1 Introduction

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IN THIS CHAPTER we begin our study of differential equations.

SECTION 1.1 presents examples of applications that lead to differential equations.

SECTION 1.2 introduces basic concepts and definitions concerning differential equations.

SECTION 1.3 presents a geometric method for dealing with differential equations that has been known for a very long time, but has become particularly useful and important with the proliferation of readily available differential equations software.

Answers to selected exercises in this chapter

1.1 Applications Leading to Differential Equations

In order to apply mathematical methods to a physical or “real life” problem, we must formulate the problem in mathematical terms; that is, we must construct a mathematical model for the problem. Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more of its derivatives. Such equations are differential equations. They are the subject of this book.

Much of calculus is devoted to learning mathematical techniques that are applied in later courses in mathematics and the sciences; you wouldn’t have time to learn much calculus if you insisted on seeing a specific application of every topic covered in the course. Similarly, much of this book is devoted to methods that can be applied in later courses. Only a relatively small part of the book is devoted to the derivation of specific differential equations from mathematical models, or relating the differential equations that we study to specific applications. In this section we mention a few such applications.

The mathematical model for an applied problem is almost always simpler than the actual situation being studied, since simplifying assumptions are usually required to obtain a mathematical problem that can be solved. For example, in modeling the motion of a falling object, we might neglect air resistance and the gravitational pull of celestial bodies other than Earth, or in modeling population growth we might assume that the population grows continuously rather than in discrete steps.

A good mathematical model has two important properties:

- It's sufficiently simple so that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the outcome of the real problem to within a useful degree of accuracy. If results predicted by the model don't agree with physical observations, the underlying assumptions of the model must be revised until satisfactory agreement is obtained.

We'll now give examples of mathematical models involving differential equations. We'll return to these problems at the appropriate times, as we learn how to solve the various types of differential equations that occur in the models.

All the examples in this section deal with functions of time, which we denote by t. If y is a function of t, y' denotes the derivative of y with respect to t; thus,

$$y' = \frac{dy}{dt}.$$

Population Growth and Decay

Although the number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t is necessarily an integer, models that use differential equations to describe the growth and decay of populations usually rest on the simplifying assumption that the number of members of the population can be regarded as a differentiable function $P = P(t)$. In most models it is assumed that the differential equation takes the form

$$P' = a(P)P, \tag{1.1.1}$$

where a is a continuous function of P that represents the rate of change of population per unit time per individual. In the Malthusian model (http://en.wikipedia.org/wiki/Thomas_Robert_Malthus), it is assumed that $a(P)$ is a constant, so (1.1.1) becomes

$$P' = aP. \tag{1.1.2}$$

(When you see a name in blue italics, just click on it for information about the person.) This model assumes that the numbers of births and deaths per unit time are both proportional to the population. The constants of proportionality are the birth rate (births per unit time per individual) and the death rate (deaths per unit time per individual); a is the birth rate minus the death rate. You learned in calculus that if c is any constant then

$$P = ce^{at} \tag{1.1.3}$$

satisfies (1.1.2), so (1.1.2) has infinitely many solutions. To select the solution of the specific problem that we're considering, we must know the population P_0 at an initial time, say $t = 0$. Setting $t = 0$ in (1.1.3) yields $c = P(0) = P_0$, so the applicable solution is

$$P(t) = P_0 e^{at}.$$

This implies that

$$\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } a > 0, \\ 0 & \text{if } a < 0; \end{cases}$$

that is, the population approaches infinity if the birth rate exceeds the death rate, or zero if the death rate exceeds the birth rate.

To see the limitations of the Malthusian model, suppose we're modeling the population of a country, starting from a time $t = 0$ when the birth rate exceeds the death rate (so $a > 0$), and the country's resources in terms of space, food supply, and other necessities of life can support the existing population. Then the prediction $P = P_0 e^{at}$ may be reasonably accurate as long as it remains within limits that the country's resources can support. However, the model must inevitably lose validity when the prediction exceeds these limits. (If nothing else, eventually there won't be enough space for the predicted population!)

This flaw in the Malthusian model suggests the need for a model that accounts for limitations of space and resources that tend to oppose the rate of population growth as the population increases. Perhaps the most famous model of this kind is the Verhulst model (http://www-history.mcs.st-and.ac.uk/Mathematicians/Verhulst.html), where (1.1.2) is replaced by

$$P' = aP(1 - \alpha P), \tag{1.1.4}$$

where α is a positive constant. As long as P is small compared to $1/\alpha$, the ratio P'/P is approximately equal to a. Therefore the growth is approximately exponential; however, as P increases, the ratio P'/P decreases as opposing factors become significant.

Equation (1.1.4) is the logistic equation. You will learn how to solve it in Section 1.2. (See Exercise 2.2. 28.) The solution is

$$P = \frac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-at}},$$

where $P_0 = P(0) > 0$. Therefore $\lim_{t\to\infty} P(t) = 1/\alpha$, independent of P_0.

Figure 1.1.1 shows typical graphs of P versus t for various values of P_0.

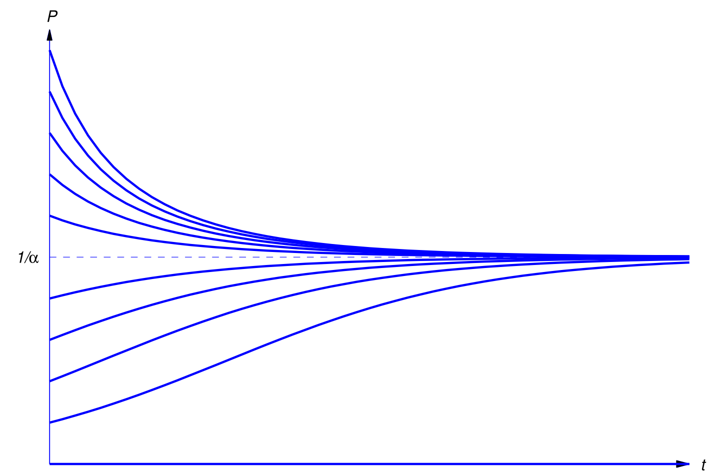

Figure 1.1.1. *Solutions of the logistic equation*

Newton’s Law of Cooling

According to Newton’s law of cooling (http://www-history.mcs.st-and.ac.uk/Mathematicians/Newton.html), the temperature of a body changes at a rate proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Thus, if T_m is the temperature of the medium and $T = T(t)$ is the temperature of the body at time t, then

$$T' = -k(T - T_m), \tag{1.1.5}$$

where k is a positive constant and the minus sign indicates; that the temperature of the body increases with time if it’s less than the temperature of the medium, or decreases if it’s greater. We’ll see in Section 4.2 that if T_m is constant then the solution of (1.1.5) is

$$T = T_m + (T_0 - T_m)e^{-kt}, \tag{1.1.6}$$

where T_0 is the temperature of the body when $t = 0$. Therefore $\lim_{t\to\infty} T(t) = T_m$, independent of T_0. (Common sense suggests this. Why?)

Figure 1.1.2 shows typical graphs of T versus t for various values of T_0.

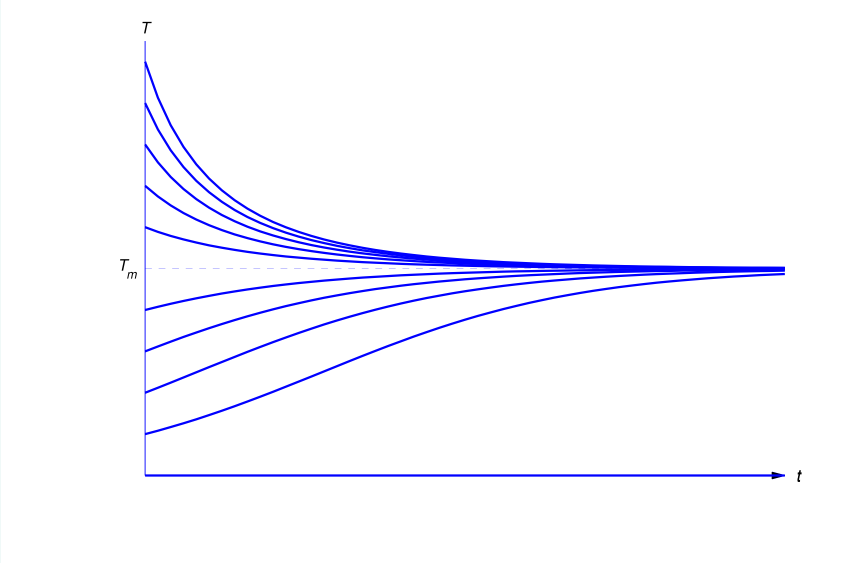

Figure 1.1.2. *Temperature according to Newton's Law of Cooling*

Assuming that the medium remains at constant temperature seems reasonable if we're considering a cup of coffee cooling in a room, but not if we're cooling a huge cauldron of molten metal in the same room. The difference between the two situations is that the heat lost by the coffee isn't likely to raise the temperature of the room appreciably, but the heat lost by the cooling metal is. In this second situation we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by (1.1.5). We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium respectively. If we assume that the total heat of the in the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0.$$

Solving this for T_m and substituting the result into (1.1.6) yields the differential equation

$$T' = -k\left(1 + \frac{a}{a_m}\right)T + k\left(T_{m0} + \frac{a}{a_m}T_0\right)$$

for the temperature of the object. After learning to solve linear first order equations, you'll be able to show (Exercise 4.2. 17) that

$$T = \frac{aT_0 + a_m T_{m0}}{a + a_m} + \frac{a_m(T_0 - T_{m0})}{a + a_m}e^{-k(1+a/a_m)t}.$$

Glucose Absorption by the Body

Glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Suppose there are G_0 units of glucose in the bloodstream when $t = 0$, and let $G = G(t)$ be the number of units in the bloodstream at time $t > 0$. Then, since the glucose being absorbed by the body is leaving the bloodstream, G satisfies the equation

$$G' = -\lambda G. \tag{1.1.7}$$

From calculus you know that if c is any constant then

$$G = ce^{-\lambda t} \tag{1.1.8}$$

satisfies (1.1.7), so (1.1.7) has infinitely many solutions. Setting $t = 0$ in (1.1.8) and requiring that $G(0) = G_0$ yields $c = G_0$, so

$$G(t) = G_0 e^{-\lambda t}.$$

Now let's complicate matters by injecting glucose intravenously at a constant rate of r units of glucose per unit of time. Then the rate of change of the amount of glucose in the bloodstream per unit time is

$$G' = -\lambda G + r, \tag{1.1.9}$$

where the first term on the right is due to the absorption of the glucose by the body and the second term is due to the injection. After you've studied Section 2.1, you'll be able to show (Exercise 2.1. 43) that the solution of (1.1.9) that satisfies $G(0) = G_0$ is

$$G = \frac{r}{\lambda} + \left(G_0 - \frac{r}{\lambda}\right) e^{-\lambda t}.$$

Graphs of this function are similar to those in Figure 1.1.2. (Why?)

Spread of Epidemics

One model for the spread of epidemics assumes that the number of people infected changes at a rate proportional to the product of the number of people already infected and the number of people who are susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then $S - I$ is the number of people who are susceptible, but not yet infected. Thus,

$$I' = rI(S - I),$$

where r is a positive constant. Assuming that $I(0) = I_0$, the solution of this equation is

$$I = \frac{SI_0}{I_0 + (S - I_0)e^{-rSt}}$$

(Exercise 2.2. 29). Graphs of this function are similar to those in Figure 1.1.1. (Why?) Since $\lim_{t\to\infty} I(t) = S$, this model predicts that all the susceptible people eventually become infected.

Newton's Second Law of Motion

According to Newton's second law of motion (http://www-history.mcs.st-and.ac.uk/Mathematicians/Newton.html), the instantaneous acceleration a of an object with constant mass m is related to the force F acting on the object by the equation $F = ma$. For simplicity, let's assume that $m = 1$ and the motion of the object is along a vertical line. Let y be the displacement of the object from some reference point on Earth's surface, measured positive upward. In many applications, there are three kinds of forces that may act on the object:

a. A force such as gravity that depends only on the position y, which we write as $-p(y)$, where $p(y) > 0$ if $y \geq 0$.

b. A force such as atmospheric resistance that depends on the position and velocity of the object, which we write as $-q(y, y')y'$, where q is a nonnegative function and we've put y' "outside" to indicate that the resistive force is always in the direction opposite to the velocity.

c. A force $f = f(t)$, exerted from an external source (such as a towline from a helicopter) that depends only on t.

In this case, Newton's second law implies that

$$y'' = -q(y, y')y' - p(y) + f(t),$$

which is usually rewritten as

$$y'' + q(y, y')y' + p(y) = f(t).$$

Since the second (and no higher) order derivative of y occurs in this equation, we say that it is a second order differential equation.

Interacting Species: Competition

Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t, and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition we would have

$$P' = aP \quad \text{and} \quad Q' = bQ, \tag{1.1.10}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (1.1.10) is replaced by

$$\begin{aligned} P' &= \quad aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. (Since negative population doesn't make sense, this system works only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$. It can be shown (Exercise 10.4. 42) that there's a positive constant ρ such that if (P_0, Q_0) is above the line L through the origin with slope ρ, then the species with population P becomes extinct in finite time, but if (P_0, Q_0) is below L, the species with population Q becomes extinct in finite time. Figure 1.1.3 illustrates this. The curves shown there are given parametrically by $P = P(t), Q = Q(t),\ t > 0$. The arrows indicate direction along the curves with increasing t.

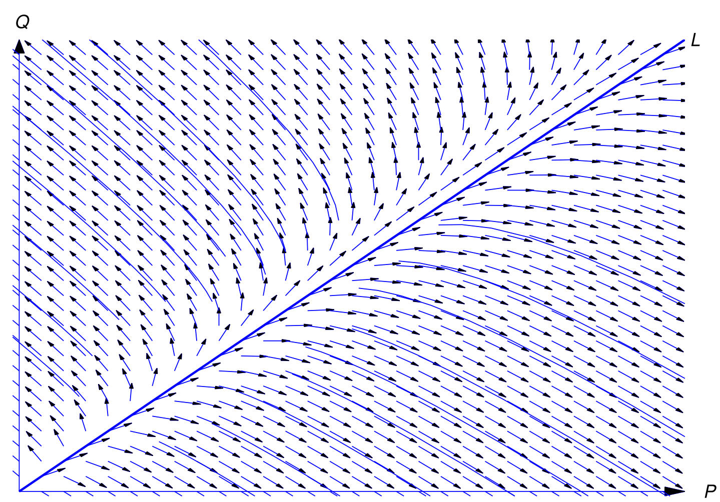

Figure 1.1.3. *Populations of competing species*

1.2 Basic Concepts

A differential equation is an equation that contains one or more derivatives of an unknown function. The order of a differential equation is the order of the highest derivative that it contains. A differential equation is an ordinary differential equation if it involves an unknown function of only one variable, or a partial differential equation if it involves partial derivatives of a function of more than one variable. For now we'll consider only ordinary differential equations, and we'll just call them differential equations.

Throughout this text, all variables and constants are real unless it's stated otherwise. We'll usually use x for the independent variable unless the independent variable is time; then we'll use t.

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x) \quad \text{or, equivalently,} \quad y' = f(x),$$

where f is a known function of x. We already know from calculus how to find functions that satisfy this kind of equation. For example, if

$$y' = x^3,$$

then

$$y = \int x^3\, dx = \frac{x^4}{4} + c,$$

where c is an arbitrary constant. If $n > 1$ we can find functions y that satisfy equations of the form

$$y^{(n)} = f(x) \tag{1.2.1}$$

by repeated integration. Again, this is a calculus problem.

Except for illustrative purposes in this section, there's no need to consider differential equations like (1.2.1).We'll usually consider differential equations that can be written as

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{1.2.2}$$

where at least one of the functions y, y', …, $y^{(n-1)}$ actually appears on the right. Here are some examples:

$$\begin{aligned}
\dst\frac{dy}{dx}-x^2 &= 0 && \text{(first order)},\\
\dst\frac{dy}{dx}+2xy^2 &= -2 && \text{(first order)},\\
\dst\frac{d^2y}{dx^2}+2\dst\frac{dy}{dx}+y &= 2x && \text{(second order)},\\
xy'''+y^2 &= \sin x && \text{(third order)},\\
y^{(n)}+xy'+3y &= x && \text{(n-th order)}.
\end{aligned}$$

Although none of these equations is written as in (1.2.2), all of them can be written in this form:

$$\begin{aligned}
y' &= x^2,\\
y' &= -2-2xy^2,\\
y'' &= 2x-2y'-y,\\
y''' &= \dst\frac{\sin x-y^2}{x},\\
y^{(n)} &= x-xy'-3y.
\end{aligned}$$

Solutions of Differential Equations

A solution of a differential equation is a function that satisfies the differential equation on some open interval; thus, y is a solution of (1.2.2) if y is n times differentiable and

$$y^{(n)}(x)=f(x,y(x),y'(x),\dots,y^{(n-1)}(x))$$

for all x in some open interval (a,b). In this case, we also say that y is a solution of $(???)$ on (a,b). Functions that satisfy a differential equation at isolated points are not interesting. For example, $y=x^2$ satisfies

$$xy'+x^2=3x$$

if and only if $x=0$ or $x=1$, but it's not a solution of this differential equation because it does not satisfy the equation on an open interval.

The graph of a solution of a differential equation is a solution curve. More generally, a curve C is said to be an integral curve of a differential equation if every function $y=y(x)$ whose graph is a segment of C is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve.

Example 1.2.1

If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{1.2.3}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{1.2.4}$$

To see this, note that the only functions whose graphs are segments of (1.2.3) are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy (1.2.4) on the open interval $(-a, a)$. However, (1.2.3) is not a solution curve of (1.2.4), since it's not the graph of a function.

Example 1.2.2

Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.2.5}$$

is a solution of

$$xy' + y = x^2 \tag{1.2.6}$$

on $(0, \infty)$ and on $(-\infty, 0)$.

Solution Substituting (1.2.5) and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into (1.2.6) yields

$$xy'(x) + y(x) = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \left(\frac{x^2}{3} + \frac{1}{x}\right) = x^2$$

for all $x \neq 0$. Therefore y is a solution of (1.2.6) on $(-\infty, 0)$ and $(0, \infty)$. However, y isn't a solution of the differential equation on any open interval that contains $x = 0$, since y is not defined at $x = 0$.

Figure 1.2.1 shows the graph of (1.2.5). The part of the graph of (1.2.5) on $(0, \infty)$ is a solution curve of (1.2.6), as is the part of the graph on $(-\infty, 0)$.

Example 1.2.3

Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2 x)e^{-x} + 2x - 4 \tag{1.2.7}$$

is a solution of

$$y'' + 2y' + y = 2x \tag{1.2.8}$$

on $(-\infty, \infty)$.

Figure 1.2.1. $y =$ \dst $\frac{x^2}{3} + \frac{1}{x}$

Solution Differentiating (1.2.7) twice yields

$$y' = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2$$

and

$$y'' = (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y = {} & (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x} \\ & + 2\left[-(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2\right] \\ & + (c_1 + c_2 x)e^{-x} + 2x - 4 \\ = {} & (1 - 2 + 1)(c_1 + c_2 x)e^{-x} + (-2 + 2)c_2 e^{-x} \\ & + 4 + 2x - 4 = 2x \end{aligned}$$

for all values of x. Therefore y is a solution of (1.2.8) on $(-\infty, \infty)$.

Example 1.2.4

Find all solutions of

$$y^{(n)} = e^{2x}. \tag{1.2.9}$$

Solution Integrating (1.2.9) yields

$$y^{(n-1)} = \frac{e^{2x}}{2} + k_1,$$

where k_1 is a constant. If $n \geq 2$, integrating again yields

$$y^{(n-2)} = \frac{e^{2x}}{4} + k_1 x + k_2.$$

If $n \geq 3$, repeatedly integrating yields

$$y = \frac{e^{2x}}{2^n} + k_1 \frac{x^{n-1}}{(n-1)!} + k_2 \frac{x^{n-2}}{(n-2)!} + \cdots + k_n, \tag{1.2.10}$$

where k_1, k_2, …, k_n are constants. This shows that every solution of (1.2.9) has the form (1.2.10) for some choice of the constants k_1, k_2, …, k_n. On the other hand, differentiating (1.2.10) n times shows that if k_1, k_2, …, k_n are arbitrary constants, then the function y in (1.2.10) satisfies (1.2.9).

Since the constants k_1, k_2, …, k_n in (1.2.10) are arbitrary, so are the constants

$$\frac{k_1}{(n-1)!}, \ \frac{k_2}{(n-2)!}, \ \cdots, \ k_n.$$

Therefore Example 1.2.4 actually shows that all solutions of (1.2.9) can be written as

$$y = \frac{e^{2x}}{2^n} + c_1 + c_2 x + \cdots + c_n x^{n-1},$$

where we renamed the arbitrary constants in (1.2.10) to obtain a simpler formula. As a general rule, arbitrary constants appearing in solutions of differential equations should be simplified if possible. You'll see examples of this throughout the text.

Initial Value Problems

In Example 1.2.4 we saw that the differential equation $y^{(n)} = e^{2x}$ has an infinite family of solutions that depend upon the n arbitrary constants c_1, c_2, …, c_n. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a solution of a differential equation that satisfies one or more specific conditions. The next example illustrates this.

Example 1.2.5

Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution At the beginning of this section we saw that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of c such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c, \quad \text{so} \quad c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.2.2 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$.

We can rewrite the problem considered in Example 1.2.5 more briefly as

$$y' = x^3, \quad y(1) = 2.$$

We call this an initial value problem. The requirement $y(1) = 2$ is an initial condition. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \tag{1.2.11}$$

is an initial value problem for a second order differential equation where y and y' are required to have specified values at $x = 0$. In general, an initial value problem for an n-th order differential equation requires y and its first $n - 1$ derivatives to have specified values at some point x_0. These requirements are the initial conditions.

Figure 1.2.2. $y = \dst\frac{x^2+7}{4}$

We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in (1.2.11). For example, we would write an initial value problem for (1.2.2) as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),\ y(x_0) = k_0,\ y'(x_0) = k_1,\ \dots,\ y^{(n-1)} = k_{n-1}. \tag{1.2.12}$$

Consistent with our earlier definition of a solution of the differential equation in (1.2.12), we say that y is a solution of the initial value problem (1.2.12) if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) that contains x_0, and y satisfies the initial conditions in (1.2.12). The largest open interval that contains x_0 on which y is defined and satisfies the differential equation is the interval of validity of y.

Example 1.2.6

In Example 1.2.5 we saw that

$$y = \frac{x^4 + 7}{4} \tag{1.2.13}$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in (1.2.13) is defined for all x, the interval of validity of this solution is $(-\infty, \infty)$.

Example 1.2.7

In Example 1.2.2 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.2.14}$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating (1.2.14) at $x = \pm 1$, you can see that (1.2.14) is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \tag{1.2.15}$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \tag{1.2.16}$$

The interval of validity of (1.2.14) as a solution of (1.2.15) is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which (1.2.14) is defined. Similarly, the interval of validity of (1.2.14) as a solution of (1.2.16) is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which (1.2.14) is defined.

Free Fall Under Constant Gravity

The term initial value problem originated in problems of motion where the independent variable is t (representing elapsed time), and the initial conditions are the position and velocity of an object at the initial (starting) time of an experiment.

Example 1.2.8

An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant g.

a. Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.

b. Solve the initial value problem derived in **(a)** to obtain the altitude as a function of time.

Solution (a) Let $y(t)$ be the altitude of the object at time t. Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to t. If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{1.2.17}$$

Solution (b) Integrating (1.2.17) twice yields

$$\begin{aligned} y' &= -gt + c_1, \\ y &= -\frac{gt^2}{2} + c_1 t + c_2. \end{aligned}$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the solution of the initial value problem (1.2.17) is

$$y = -\frac{gt^2}{2} + v_0 t + y_0.$$

1.2 Exercises

1. Find the order of the equation.

(a) \dst$\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\ \frac{d^3y}{dx^3} + x = 0$	**(b)** $y'' - 3y' + 2y = x^7$
(c) $y' - y^7 = 0$	**(d)** $y''y - (y')^2 = 2$

Show answer

2. Verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

a. $y = ce^{2x}; \quad y' = 2y$

b. $y =$ \dst$\frac{x^2}{3} + \frac{c}{x}; \quad xy' + y = x^2$

c. $y =$ \dst$\frac{1}{2} + ce^{-x^2}; \quad y' + 2xy = x$

d. $y = (1 + ce^{-x^2/2}); (1 - ce^{-x^2/2})^{-1} \quad 2y' + x(y^2 - 1) = 0$

e. $y =$ \dst$\tan\left(\frac{x^3}{3} + c\right); \quad y' = x^2(1 + y^2)$

f. $y = (c_1 + c_2x)e^x + \sin x + x^2; \quad y'' - 2y' + y = -2\cos x + x^2 - 4x + 2$

g. $y = c_1e^x + c_2x +$ \dst$\frac{2}{x}; \quad (1 - x)y'' + xy' - y = 4(1 - x - x^2)x^{-3}$

h. $y = x^{-1/2}(c_1 \sin x + c_2 \cos x) + 4x + 8; \ x^2y'' + xy' +$ \dst$(x^2 - \frac{1}{4})y = 4x^3 + 8x^2 + 3x - 2$

3. Find all solutions of the equation.

(a) $y' = -x$	**(b)** $y' = -x\sin x$
(c) $y' = x\ln x$	**(d)** $y'' = x\cos x$
(e) $y'' = 2xe^x$	**(f)** $y'' = 2x + \sin x + e^x$
(g) $y''' = -\cos x$	**(h)** $y''' = -x^2 + e^x$
(i) $y''' = 7e^{4x}$	

Show answer

4. Solve the initial value problem.

a. $y' = -xe^x, \quad y(0) = 1$

b. $\backslash\mathrm{dst} y' = x\sin x^2, \quad y\left(\sqrt{\frac{\pi}{2}}\right) = 1$

c. $y' = \tan x, \quad y(\pi/4) = 3$

d. $y'' = x^4, \quad y(2) = -1, \quad y'(2) = -1$

e. $y'' = xe^{2x}, \quad y(0) = 7, \quad y'(0) = 1$

f. $y'' = -x\sin x, \quad y(0) = 1, \quad y'(0) = -3$

g. $y''' = x^2e^x, \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 3$

h. $y''' = 2 + \sin 2x, \quad y(0) = 1, \quad y'(0) = -6, \quad y''(0) = 3$

i. $y''' = 2x + 1, \quad y(2) = 1, \quad y'(2) = -4, \quad y''(2) = 7$

Show answer

5. Verify that the function is a solution of the initial value problem.

a. $y = x\cos x; \quad y' = \cos x - y\tan x, \quad y(\pi/4) = \backslash\mathrm{dst}\frac{\pi}{4\sqrt{2}}$

b. $\backslash\mathrm{dst} y = \frac{1+2\ln x}{x^2} + \frac{1}{2}; \quad y' = \frac{x^2-2x^2y+2}{x^3}, \quad y(1) = \frac{3}{2}$

c. $y = \backslash\mathrm{dst}\tan\left(\frac{x^2}{2}\right); \quad y' = x(1+y^2), \quad y(0) = 0$

d. $\backslash\mathrm{dst} y = \frac{2}{x-2}; \quad y' = \frac{-y(y+1)}{x}, \quad y(1) = -2$

6. Verify that the function is a solution of the initial value problem.

a. $y = x^2(1+\ln x); \quad y'' = \backslash\mathrm{dst}\frac{3xy'-4y}{x^2}, \quad y(e) = 2e^2, \quad y'(e) = 5e$

b. $y = \backslash\mathrm{dst}\frac{x^2}{3} + x - 1; \quad y'' = \backslash\mathrm{dst}\frac{x^2-xy'+y+1}{x^2}, \quad y(1) = \backslash\mathrm{dst}\frac{1}{3}, \quad y'(1) = \backslash\mathrm{dst}\frac{5}{3}$

c. $y = (1+x^2)^{-1/2}; \quad y'' = \backslash\mathrm{dst}\frac{(x^2-1)y - x(x^2+1)y'}{(x^2+1)^2}, \quad y(0) = 1,$

$y'(0) = 0$

d. $y = \backslash\mathrm{dst}\frac{x^2}{1-x}; \quad y'' = \backslash\mathrm{dst}\frac{2(x+y)(xy'-y)}{x^3}, \quad y(1/2) = 1/2, \quad y'(1/2) = 3$

7. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. Take $g = 32$ ft/sec^2.

a. Find the highest altitude attained by the object.

b. Determine how long it takes for the object to fall to the ground.

Show answer

8. Let a be a nonzero real number.

a. Verify that if c is an arbitrary constant then

$$y = (x-c)^a \tag{A}$$

is a solution of

$$y' = ay^{(a-1)/a} \tag{B}$$

on (c, ∞).

b. Suppose $a < 0$ or $a > 1$. Can you think of a solution of (B) that isn't of the form (A)?

Show answer

9. Verify that

$$y = \begin{cases} e^x - 1, & x \geq 0, \\ 1 - e^{-x}, & x < 0, \end{cases}$$

is a solution of

$$y' = |y| + 1$$

on $(-\infty, \infty)$. *Hint: Use the definition of derivative at* $x = 0$.

10. a. Verify that if c is any real number then

$$y = c^2 + cx + 2c + 1 \tag{A}$$

satisfies

$$y' = \frac{-(x+2) + \sqrt{x^2 + 4x + 4y}}{2} \tag{B}$$

on some open interval. Identify the open interval.

b. Verify that

$$y_1 = \frac{-x(x+4)}{4}$$

also satisfies (B) on some open interval, and identify the open interval. (Note that y_1 can't be obtained by selecting a value of c in (A).)

Show answer

1.3 Direction Fields for First Order Equations

It's impossible to find explicit formulas for solutions of some differential equations. Even if there are such formulas, they may be so complicated that they're useless. In this case we may resort to graphical or numerical methods to get some idea of how the solutions of the given equation behave.

In Section 2.3 we'll take up the question of existence of solutions of a first order equation

$$y' = f(x, y). \tag{1.3.1}$$

In this section we'll simply assume that (1.3.1) has solutions and discuss a graphical method for approximating them. In Chapter 3 we discuss numerical methods for obtaining approximate solutions of (1.3.1).

Recall that a solution of (1.3.1) is a function $y = y(x)$ such that

$$y'(x) = f(x, y(x))$$

for all values of x in some interval, and an integral curve is either the graph of a solution or is made up of segments that are graphs of solutions. Therefore, not being able to solve (1.3.1) is equivalent to not knowing the equations of integral curves of (1.3.1). However, it's easy to calculate the slopes of these curves. To be specific, the slope of an integral curve of (1.3.1) through a given point (x_0, y_0) is given by the number $f(x_0, y_0)$. This is the basis of the method of direction fields.

If f is defined on a set R, we can construct a direction field for (1.3.1) in R by drawing a short line segment through each point (x, y) in R with slope $f(x, y)$. Of course, as a practical matter, we can't actually draw line segments through every point in R; rather, we must select a finite set of points in R. For example, suppose f is defined on the closed rectangular region

$$R : \{a \le x \le b, c \le y \le d\}.$$

Let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be equally spaced points in $[a, b]$ and

$$c = y_0 < y_1 < \cdots < y_n = d$$

be equally spaced points in $[c, d]$. We say that the points

$$(x_i, y_j), \quad 0 \le i \le m, \quad 0 \le j \le n,$$

form a rectangular grid (Figure 1.3.1). Through each point in the grid we draw a short line segment with slope $f(x_i, y_j)$. The result is an approximation to a direction field for (1.3.1) in R. If the grid points are sufficiently numerous and close together, we can draw approximate integral curves of (1.3.1) by drawing curves through points in the grid tangent to the line segments associated with the points in the grid.

Figure 1.3.1. *A rectangular grid*

Unfortunately, approximating a direction field and graphing integral curves in this way is too tedious to be done effectively by hand. However, there is software for doing this. As you'll see, the combination of direction fields and integral curves gives useful insights into the behavior of the solutions of the differential equation even if we can't obtain exact solutions.

We'll study numerical methods for solving a single first order equation (1.3.1) in Chapter 3. These methods can be used to plot solution curves of (1.3.1) in a rectangular region R if f is continuous on R. Figures 1.3.2, 1.3.3, and 1.3.4 show direction fields and solution curves for the differential equations

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y' = 1 + xy^2, \quad \text{and} \quad y' = \frac{x - y}{1 + x^2},$$

which are all of the form (1.3.1) with f continuous for all (x, y).

Figure 1.3.2. *A direction field and integral curves for* $y = \dst \frac{x^2-y^2}{1+x^2+y^2}$

Figure 1.3.3. *A direction field and integral curves for* $y' = 1 + xy^2$

Figure 1.3.4. *A direction and integral curves for* $y' = \dst \frac{x-y}{1+x^2}$

The methods of Chapter 3 won't work for the equation

$$y' = -x/y \tag{1.3.2}$$

if R contains part of the x-axis, since $f(x, y) = -x/y$ is undefined when $y = 0$. Similarly, they won't work for the equation

$$y' = \frac{x^2}{1 - x^2 - y^2} \tag{1.3.3}$$

if R contains any part of the unit circle $x^2 + y^2 = 1$, because the right side of (1.3.3) is undefined if $x^2 + y^2 = 1$. However, (1.3.2) and (1.3.3) can written as

$$y' = \frac{A(x, y)}{B(x, y)} \tag{1.3.4}$$

where A and B are continuous on any rectangle R. Because of this, some differential equation software is based on numerically solving pairs of equations of the form

$$\frac{dx}{dt} = B(x, y), \quad \frac{dy}{dt} = A(x, y) \tag{1.3.5}$$

where x and y are regarded as functions of a parameter t. If $x = x(t)$ and $y = y(t)$ satisfy these equations, then

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{A(x, y)}{B(x, y)},$$

so $y = y(x)$ satisfies (1.3.4).

Eqns. (1.3.2) and (1.3.3) can be reformulated as in (1.3.4) with

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x$$

and

$$\frac{dx}{dt} = 1 - x^2 - y^2, \quad \frac{dy}{dt} = x^2,$$

respectively. Even if f is continuous and otherwise “nice” throughout R, your software may require you to reformulate the equation $y' = f(x, y)$ as

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f(x, y),$$

which is of the form (1.3.5) with $A(x, y) = f(x, y)$ and $B(x, y) = 1$.

Figure 1.3.5 shows a direction field and some integral curves for (1.3.2). As we saw in Example 1.2.1 and will verify again in Section 2.2, the integral curves of (1.3.2) are circles centered at the origin.

Figure 1.3.5. *A direction field and integral curves for* $y' = -\backslash\mathrm{dst}\frac{x}{y}$

Figure 1.3.6 shows a direction field and some integral curves for (1.3.3). The integral curves near the top and bottom are solution curves. However, the integral curves near the middle are more complicated. For example, Figure 1.3.7 shows the integral curve through the origin. The vertices of the dashed rectangle are on the circle $x^2 + y^2 = 1$ ($a \approx .846$, $b \approx .533$), where all integral curves of (1.3.3) have infinite slope. There are three solution curves of (1.3.3) on the integral curve in the figure: the segment above the level $y = b$ is the graph of a solution on $(-\infty, a)$, the segment below the level $y = -b$ is the graph of a solution on $(-a, \infty)$, and the segment between these two levels is the graph of a solution on $(-a, a)$.

Using Technology

As you study from this book, you’ll often be asked to use computer software and graphics. Exercises with this intent are marked as C (computer or calculator required), C/G (computer and/or graphics required), or L (laboratory work requiring software and/or graphics). Often you may not completely understand how the software does what it does. This is similar to the situation most people are in when they drive automobiles or watch television, and it doesn’t decrease the value of using modern technology as an aid to learning. Just be careful that you use the technology as a supplement to thought rather than a substitute for it.

Figure 1.3.6. *A direction field and integral curves for* $y' = \dst\frac{x^2}{1-x^2-y^2}$

Figure 1.3.7.

1.3 Exercises

In Exercises 1–11 a direction field is drawn for the given equation. Sketch some integral curves.

1 *A direction field for* $y' = \dst\frac{x}{y}$

2 *A direction field for* $\dst y' = \dst\frac{2xy^2}{1+x^2}$

3 *A direction field for* $y' = x^2(1+y^2)$

4 *A direction field for* $y' = \dst\frac{1}{1+x^2+y^2}$

5 *A direction field for* $y' = -(2xy^2+y^3)$

6 *A direction field for* $y' = (x^2+y^2)^{1/2}$

7 *A direction field for* $y' = \sin xy$

8 *A direction field for* $y' = e^{xy}$

9 *A direction field for* $y' = (x-y^2)(x^2-y)$

10 *A direction field for* $y' = x^3y^2+xy^3$

11 *A direction field for* $y' = \sin(x-2y)$

In Exercises 12-22 construct a direction field and plot some integral curves in the indicated rectangular region.

12. C/G $y' = y(y-1); \quad \{-1 \le x \le 2, \ -2 \le y \le 2\}$

13. C/G $y' = 2 - 3xy; \quad \{-1 \le x \le 4, \ -4 \le y \le 4\}$

14. C/G $y' = xy(y-1); \quad \{-2 \le x \le 2, \ -4 \le y \le 4\}$

15. C/G $y' = 3x + y; \quad \{-2 \le x \le 2, \ 0 \le y \le 4\}$

16. C/G $y' = y - x^3; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

17. C/G $y' = 1 - x^2 - y^2; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

18. C/G $y' = x(y^2 - 1); \quad \{-3 \le x \le 3, \ -3 \le y \le 2\}$

19. C/G $y' = \backslash\mathrm{dst}\frac{x}{y(y^2-1)}; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

20. C/G $y' = \backslash\mathrm{dst}\frac{xy^2}{y-1}; \quad \{-2 \le x \le 2, \ -1 \le y \le 4\}$

21. C/G $y' = \backslash\mathrm{dst}\frac{x(y^2-1)}{y}; \quad \{-1 \le x \le 1, \ -2 \le y \le 2\}$

22. C/G $y' = -\backslash\mathrm{dst}\frac{x^2+y^2}{1-x^2-y^2}; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

23. L By suitably renaming the constants and dependent variables in the equations

$$T' = -k(T - T_m) \tag{A}$$

and

$$G' = -\lambda G + r \tag{B}$$

discussed in Section 1.2 in connection with Newton's law of cooling and absorption of glucose in the body, we can write both as

$$y' = -ay + b, \tag{C}$$

where a is a positive constant and b is an arbitrary constant. Thus, (A) is of the form (C) with $y = T$, $a = k$, and $b = kT_m$, and (B) is of the form (C) with $y = G$, $a = \lambda$, and $b = r$. We'll encounter equations of the form (C) in many other applications in Chapter 2.

Choose a positive a and an arbitrary b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ c \le y \le d\}$$

of the ty-plane. Vary T, c, and d until you discover a common property of all the solutions of (C). Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b.

24. L By suitably renaming the constants and dependent variables in the equations

$$P' = aP(1 - \alpha P) \tag{A}$$

and

$$I' = rI(S - I) \tag{B}$$

discussed in Section 1.1 in connection with Verhulst's population model and the spread of an epidemic, we can write both in the form

$$y' = ay - by^2, \tag{C}$$

where a and b are positive constants. Thus, (A) is of the form (C) with $y = P$, $a = a$, and $b = a\alpha$, and (B) is of the form (C) with $y = I$, $a = rS$, and $b = r$. In Chapter 2 we'll encounter equations of the form (C) in other applications..

a. Choose positive numbers a and b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ 0 \le y \le d\}$$

of the ty-plane. Vary T and d until you discover a common property of all solutions of (C) with $y(0) > 0$. Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b.

b. Choose positive numbers a and b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ c \le y \le 0\}$$

of the ty-plane. Vary a, b, T and c until you discover a common property of all solutions of (C) with $y(0) < 0$.

You can verify your results later by doing Exercise 2.2. 27.

Chapter 1 Introduction

1.2 Basic Concepts

1 **(a)** 3 **(b)** 2 **(c)** 1 **(d)** 2

3 **(a)** $y=-$\dst$\frac{x^2}{2}+c$ **(b)** $y=x\cos x-\sin x+c$

(c) $y=$\dst$\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$

(d) $y=-x\cos x+2\sin x+c_1+c_2x$

(e) $y=(2x-4)e^x+c_1+c_2x$ **(f)** $y=$\dst$\frac{x^3}{3}-\sin x+e^x+c_1+c_2x$

(g) $y=\sin x+c_1+c_2x+c_3x^2$

(h) $y=-$\dst$\frac{x^5}{60}+e^x+c_1+c_2x+c_3x^2$

(i) $y=$\dst$\frac{7}{64}e^{4x}+c_1+c_2x+c_3x^2$

4 **(a)** $y = -(x-1)e^x$ **(b)** $y = 1 - \dst\frac{1}{2}\cos x^2$ **(c)** $y = 3 - \ln(\sqrt{2}\cos x)$

(d) **(f)** $y = x\sin x + 2\cos x - 3x - 1$ **(g)** $y = (x^2 - 6x + 12)e^x + \dst\frac{x^2}{2} - 8x - 11$

y

$= \dst -\frac{47}{15} - \frac{37}{5}(x-2) + \frac{x^5}{30}$

(e) y

$= \dst\frac{1}{4}xe^{2x} - \frac{1}{4}e^{2x} + \frac{29}{4}$

(h) y

$= \dst\frac{x^3}{3} + \frac{\cos 2x}{6} + \frac{7}{4}x^2 - 6x + \frac{7}{8}$

(i) y

$= \dst\frac{x^4}{12} + \frac{x^3}{6} + \frac{1}{2}(x-2)^2 - \frac{26}{3}(x-2) - \frac{5}{3}$

7 **(a)** 576 ft **(b)** 10 s

8 **(b)** $y = 0$

10 **(a)** $(-2c-2, \infty)$ $(-\infty, \infty)$