{-5t}(1+6t) \\ -2e^{t}(1-6t)-e^{-5t}(1-12t) \\ e^{t}(1+12t)-e^{-5t}(1+6t) \end{array}\right]$

10 \dst $\frac{1}{3}\left[\begin{array}{c} 2e^{t} \\ e^{t} \\ 2e^{t} \end{array}\right]$

11 \dst $\left[\begin{array}{c} t\sin t \\ 0 \end{array}\right]$

12 \dst $-\left[\begin{array}{c} t^{2} \\ 2t \end{array}\right]$

13 \dst $(t-1)\left(\ln|t-1|+t\right)\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$

14 \dst $\frac{1}{9}\left[\begin{array}{c} 5e^{2t}-e^{-3t} \\ e^{3t}-5e^{-2t} \end{array}\right]$

15 \dst $\frac{1}{4t}\left[\begin{array}{c} 2t^{3}\ln|t|+t^{3}(t+2) \\ 2\ln|t|+3t-2 \end{array}\right]$

16 \dst $\frac{1}{2}\left[\begin{array}{c} te^{-t}(t+2)+(t^{3}-2) \\ te^{t}(t-2)+(t^{3}+2) \end{array}\right]$

17 \dst$-\begin{bmatrix} t \\ t \\ t \end{bmatrix}$

18 \dst$\frac{1}{4}\begin{bmatrix} -3e^t \\ 1 \\ e^{-t} \end{bmatrix}$

19 $\begin{bmatrix} 2t^2+t \\ t \\ -t \end{bmatrix}$

20 \dst$\frac{e^t}{4t}\begin{bmatrix} 2t+1 \\ 2t-1 \\ 2t+1 \end{bmatrix}$

22 **(a)** \dst$\mathbf{y}' = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -P_n(t)/P_0(t) & -P_{n-1}/P_0(t) & \cdots & -P_1(t)/P_0(t) \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ F(t)/P_0(t) \end{bmatrix}.$

(b)
\dst$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$