9 Plane Waves in Lossless Media

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9.1 Maxwell's Equations in Differential Phasor Form

In this section, we derive the phasor form of Maxwell's Equations from the general time-varying form of these equations. Here we are interested exclusively in the differential ("point") form of these equations. It is assumed that the reader is comfortable with phasor representation and its benefits; if not, a review of Section 1.5 is recommended before attempting this section.

Maxwell's Equations in differential time-domain form are Gauss' Law (Section 5.7):

$$\nabla \cdot \mathbf{D} = \rho_v \tag{9.1}$$

the Maxwell-Faraday Equation (MFE; Section 8.8):

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{9.2}$$

Gauss' Law for Magnetism (GSM; Section 7.3):

$$\nabla \cdot \mathbf{B} = 0 \tag{9.3}$$

and Ampere's Law (Section 8.9):

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \tag{9.4}$$

We begin with Gauss's Law (Equation 9.1). We define $\widetilde{\mathbf{D}}$ and $\tilde{\rho}_v$ as phasor quantities through the usual relationship:

$$\mathbf{D} = \mathrm{Re}\left\{\widetilde{\mathbf{D}}e^{j\omega t}\right\} \tag{9.5}$$

and

$$\rho_v = \mathrm{Re}\left\{\tilde{\rho}_v e^{j\omega t}\right\} \tag{9.6}$$

Substituting these expressions into Equation 9.1:

$$\nabla \cdot \left[\text{Re}\left\{ \widetilde{\mathbf{D}} e^{j\omega t} \right\} \right] = \text{Re}\left\{ \tilde{\rho}_v e^{j\omega t} \right\} \tag{9.7}$$

Divergence is a real-valued linear operator. Therefore, we may exchange the order of the “Re” and “$\nabla\cdot$” operations (this is “Claim 2” from Section 1.5):

$$\text{Re}\left\{ \nabla \cdot \left[\widetilde{\mathbf{D}} e^{j\omega t} \right] \right\} = \text{Re}\left\{ \tilde{\rho}_v e^{j\omega t} \right\} \tag{9.8}$$

Next, we note that the differentiation associated with the divergence operator is with respect to position and not with respect to time, so the order of operations may be further rearranged as follows:

$$\text{Re}\left\{ \left[\nabla \cdot \widetilde{\mathbf{D}} \right] e^{j\omega t} \right\} = \text{Re}\left\{ \tilde{\rho}_v e^{j\omega t} \right\} \tag{9.9}$$

Finally, we note that the equality of the left and right sides of the above equation implies the equality of the associated phasors (this is “Claim 1” from Section 1.5); thus,

$$\boxed{\nabla \cdot \widetilde{\mathbf{D}} = \tilde{\rho}_v} \tag{9.10}$$

In other words, the differential form of Gauss’ Law for phasors is identical to the differential form of Gauss’ Law for physical time-domain quantities.

The same procedure applied to the MFE is only a little more complicated. First, we establish the phasor representations of the electric and magnetic fields:

$$\mathbf{E} = \text{Re}\left\{ \widetilde{\mathbf{E}} e^{j\omega t} \right\} \tag{9.11}$$

$$\mathbf{B} = \text{Re}\left\{ \widetilde{\mathbf{B}} e^{j\omega t} \right\} \tag{9.12}$$

After substitution into Equation 9.2:

$$\nabla \times \left[\text{Re}\left\{ \widetilde{\mathbf{E}} e^{j\omega t} \right\} \right] = -\frac{\partial}{\partial t} \left[\text{Re}\left\{ \widetilde{\mathbf{B}} e^{j\omega t} \right\} \right] \tag{9.13}$$

Both curl and time-differentiation are real-valued linear operations, so we are entitled to change the order of operations as follows:

$$\operatorname{Re}\left\{\nabla \times \left[\widetilde{\mathbf{E}}e^{j\omega t}\right]\right\} = -\operatorname{Re}\left\{\frac{\partial}{\partial t}\left[\widetilde{\mathbf{B}}e^{j\omega t}\right]\right\} \tag{9.14}$$

On the left, we note that the time dependence $e^{j\omega t}$ can be pulled out of the argument of the curl operator, since it does not depend on position:

$$\operatorname{Re}\left\{\left[\nabla \times \widetilde{\mathbf{E}}\right]e^{j\omega t}\right\} = -\operatorname{Re}\left\{\frac{\partial}{\partial t}\left[\widetilde{\mathbf{B}}e^{j\omega t}\right]\right\} \tag{9.15}$$

On the right, we note that $\widetilde{\mathrm{B}}$ is constant with respect to time (because it is a phasor), so:

$$\begin{aligned}\operatorname{Re}\left\{\left[\nabla \times \widetilde{\mathbf{E}}\right]e^{j\omega t}\right\} &= -\operatorname{Re}\left\{\widetilde{\mathbf{B}}\frac{\partial}{\partial t}e^{j\omega t}\right\} \\ &= -\operatorname{Re}\left\{\widetilde{\mathbf{B}}j\omega e^{j\omega t}\right\} \\ &= \operatorname{Re}\left\{\left[-j\omega\widetilde{\mathbf{B}}\right]e^{j\omega t}\right\}\end{aligned} \tag{9.16}$$

And so we have found:

$$\boxed{\nabla \times \widetilde{\mathbf{E}} = -j\omega\widetilde{\mathbf{B}}} \tag{9.17}$$

Let’s pause for a moment to consider the above result. In the general time-domain version of the MFE, we must take spatial derivatives of the electric field and time derivatives of the magnetic field. In the phasor version of the MFE, the time derivative operator has been replaced with multiplication by $j\omega$. This is a tremendous simplification since the equations now involve differentiation over position only. Furthermore, no information is lost in this simplification – for a reminder of why that is, see the discussion of Fourier Analysis at the end of Section 1.5. Without this kind of simplification, much of what is now considered “basic” engineering electromagnetics would be intractable.

The procedure for conversion of the remaining two equations is very similar, yielding:

$$\boxed{\nabla \cdot \widetilde{\mathbf{B}} = 0} \tag{9.18}$$

$$\boxed{\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\widetilde{\mathbf{D}}} \tag{9.19}$$

The details are left as an exercise for the reader.

The differential form of Maxwell's Equations (Equations 9.10, 9.17, 9.18, and 9.19) involve operations on the phasor representations of the physical quantities. These equations have the advantage that differentiation with respect to time is replaced by multiplication by $j\omega$.

9.2 Wave Equations for Source-Free and Lossless Regions

Electromagnetic waves are solutions to a set of coupled differential simultaneous equations – namely, Maxwell's Equations. The general solution to these equations includes constants whose values are determined by the applicable electromagnetic boundary conditions. However, this direct approach can be difficult and is often not necessary. In unbounded homogeneous regions that are "source free" (containing no charges or currents), a simpler approach is possible. In this section, we reduce Maxwell's Equations to *wave equations* that apply to the electric and magnetic fields in this simpler category of scenarios. Before reading further, the reader should consider a review of Section 1.3 (noting in particular Equation 1.1) and Section 3.6 (wave equations for voltage and current on a transmission line). This section seeks to develop the analogous equations for electric and magnetic waves.

We can get the job done using the differential "point" phasor form of Maxwell's Equations, developed in Section 9.1. Here they are:

$$\nabla \cdot \widetilde{\mathbf{D}} = \tilde{\rho}_v \tag{9.20}$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\widetilde{\mathbf{B}} \tag{9.21}$$

$$\nabla \cdot \widetilde{\mathbf{B}} = 0 \tag{9.22}$$

$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\widetilde{\mathbf{D}} \tag{9.23}$$

In a source-free region, there is no net charge and no current, hence $\tilde{\rho}_v = 0$ and $\widetilde{\mathbf{J}} = 0$ in the present analysis. The above equations become

$$\nabla \cdot \widetilde{\mathbf{D}} = 0 \tag{9.24}$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\widetilde{\mathbf{B}} \tag{9.25}$$

$$\nabla \cdot \widetilde{\mathbf{B}} = 0 \tag{9.26}$$

$$\nabla \times \widetilde{\mathbf{H}} = +j\omega\widetilde{\mathbf{D}} \tag{9.27}$$

Next, we recall that $\widetilde{\mathbf{D}} = \epsilon\widetilde{\mathbf{E}}$ and that ϵ is a real-valued constant for a medium that is homogeneous, isotropic, and linear (Section 2.8). Similarly, $\widetilde{\mathbf{B}} = \mu\tilde{\mathbf{H}}$ and μ is a real-valued constant. Thus, under these conditions, it is sufficient to consider *either* $\widetilde{\mathbf{D}}$ or $\widetilde{\mathbf{E}}$ and *either* $\widetilde{\mathbf{B}}$ or $\widetilde{\mathbf{H}}$. The choice is arbitrary, but in engineering applications it is customary to use $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$. Eliminating the now-redundant quantities $\widetilde{\mathbf{D}}$ and $\widetilde{\mathbf{B}}$, the above equations become

$$\nabla \cdot \widetilde{\mathbf{E}} = 0 \tag{9.28}$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}} \tag{9.29}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0 \tag{9.30}$$

$$\nabla \times \widetilde{\mathbf{H}} = +j\omega\epsilon\widetilde{\mathbf{E}} \tag{9.31}$$

It is important to note that requiring the region of interest to be source-free precludes the possibility of loss in the medium. To see this, let's first be clear about what we mean by "loss." For an electromagnetic wave, loss is observed as a reduction in the magnitude of the electric and magnetic field with increasing distance. This reduction is due to the dissipation of power in the medium. This occurs when the conductivity σ is greater than zero because Ohm's Law for Electromagnetics ($\widetilde{\mathbf{J}} = \sigma\widetilde{\mathbf{E}}$; Section 6.3) requires that power in the electric field be transferred into conduction current, and is thereby lost to the wave (Section 6.6). When we required $\mathbf{J}$ to be zero above, we precluded this possibility; that is, we implicitly specified $\sigma = 0$. The fact that the constitutive parameters μ and ϵ appear in Equations 9.28–9.31, but σ does not, is further evidence of this.

Equations 9.28–9.31 are Maxwell's Equations for a region comprised of isotropic, homogeneous, and source-free material. Because there can be no conduction current in a source-free region, these equations apply only to material that is lossless (i.e., having negligible σ).

Before moving on, one additional disclosure is appropriate. It turns out that there actually *is* a way to use Equations 9.28–9.31 for regions in which loss is significant. This requires a redefinition of ϵ as a complex-valued quantity. We shall not consider this technique in this section. We mention this simply because one should be aware that if permittivity appears as a complex-valued quantity, then the imaginary part represents loss.

To derive the wave equations we begin with the MFE, Equation 9.29. Taking the curl of both sides of the equation we obtain

$$\begin{aligned}\nabla \times \left(\nabla \times \widetilde{\mathbf{E}}\right) &= \nabla \times \left(-j\omega\mu\widetilde{\mathbf{H}}\right) \\ &= -j\omega\mu \left(\nabla \times \widetilde{\mathbf{H}}\right)\end{aligned} \tag{9.32}$$

On the right we can eliminate $\nabla \times \widetilde{\mathbf{H}}$ using Equation 9.31:

$$\begin{aligned}-j\omega\mu \left(\nabla \times \widetilde{\mathbf{H}}\right) &= -j\omega\mu \left(+j\omega\epsilon\widetilde{\mathbf{E}}\right) \\ &= +\omega^2\mu\epsilon\widetilde{\mathbf{E}}\end{aligned} \tag{9.33}$$

On the left side of Equation 9.32, we apply the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla \left(\nabla \cdot \mathbf{A}\right) - \nabla^2\mathbf{A} \tag{9.34}$$

which in this case is

$$\nabla \times \nabla \times \widetilde{\mathbf{E}} = \nabla \left(\nabla \cdot \widetilde{\mathbf{E}}\right) - \nabla^2\widetilde{\mathbf{E}} \tag{9.35}$$

We may eliminate the first term on the right using Equation 9.28, yielding

$$\nabla \times \nabla \times \widetilde{\mathbf{E}} = -\nabla^2\widetilde{\mathbf{E}} \tag{9.36}$$

Substituting these results back into Equation 9.32 and rearranging terms we have

$$\nabla^2\widetilde{\mathbf{E}} + \omega^2\mu\epsilon\widetilde{\mathbf{E}} = 0 \tag{9.37}$$

This is the wave equation for $\widetilde{\mathbf{E}}$. Note that it is a homogeneous (in the mathematical sense of the word) differential equation, which is expected since we have derived it for a source-free region.

It is common to make the following definition

$$\boxed{\beta \triangleq \omega\sqrt{\mu\epsilon}} \tag{9.38}$$

so that Equation 9.37 may be written

$$\boxed{\nabla^2\widetilde{\mathbf{E}} + \beta^2\widetilde{\mathbf{E}} = 0} \tag{9.39}$$

Why go the the trouble of defining β? One reason is that β conveniently captures the contribution of the frequency, permittivity, and permeability all in one constant. Another reason is to emphasize the connection to the parameter β appearing in transmission line theory (see Section 3.8 for a reminder). It should be clear that β is a *phase propagation constant*, having units of 1/m (or rad/m, if you prefer), and indicates the rate at which the phase of the propagating wave progresses with distance.

The wave equation for $\widetilde{\mathbf{H}}$ is obtained using essentially the same procedure, which is left as an exercise for the reader. It should be clear from the duality apparent in Equations 9.28-9.31 that the result will be very similar. One finds:

$$\boxed{\nabla^2\widetilde{\mathbf{H}} + \beta^2\widetilde{\mathbf{H}} = 0} \tag{9.40}$$

Equations 9.39 and 9.40 are the wave equations for $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$, respectively, for a region comprised of isotropic, homogeneous, lossless, and source-free material.

Looking ahead, note that $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are solutions to the *same* homogeneous differential equation. Consequently, $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ cannot be different by more than a constant factor and a direction. In fact, we can also determine something about the factor simply by examining the units involved: Since $\widetilde{\mathbf{E}}$ has units of V/m and $\widetilde{\mathbf{H}}$ has units of A/m, this factor will be expressible in units of the ratio of V/m to A/m, which is Ω. This indicates that the factor will be an impedance. This factor is known as the *wave impedance* and will be addressed in Section 9.5. This impedance is analogous the characteristic impedance of a transmission line (Section 3.7).

Additional Reading:

- "Wave Equation (https://en.wikipedia.org/wiki/Wave_equation)" on Wikipedia.
- "Electromagnetic Wave Equation (https://en.wikipedia.org/wiki/Electromagnetic_wave_equation)" on Wikipedia.

9.3 Types of Waves

Solutions to the electromagnetic wave equations (Section 9.2) exist in a variety of forms, representing different types of waves. It is useful to identify three particular geometries for unguided waves. Each of these geometries is defined by the shape formed by surfaces of constant phase, which we refer to as *phasefronts*. (Keep in mind the analogy between electromagnetic waves and sound waves (described in Section 1.3), and note that sound waves also exhibit these geometries.)

A *spherical wave* has phasefronts that form concentric spheres, as shown in Figure 9.1. Waves are well-modeled as spherical when the dimensions of the source of the wave are small relative to the scale at which the wave is observed. For example, the wave radiated by an antenna having dimensions of

10 cm, when observed in free space over a scale of 10 km, appears to have phasefronts that are very nearly spherical. Note that the magnitude of the field on a phasefront of a spherical wave may vary significantly, but it is the shape of phasefronts that make it a spherical wave.

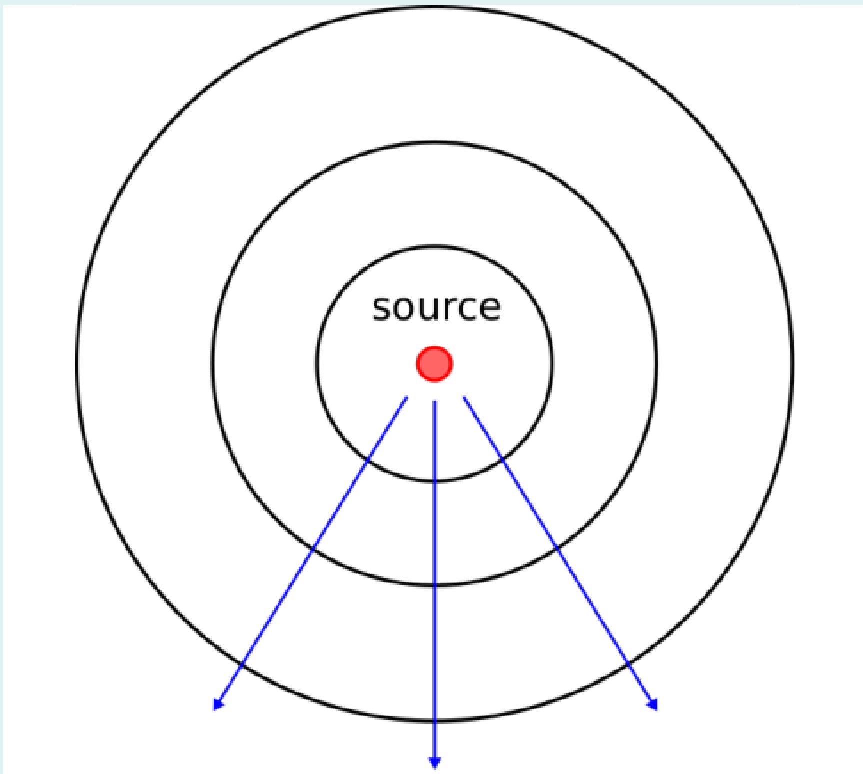

Figure 9.1. *The phasefronts of a spherical wave form concentric spheres.*

A *cylindrical wave* exhibits phasefronts that form concentric cylinders, as shown in Figure 9.2. Said differently, the phasefronts of a cylindrical wave are circular in one dimension, and planar in the perpendicular direction. A cylindrical wave is often a good description of the wave that emerges from a line-shaped source.

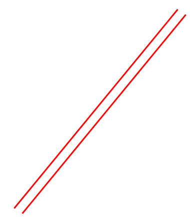

Figure 9.2. *The phasefronts of a cylindrical wave form concentric cylinders.*

A *plane wave* exhibits phasefronts that are planar, with planes that are parallel to each other as shown in Figure 9.3. There are two conditions in which waves are well-modeled as plane waves. First, some structures give rise to waves that appear to have planar phasefronts over a limited area; a good example is the wave radiated by a parabolic reflector, as shown in Figure 9.4. Second, all waves are well-modeled as plane waves when observed over a small region located sufficiently far from the source. In particular, spherical waves are “locally planar” in the sense that they are well-modeled as planar when observed over a small portion of the spherical phasefront, as shown in Figure 9.5. An analogy is that the Earth seems “locally flat” to an observer on the ground, even though it is clearly spherical to an observer in orbit. The “locally planar” approximation is often employed because it is broadly applicable and simplifies analysis.

Figure 9.3. *The phasefronts of a plane wave form parallel planes.*

Figure 9.4. *Plane waves formed in the region in front of a parabolic reflector antenna.*

Figure 9.5. *“Locally planar” approximation of a spherical wave over a limited area.*

Most waves are well-modeled as spherical, cylindrical, or plane waves.

Plane waves (having planar phasefronts) are of particular importance due to wide applicability of the “locally planar” approximation.

9.4 Uniform Plane Waves: Derivation

Section 9.2 showed how Maxwell’s Equations could be reduced to a pair of phasor-domain “wave equations,” namely:

$$\nabla^2\widetilde{\mathbf{E}} + \beta^2\widetilde{\mathbf{E}} = 0 \tag{9.41}$$

$$\nabla^2\widetilde{\mathbf{H}} + \beta^2\widetilde{\mathbf{H}} = 0 \tag{9.42}$$

where $\beta = \omega\sqrt{\mu\epsilon}$, assuming unbounded homogeneous, isotropic, lossless, and source-free media. In this section, we solve these equations for the special case of a uniform plane wave. A uniform plane wave is one for which both $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ have constant magnitude and phase in a specified plane. Despite being a special case, the solution turns out to be broadly applicable, appearing as a common building block in many practical and theoretical problems in unguided propagation (as explained in Section 9.3), as well as in more than a few transmission line and waveguide problems.

To begin, let us assume that the plane over which $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ have constant magnitude and phase is a plane of constant z. First, note that we may make this assumption with no loss of generality. For example, we could alternatively select a plane of constant y, solve the problem, and then simply exchange variables to get a solution for planes of constant z (or x).[1] Furthermore, the solution for any planar orientation not corresponding to a plane of constant x, y, or z may be similarly obtained by a rotation of coordinates, since the physics of the problem does not depend on the orientation of this plane – if it does, then the medium is not isotropic!

We may express the constraint that the magnitude and phase of $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are constant over a plane that is perpendicular to the z axis as follows:

$$\frac{\partial}{\partial x}\widetilde{\mathbf{E}} = \frac{\partial}{\partial y}\widetilde{\mathbf{E}} = \frac{\partial}{\partial x}\widetilde{\mathbf{H}} = \frac{\partial}{\partial y}\widetilde{\mathbf{H}} = 0 \tag{9.43}$$

Let us identify the Cartesian components of each of these fields as follows:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z \text{ , and} \tag{9.44}$$
$$\widetilde{\mathbf{H}} = \hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{y}}\widetilde{H}_y + \hat{\mathbf{z}}\widetilde{H}_z \tag{9.45}$$

Now Equation 9.43 may be interpreted in detail for $\widetilde{\mathbf{E}}$ as follows:

$$\frac{\partial}{\partial x}\widetilde{E}_x = \frac{\partial}{\partial x}\widetilde{E}_y = \frac{\partial}{\partial x}\widetilde{E}_z = 0 \tag{9.46}$$

$$\frac{\partial}{\partial y}\widetilde{E}_x = \frac{\partial}{\partial y}\widetilde{E}_y = \frac{\partial}{\partial y}\widetilde{E}_z = 0 \tag{9.47}$$

and for $\widetilde{\mathbf{H}}$ as follows:

$$\frac{\partial}{\partial x}\widetilde{H}_x = \frac{\partial}{\partial x}\widetilde{H}_y = \frac{\partial}{\partial x}\widetilde{H}_z = 0 \tag{9.48}$$

$$\frac{\partial}{\partial y}\widetilde{H}_x = \frac{\partial}{\partial y}\widetilde{H}_y = \frac{\partial}{\partial y}\widetilde{H}_z = 0 \tag{9.49}$$

The wave equation for $\widetilde{\mathbf{E}}$ (Equation 9.41) written explicitly in Cartesian coordinates is

$$\begin{aligned}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]&\left(\hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z\right) \\ &+\beta^2\left(\hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z\right) = 0\end{aligned} \tag{9.50}$$

Decomposing this equation into separate equations for each of the three coordinates, we obtain the following:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\widetilde{E}_x + \beta^2\widetilde{E}_x = 0 \tag{9.51}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\widetilde{E}_y + \beta^2\widetilde{E}_y = 0 \tag{9.52}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\widetilde{E}_z + \beta^2\widetilde{E}_z = 0 \tag{9.53}$$

Applying the constraints of Equations 9.46 and 9.47, we note that many of the terms in Equations 9.51–9.53 are zero. We are left with:

$$\frac{\partial^2}{\partial z^2}\widetilde{E}_x + \beta^2\widetilde{E}_x = 0 \tag{9.54}$$

$$\frac{\partial^2}{\partial z^2}\widetilde{E}_y + \beta^2\widetilde{E}_y = 0 \tag{9.55}$$

$$\frac{\partial^2}{\partial z^2}\widetilde{E}_z + \beta^2\widetilde{E}_z = 0 \tag{9.56}$$

Now we will show that Equation 9.43 also implies that $\widetilde{E}_z$ must be zero. To show this, we use Ampere's Law for a source-free region (Section 9.2):

$$\nabla \times \widetilde{\mathbf{H}} = +j\omega\epsilon\widetilde{\mathbf{E}} \tag{9.57}$$

and take the dot product with $\hat{\mathbf{z}}$ on both sides:

$$\begin{aligned} \hat{\mathbf{z}} \cdot \left(\nabla \times \widetilde{\mathbf{H}}\right) &= +j\omega\epsilon\widetilde{E}_z \\ \frac{\partial}{\partial y}\widetilde{H}_x - \frac{\partial}{\partial x}\widetilde{H}_y &= +j\omega\epsilon\widetilde{E}_z \end{aligned} \tag{9.58}$$

Again applying the constraints of Equation 9.43, the left side of Equation 9.58 must be zero; therefore, $\widetilde{E}_z = 0$. The exact same procedure applied to $\widetilde{\mathbf{H}}$ (using the Maxwell-Faraday Equation; also given in Section 9.2) reveals that $\widetilde{H}_z$ is also zero.[2] Here is what we have found:

> If a wave is uniform over a plane, then the electric and magnetic field vectors must lie in this plane.

This conclusion is a direct consequence of the fact that Maxwell's Equations require the electric field to be proportional to the curl of the magnetic field and vice-versa.

The general solution to Equation 9.54 is:

$$\widetilde{E}_x = E_{x0}^{+}e^{-j\beta z} + E_{x0}^{-}e^{+j\beta z} \tag{9.59}$$

where E_{x0}^{+} and E_{x0}^{-} are complex-valued constants. The values of these constants are determined by boundary conditions – possibly sources – outside the region of interest. Since in this section we are limiting our scope to source-free and homogeneous regions, we may for the moment consider the values of E_{x0}^{+} and E_{x0}^{-} to be arbitrary, since any values will satisfy the associated wave equation.[3]

Similarly we have for $\widetilde{E}_y$:

$$\widetilde{E}_y = E_{y0}^{+}e^{-j\beta z} + E_{y0}^{-}e^{+j\beta z} \tag{9.60}$$

where E_{y0}^{+} and E_{y0}^{-} are again arbitrary constants. Summarizing, we have found

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y \tag{9.61}$$

where $\widetilde{E}_x$ and $\widetilde{E}_y$ are given by Equations 9.59 and 9.60, respectively.

Note that Equations 9.59 and 9.60 are essentially the *same* equations encountered in the study of waves in lossless transmission lines; for a reminder, see Section 3.6. Specifically, factors containing $e^{-j\beta z}$ describe propagation in the $+z$ direction, whereas factors containing $e^{+j\beta z}$ describe propagation in the $-z$ direction. We conclude:

> If a wave is uniform over a plane, then possible directions of propagation are the two directions perpendicular to the plane.

Since we previously established that the electric and magnetic field vectors must lie in the plane, we also conclude:

> The direction of propagation is perpendicular to the electric and magnetic field vectors.

This conclusion turns out to be generally true; i.e., it is not limited to uniform plane waves. Although we will not provide a rigorous proof of this, one way to see that this is true is to imagine that any type of wave can be interpreted as the sum (formally, a *linear combination*) of uniform plane waves, so perpendicular orientation of the field vectors with respect to the direction of propagation is inescapable.

The same procedure yields the uniform plane wave solution to the wave equation for $\widetilde{\mathbf{H}}$, which is

$$\widetilde{\mathbf{H}} = \hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{y}}\widetilde{H}_y \tag{9.62}$$

where

$$\widetilde{H}_x = H_{x0}^{+} e^{-j\beta z} + H_{x0}^{-} e^{+j\beta z} \tag{9.63}$$

$$\widetilde{H}_y = H_{y0}^{+} e^{-j\beta z} + H_{y0}^{-} e^{+j\beta z} \tag{9.64}$$

and where H_{x0}^{+}, H_{x0}^{-}, H_{y0}^{+} and H_{y0}^{-} are arbitrary constants. Note that the solution is essentially the same as that for $\widetilde{\mathbf{E}}$, with the sole difference being that the arbitrary constants may apparently have different values.

To this point, we have seen no particular relationship between the electric and magnetic fields, and it may appear that the electric and magnetic fields are independent of each other. However, Maxwell's Equations – specifically, the two curl equations – make it clear that there must be a relationship between these fields. Subsequently the arbitrary constants in the solutions for $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ must also be related. In fact, there are two considerations here:

- The *magnitude and phase* of $\widetilde{\mathbf{E}}$ must be related to the magnitude and phase of $\widetilde{\mathbf{H}}$. Since both fields are solutions to the same differential (wave) equation, they may differ by no more than a multiplicative constant. Since the units of $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are V/m and A/m respectively, this constant must be expressible in units of V/m divided by A/m; i.e., in units of Ω, an impedance.
- The *direction* of $\widetilde{\mathbf{E}}$ must be related to direction of $\widetilde{\mathbf{H}}$.

Let us now address these considerations. Consider an electric field that points in one of the cardinal directions – let's say $+\hat{\mathbf{x}}$ – and make the definition $E_0 \triangleq E_{x0}^{+}$ for notational convenience. Then the electric field intensity may be written as follows:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}} E_0 e^{-j\beta z} \tag{9.65}$$

Again, there is no loss of generality here since the coordinate system could be rotated in such a way that *any* uniform plane could be described in this way.

We may now determine $\widetilde{\mathbf{H}}$ from $\widetilde{\mathbf{E}}$ using the Maxwell-Faraday Equation (Section 9.2):

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}} \tag{9.66}$$

Solving this equation for $\widetilde{\mathbf{H}}$, we find:

$$\widetilde{\mathbf{H}} = \frac{\nabla \times \widetilde{\mathbf{E}}}{-j\omega\mu} = \frac{\nabla \times \left[\hat{\mathbf{x}} E_0 e^{-j\beta z}\right]}{-j\omega\mu} \tag{9.67}$$

Now let us apply the curl operator. The complete expression for the curl operator in Cartesian coordinates is given in Section B.2. Here let us consider one component at a time, starting with the $\hat{\mathbf{x}}$ component:

$$\hat{\mathbf{x}} \cdot \left(\nabla \times \widetilde{\mathbf{E}}\right) = \frac{\partial \widetilde{E}_z}{\partial y} - \frac{\partial \widetilde{E}_y}{\partial z} \tag{9.68}$$

Since $\widetilde{E}_y = \widetilde{E}_z = 0$, the above expression is zero and subsequently $\widetilde{H}_x = 0$. Next, the $\hat{\mathbf{y}}$ component:

$$\hat{\mathbf{y}} \cdot \left(\nabla \times \widetilde{\mathbf{E}}\right) = \frac{\partial \widetilde{E}_x}{\partial z} - \frac{\partial \widetilde{E}_z}{\partial x} \tag{9.69}$$

Here $\widetilde{E}_z = 0$, so we have simply

$$\hat{\mathbf{y}} \cdot \left(\nabla \times \widetilde{\mathbf{E}}\right) = \frac{\partial \widetilde{E}_x}{\partial z} \tag{9.70}$$

It is not necessary to repeat this procedure for $\widetilde{H}_z$, since we know in advance that $\widetilde{\mathbf{H}}$ must be perpendicular to the direction of propagation and subsequently $\widetilde{H}_z = 0$. Returning to Equation 9.67, we obtain:

$$\begin{aligned} \widetilde{\mathbf{H}} &= \hat{\mathbf{y}} \frac{1}{-j\omega\mu} \frac{\partial}{\partial z} \widetilde{E}_x \\ &= \hat{\mathbf{y}} \frac{1}{-j\omega\mu} \frac{\partial}{\partial z} \left(E_0 e^{-j\beta z} \right) \\ &= \hat{\mathbf{y}} \frac{1}{-j\omega\mu} \left(-j\beta E_0 e^{-j\beta z} \right) \\ &= \hat{\mathbf{y}} \frac{\beta}{\omega\mu} E_0 e^{-j\beta z} \end{aligned} \tag{9.71}$$

Note that $\widetilde{\mathbf{H}}$ points in the $+\hat{\mathbf{y}}$ direction. So, as expected, both $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are perpendicular to the direction of propagation. However, we have now found a more specific relationship: $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are perpendicular *to each other*. Just as $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, we see that $\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}$ points in the direction of propagation. This is illustrated in Figure 9.6. Summarizing:

$\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{H}}$, and the direction of propagation $\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}$ are mutually perpendicular.

Figure 9.6. *Relationship between the electric field direction, magnetic field direction, and direction of propagation.*

Now let us resolve the question of the factor relating $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$. The factor is now seen to be $\beta/\omega\mu$ in Equation 9.71, which can be simplified as follows:

$$\frac{\beta}{\omega\mu} = \frac{\omega\sqrt{\mu\epsilon}}{\omega\mu} = \frac{1}{\sqrt{\mu/\epsilon}} \tag{9.72}$$

The factor $\sqrt{\mu/\epsilon}$ appearing above has units of Ω is known variously as the *wave impedance* or the *intrinsic impedance of the medium*. Assigning this quantity the symbol "η," we have:

$$\boxed{\eta \triangleq \frac{\widetilde{E}_x}{\widetilde{H}_y} = \sqrt{\frac{\mu}{\epsilon}}} \tag{9.73}$$

The ratio of the electric field intensity to the magnetic field intensity is the wave impedance η (Equation 9.73; units of Ω). In lossless media, η is determined by the ratio of permeability of the medium to the permittivity of the medium.

The wave impedance in free space, assigned the symbol η_0, is

$$\eta_0 \triangleq \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 377\ \Omega. \tag{9.74}$$

Wrapping up our solution, we find that if $\widetilde{\mathbf{E}}$ is as given by Equation 9.65, then

$$\widetilde{\mathbf{H}} = \hat{\mathbf{y}}\frac{E_0}{\eta}e^{-j\beta z} \tag{9.75}$$

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1. By the way, this is a highly-recommended exercise for the student. ↩
2. Showing this is a highly-recommended exercise for the reader. ↩
3. The reader is encouraged to confirm that these are solutions by substitution into the associated wave equation. ↩

9.5 Uniform Plane Waves: Characteristics

In Section 9.4, expressions for the electric and magnetic fields are determined for a uniform plane wave in lossless media. If the planar phasefront is perpendicular to the z axis, then waves may propagate in either the $+\hat{\mathbf{z}}$ direction of the $-\hat{\mathbf{z}}$ direction. If we consider only the former, and select $\widetilde{\mathbf{E}}$ to point in the $+\hat{\mathbf{x}}$ direction, then we find

$$\widetilde{\mathbf{E}} = +\hat{\mathbf{x}}E_0e^{-j\beta z} \tag{9.76}$$

$$\widetilde{\mathbf{H}} = +\hat{\mathbf{y}}\frac{E_0}{\eta}e^{-j\beta z} \tag{9.77}$$

where $\beta = \omega\sqrt{\mu\epsilon}$ is the phase propagation constant, $\eta = \sqrt{\mu/\epsilon}$ is the wave impedance, and E_0 is a complex-valued constant associated with the magnitude and phase of the source. This result is in fact completely general for uniform plane waves, since any other possibility may be obtained by simply rotating coordinates. In fact, this is pretty easy because (as determined in Section 9.4) $\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{H}}$, and the direction of propagation are mutually perpendicular, with the direction of propagation pointing in the same direction as $\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}$.

In this section, we identify some important characteristics of uniform plane waves, including wavelength and phase velocity. Chances are that much of what appears here will be familiar to the reader; if not, a quick review of Sections 1.3 ("Fundamentals of Waves") and 3.8 ("Wave Propagation on a Transmission Line") are recommended.

First, recall that $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are phasors representing physical (real-valued) fields and are *not* the field values themselves. The actual, physical electric field intensity is

$$\begin{aligned}\mathbf{E} &= \mathrm{Re}\left\{\widetilde{\mathbf{E}}e^{j\omega t}\right\} \\ &= \mathrm{Re}\left\{\hat{\mathbf{x}}E_0 e^{-j\beta z}e^{j\omega t}\right\} \\ &= \hat{\mathbf{x}}\left|E_0\right|\cos\left(\omega t - \beta z + \psi\right)\end{aligned} \tag{9.78}$$

where ψ is the phase of E_0. Similarly:

$$\mathbf{H} = \hat{\mathbf{y}}\frac{|E_0|}{\eta}\cos\left(\omega t - \beta z + \psi\right) \tag{9.79}$$

This result is illustrated in Figure 9.7. Note that both $\mathbf{E}$ and $\mathbf{H}$ (as well as their phasor representations) have the same phase and have the same frequency and position dependence $\cos\left(\omega t - \beta z + \psi\right)$. Since β is a real-valued constant for lossless media, *only* the phase, and not the magnitude, varies with z. If $\omega \rightarrow 0$, then $\beta \rightarrow 0$ and the fields no longer depend on z; in this case, the field is not propagating. For $\omega > 0$, $\cos\left(\omega t - \beta z + \psi\right)$ is periodic in z; specifically, it has the same value each time z increases or decreases by $2\pi/\beta$. This is, by definition, the wavelength λ:

$$\lambda \triangleq \frac{2\pi}{\beta} \tag{9.80}$$

Figure 9.7. *Relationship between the electric field direction, magnetic field direction, and direction of propagation (here, $+\hat{\mathbf{z}}$).*

If we observe this wave at some fixed point (i.e., hold z constant), we find that the electric and magnetic fields are also periodic in time; specifically, they have the same value each time t increases by $2\pi/\omega$. We may characterize the speed at which the wave travels by comparing the distance required to experience 2π of phase rotation at a fixed time, which is $1/\beta$; to the time it takes to experience 2π of phase rotation at a fixed location, which is $1/\omega$. This is known as the *phase velocity*[1] v_p:

$$v_p \triangleq \frac{1/\beta}{1/\omega} = \frac{\omega}{\beta} \tag{9.81}$$

Note that v_p has the units expected from its definition, namely (rad/s)/(rad/m) $=$ m/s. If we make the substitution $\beta = \omega\sqrt{\mu\epsilon}$, we find

$$v_p = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} \tag{9.82}$$

Note that v_p, like the wave impedance η, depends only on material properties. For example, the phase velocity of an electromagnetic wave in free space, given the special symbol c, is

$$c \triangleq v_p\big|_{\mu=\mu_0,\epsilon=\epsilon_0} = \frac{1}{\sqrt{\mu_0\epsilon_0}} \cong 3.00 \times 10^8 \text{ m/s} \tag{9.83}$$

This constant is commonly referred to as the *speed of light*, but in fact it is the phase velocity of an electromagnetic field at *any* frequency (not just optical frequencies) in free space. Since the permittivity ϵ and permeability μ of any material is greater than that of a vacuum, v_p in any material is less than the phase velocity in free space. Summarizing:

Phase velocity is the speed at which any point of constant phase appears to travel along the direction of propagation.

Phase velocity is maximum ($= c$) in free space, and slower by a factor of $1/\sqrt{\mu_r\epsilon_r}$ in any other lossless medium.

Finally, we note the following relationship between frequency f, wavelength, and phase velocity:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega/v_p} = \frac{2\pi}{2\pi f/v_p} = \frac{v_p}{f} \tag{9.84}$$

Thus, given any two of the parameters f, λ, and v_p, we may solve for the remaining quantity. Also note that as a consequence of the inverse-proportional relationship between λ and v_p, we find:

At a given frequency, the wavelength in any material is shorter than the wavelength in free space.

Example

Example 9.1

Wave propagation in a lossless dielectric.

Polyethylene is a low-loss dielectric having $\epsilon_r \cong 2.3$. What is the phase velocity in polyethylene? What is wavelength in polyethylene? The frequency of interest is 1 GHz.

Solution. Low-loss dielectrics exhibit $\mu_r \cong 1$ and $\sigma \approx 0$. Therefore the phase velocity is

$$v_p = \frac{1}{\sqrt{\mu_0 \epsilon_r \epsilon_0}} = \frac{c}{\sqrt{\epsilon_r}} \cong 1.98 \times 10^8 \text{ m/s} \tag{9.85}$$

i.e., very nearly two-thirds of the speed of light in free space. The wavelength at 1 GHz is

$$\lambda = \frac{v_p}{f} \cong 19.8 \text{ cm} \tag{9.86}$$

Again, about two-thirds of a wavelength at the same frequency in free space.

Returning to polarization and magnitude, it is useful to note that Equation 9.79 could be written in terms of **E** as follows:

$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{z}} \times \mathbf{E} \tag{9.87}$$

i.e., $\mathbf{H}$ is perpendicular to both the direction of propagation (here, $+\hat{\mathbf{z}}$) and $\mathbf{E}$, with magnitude that is different from $\mathbf{E}$ by the wave impedance η. This simple expression is very useful for quickly determining $\mathbf{H}$ given $\mathbf{E}$ when the direction of propagation is known. In fact, it is so useful that it commonly explicitly generalized as follows:

$$\boxed{\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}} \tag{9.88}$$

where $\hat{\mathbf{k}}$ (here, $+\hat{\mathbf{z}}$) is the direction of propagation. Similarly, given $\mathbf{H}$, $\hat{\mathbf{k}}$, and η, we may find $\mathbf{E}$ using

$$\boxed{\mathbf{E} = -\eta \hat{\mathbf{k}} \times \mathbf{H}} \tag{9.89}$$

These spatial relationships can be readily verified using Figure 9.7.

Equations 9.88 and 9.89 are known as the *plane wave relationships*.

The plane wave relationships apply equally well to the phasor representations of $\mathbf{E}$ and $\mathbf{H}$; i.e.,

$$\widetilde{\mathbf{E}} = -\eta\hat{\mathbf{k}} \times \widetilde{\mathbf{H}} \tag{9.90}$$

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\hat{\mathbf{k}} \times \widetilde{\mathbf{E}} \tag{9.91}$$

These equations can be readily verified by applying the definition of the associated phasors (e.g., Equation 9.78). It also turns out that these relationships apply at each point in space, even if the waves are not planar, uniform, or in lossless media.

Example

Example 9.2

Analysis of a radially-directed plane wave.

Consider the scenario illustrated in Figure 9.8. Here a uniform plane wave having frequency $f = 3$ GHz is propagating along a path of constant ϕ, where is ϕ is known but not specified. The phase of the electric field is $\pi/2$ radians at $\rho = 0$ (the origin) and $t = 0$. The material is an effectively unbounded region of free space. The electric field is oriented in the $+\hat{\mathbf{z}}$ direction and has peak magnitude of 1 mV/m. Find (a) the electric field intensity in phasor representation, (b) the magnetic field intensity in phasor representation, and (c) the actual, physical electric field along the radial path.

Solution: First, realize that this "radially-directed" plane wave is in fact a plane wave, and not a cylindrical wave. It may well we be that if we "zoom out" far enough, we are able to perceive a cylindrical wave (for more on this idea, see Section 9.3), or it might simply be that wave is *exactly* planar, and cylindrical coordinates just happen to be a convenient coordinate system for this application. In either case, the direction of propagation $\hat{\mathbf{k}} = +\hat{\rho}$ and the solution to this example will be the same.

Here's the phasor representation for the electric field intensity of a uniform plane wave in a lossless medium, analogous to Equation 9.76:

$$\widetilde{\mathbf{E}} = +\hat{\mathbf{z}}E_0 e^{-j\beta\rho} \tag{9.92}$$

From the problem statement, $|E_0| = 1$ mV/m. Also from the problem statement, the phase of E_0 is $\pi/2$ radians; in fact, we could just write $E_0 = +j\,|E_0|$. Thus, the answer to (a) is

$$\widetilde{\mathbf{E}} = +\hat{\mathbf{z}}j\,|E_0|\,e^{-j\beta\rho} \tag{9.93}$$

where the propagation constant $\beta = \omega\sqrt{\mu\epsilon} = 2\pi f\sqrt{\mu_0\epsilon_0} \cong 62.9$ rad/m.

The answer to (b) is easiest to obtain from the plane wave relationship:

$$\begin{aligned}\widetilde{\mathbf{H}} &= \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E} \\ &= \frac{1}{\eta}\hat{\rho} \times \left(+\hat{\mathbf{z}}j\,|E_0|\,e^{-j\beta\rho}\right) \\ &= -\hat{\phi}\frac{j\,|E_0|}{\eta}e^{-j\beta\rho}\end{aligned} \tag{9.94}$$

where η here is $\sqrt{\mu_0/\epsilon_0} \cong 377\ \Omega$. Thus, the answer to part (b) is

$$\widetilde{\mathbf{H}} = -\hat{\phi} j H_0 e^{-j\beta\rho} \tag{9.95}$$

where $H_0 \cong 2.65\ \mu$A/m. At this point you should check vector directions: $\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}$ should point in the direction of propagation $+\hat{\rho}$. Here we find $\hat{\mathbf{z}} \times \left(-\hat{\phi}\right) = +\hat{\rho}$, as expected.

The answer to (c) is obtained by applying the defining relationship for phasors to the answer to part (a):

$$\begin{aligned} \mathbf{E} &= \text{Re}\left\{\widetilde{\mathbf{E}} e^{j\omega t}\right\} \\ &= \text{Re}\left\{\left(+\hat{\mathbf{z}} j \left|E_0\right| e^{-j\beta\rho}\right) e^{j\omega t}\right\} \\ &= +\hat{\mathbf{z}} \left|E_0\right| \text{Re}\left\{e^{j\pi/2} e^{-j\beta\rho} e^{j\omega t}\right\} \\ &= +\hat{\mathbf{z}} \left|E_0\right| \cos\left(\omega t - \beta\rho + \frac{\pi}{2}\right) \end{aligned} \tag{9.96}$$

Figure 9.8. *A radially-directed plane wave.*

Additional Reading:

- "Electromagnetic radiation (https://en.wikipedia.org/wiki/Electromagnetic_radiation)" on Wikipedia.

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1. We acknowledge that this is a misnomer, since velocity is properly defined as speed in a *specified* direction, and v_p by itself does not specify direction. In fact, the phase velocity in this case is properly said to be $+\hat{\mathbf{z}} v_p$. Nevertheless, we adopt the prevailing terminology. ↩

9.6 Wave Polarization

Polarization refers to the orientation of the electric field vector. For waves, the term "polarization" refers specifically to the orientation of this vector with increasing distance along the direction of propagation, or, equivalently, the orientation of this vector with increasing time at a fixed point in space. The relevant concepts are readily demonstrated for uniform plane waves, as shown in this section. A review of Section 9.5 ("Uniform Plane Waves: Characteristics") is recommended before reading further.

To begin, consider the following uniform plane wave, described in terms of the phasor representation of its electric field intensity:

$$\widetilde{\mathbf{E}}_x = \hat{\mathbf{x}} E_x e^{-j\beta z} \tag{9.97}$$

Here E_x is a complex-valued constant representing the magnitude and phase of the wave, and β is the positive real-valued propagation constant. Therefore, this wave is propagating in the $+\hat{\mathbf{z}}$ direction in lossless media. This wave is said to exhibit *linear* polarization (and “linearly polarized”) because the electric field always points in the same direction, namely $+\hat{\mathbf{x}}$. Now consider the wave

$$\widetilde{\mathbf{E}}_y = \hat{\mathbf{y}} E_y e^{-j\beta z} \tag{9.98}$$

Note that $\widetilde{\mathbf{E}}_y$ is identical to $\widetilde{\mathbf{E}}_x$ except that the electric field vector now points in the $+\hat{\mathbf{y}}$ direction and has magnitude and phase that is different by the factor E_y/E_x. This wave too is said to exhibit linear polarization, because, again, the direction of the electric field is constant with both time and position. In fact, *all* linearly-polarized uniform plane waves propagating in the $+\hat{\mathbf{z}}$ direction in lossless media can be described as follows:

$$\widetilde{\mathbf{E}} = \hat{\rho} E_\rho e^{-j\beta z} \tag{9.99}$$

This is so because $\hat{\rho}$ could be $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, or any other direction that is perpendicular to $\hat{\mathbf{z}}$. If one is determined to use Cartesian coordinates, the above expression may be rewritten using (Section 4.3)

$$\hat{\rho} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \cos\phi \tag{9.100}$$

yielding

$$\widetilde{\mathbf{E}} = \left(\hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \cos\phi\right) E_\rho e^{-j\beta z} \tag{9.101}$$

When written in this form, $\phi = 0$ corresponds to $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_x$, $\phi = \pi/2$ corresponds to $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_y$, and any other value of ϕ corresponds to some other constant orientation of the electric field vector; see Figure 9.9 for an example.

Figure 9.9. *Linear polarization. Here $\mathbf{E}_x$ is shown in blue, $\mathbf{E}_y$ is shown in green, and $\mathbf{E}$ is shown in red with black vector symbols. In this example the phases of E_x and E_y are zero and $\phi = -\pi/4$.*

by RJB1 (Modified)

A wave is said to exhibit linear polarization if the direction of the electric field vector does not vary with either time or position.

Linear polarization arises when the source of the wave is linearly polarized. A common example is the wave radiated by a straight wire antenna, such as a dipole or a monopole. Linear polarization may also be created by passing a plane wave through a *polarizer*; this is particularly common at optical frequencies (see “Additional Reading” at the end of this section).

A commonly-encountered alternative to linear polarization is *circular polarization*. For an explanation, let us return to the linearly-polarized plane waves $\widetilde{\mathbf{E}}_x$ and $\widetilde{\mathbf{E}}_y$ defined earlier. If both of these waves exist simultaneously, then the total electric field intensity is simply the sum:

$$\begin{aligned}\widetilde{\mathbf{E}} &= \widetilde{\mathbf{E}}_x + \widetilde{\mathbf{E}}_y \\ &= \left(\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y\right) e^{-j\beta z}\end{aligned} \tag{9.102}$$

If the phase of E_x and E_y is the same, then $E_x = E_\rho \cos\phi$, $E_y = E_\rho \sin\phi$, and the above expression is essentially the same as Equation 9.101. In this case, $\widetilde{\mathbf{E}}$ is linearly polarized. But what if the phases of E_x and E_y are different? In particular, let’s consider the following case. Let $E_x = E_0$, some complex-valued constant, and let $E_y = +jE_0$, which is E_0 phase-shifted by $+\pi/2$ radians. With no further math, it is apparent that $\widetilde{\mathbf{E}}_x$ and $\widetilde{\mathbf{E}}_y$ are different only in that one is phase-shifted by $\pi/2$ radians relative to the other. For the physical (real-valued) fields, this means that $\mathbf{E}_x$ has maximum magnitude when $\mathbf{E}_y$ is zero and vice versa. As a result, the direction of $\mathbf{E} = \mathbf{E}_x + \mathbf{E}_y$ will rotate in the $x-y$ plane, as shown in Figure 9.10

Figure 9.10. *A circularly-polarized wave (in red, with black vector symbols) resulting from the addition of orthogonal linearly-polarized waves (shown in green and blue) that are phase-shifted by $\pi/2$ radians.*

by Dave3457

The rotation of the electric field vector can also be identified mathematically. When $E_x = E_0$ and $E_y = +jE_0$, Equation 9.102 can be written:

$$\widetilde{\mathbf{E}} = \left(\hat{\mathbf{x}} + j\hat{\mathbf{y}}\right) E_0 e^{-j\beta z} \tag{9.103}$$

Now reverting from phasor notation to the physical field:

$$\begin{aligned}\mathbf{E} &= \mathrm{Re}\left\{\widetilde{\mathbf{E}}e^{j\omega t}\right\} \\ &= \mathrm{Re}\left\{\left(\hat{\mathbf{x}} + j\hat{\mathbf{y}}\right) E_0 e^{-j\beta z} e^{j\omega t}\right\} \\ &= \hat{\mathbf{x}}\left|E_0\right| \cos\left(\omega t - \beta z\right) + \hat{\mathbf{y}}\left|E_0\right| \cos\left(\omega t - \beta z + \frac{\pi}{2}\right)\end{aligned} \tag{9.104}$$

As anticipated, we see that both $\mathbf{E}_x$ and $\mathbf{E}_y$ vary sinusoidally, but are $\pi/2$ radians out of phase resulting in rotation in the plane perpendicular to the direction of propagation.

In the example above, the electric field vector rotates either clockwise or counter-clockwise relative to the direction of propagation. The direction of this rotation can be identified by pointing the thumb of the left hand in the direction of propagation; in this case, the fingers of the left hand curl in the direction of rotation. For this reason, this particular form of circular polarization is known as *left circular* (or ”left-hand” circular) polarization (LCP). If we instead had chosen $E_y = -jE_0 = -jE_x$, then the direction of $\mathbf{E}$ rotates in the opposite direction, giving rise to *right circular* (or “right-hand” circular) polarization (RCP). These two conditions are illustrated in Figure 9.11.

Figure 9.11. *Left-circular polarization (LCP; top) and right-circular polarization (RCP; bottom).*

by Dave3457

A wave is said to exhibit circular polarization if the electric field vector rotates with constant magnitude. Left- and right-circular polarizations may be identified by the direction of rotation with respect to the direction of propagation.

In engineering applications, circular polarization is useful when the relative orientations of transmit and receive equipment is variable and/or when the medium is able rotate the electric field vector. For example, radio communications involving satellites in non-geosynchronous orbits typically employ circular polarization. In particular, satellites of the U.S. Global Positioning System (GPS) transmit circular polarization because of the variable geometry of the space-to-earth radio link and the tendency of the Earth’s ionosphere to rotate the electric field vector through a mechanism known *Faraday rotation* (sometimes called the “Faraday effect”). If GPS were instead to transmit using a linear polarization, then a receiver would need to continuously adjust the orientation of its antenna in order to optimally receive the signal. Circularly-polarized radio waves can be generated (or received) using pairs of perpendicularly-oriented dipoles that are fed the same signal but with a 90° phase shift, or alternatively by using an antenna that is intrinsically circularly-polarized, such as a helical antenna (see “Additional Reading” at the end of this section).

Linear and circular polarization are certainly not the only possibilities. *Elliptical polarization* results when E_x and E_y do not have equal magnitude. Elliptical polarization is typically not an intended condition, but rather is most commonly observed as a degradation in a system that is nominally linearly- or circularly-polarized. For example, most antennas that are said to be “circularly polarized” instead produce circular polarization only in one direction and various degrees of elliptical polarization in all other directions.

Additional Reading:

- "Polarization (waves) (https://en.wikipedia.org/wiki/Polarization_(waves))" on Wikipedia.
- "Dipole antenna (https://en.wikipedia.org/wiki/Dipole_antenna)" on Wikipedia.
- "Polarizer (https://en.wikipedia.org/wiki/Polarizer)" on Wikipedia.
- "Faraday effect (https://en.wikipedia.org/wiki/Faraday_effect)" on Wikipedia.
- "Helical antenna (https://en.wikipedia.org/wiki/Helical_antenna)" on Wikipedia.

9.7 Wave Power in a Lossless Medium

In many applications involving electromagnetic waves, one is less concerned with the instantaneous values of the electric and magnetic fields than the *power* associated with the wave. In this section, we address the issue of how much power is conveyed by an electromagnetic wave in a lossless medium. The relevant concepts are readily demonstrated in the context of uniform plane waves, as shown in this section. A review of Section 9.5 ("Uniform Plane Waves: Characteristics") is recommended before reading further.

Consider the following uniform plane wave, described in terms of the phasor representation of its electric field intensity:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}} E_0 e^{-j\beta z} \tag{9.105}$$

Here E_0 is a complex-valued constant associated with the source of the wave, and β is the positive real-valued propagation constant. Therefore, the wave is propagating in the $+\hat{\mathbf{z}}$ direction in lossless media.

The first thing that should be apparent is that the amount of power conveyed by this wave is infinite. The reason is as follows. If the power passing through any finite area is greater than zero, then the total power must be infinite because, for a uniform plane wave, the electric and magnetic field intensities are constant over a plane of infinite area. In practice, we never encounter this situation because all practical plane waves are only "locally planar" (see Section 9.3 for a refresher on this idea). Nevertheless, we seek some way to express the power associated with such waves.

The solution is not to seek total power, but rather power per unit area. This quantity is known as the *spatial power density*, or simply "*power density*," and has units of W/m^2.[1] Then, if we are interested in total power passing through some finite area, then we may simply integrate the power density over this area. Let's skip to the answer, and then consider where this answer comes from. It turns out that the *instantaneous* power density of a uniform plane wave is the magnitude of the *Poynting vector*

$$\mathbf{S} \triangleq \mathbf{E} \times \mathbf{H} \tag{9.106}$$

Note that this equation is dimensionally correct; i.e. the units of $\mathbf{E}$ (V/m) times the units of $\mathbf{H}$ (A/m) yield the units of spatial power density (V·A/m^2, which is W/m^2). Also, the direction of $\mathbf{E}\times\mathbf{H}$ is in the direction of propagation (reminder: Section 9.5), which is the direction in which the power is expected to flow. Thus, we have some compelling evidence that $|\mathbf{S}|$ is the power density we seek. However, this is not proof – for that, we require the *Poynting Theorem*, which is a bit outside the scope of the present section, but is addressed in the "Additional Reading" at the end of this section.

A bit later we're going to need to know $\mathbf{S}$ for a uniform plane wave, so let's work that out now. From the plane wave relationships (Section 9.5) we find that the magnetic field intensity associated with the electric field in Equation 9.105 is

$$\widetilde{\mathbf{H}} = \hat{\mathbf{y}}\frac{E_0}{\eta}e^{-j\beta z} \tag{9.107}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the real-valued impedance of the medium. Let ψ be the phase of E_0; i.e., $E_0 = |E_0|e^{j\psi}$. Then

$$\begin{aligned}\mathbf{E} &= \mathrm{Re}\left\{\widetilde{\mathbf{E}}e^{j\omega t}\right\}\\ &= \hat{\mathbf{x}}\,|E_0|\cos\left(\omega t - \beta z + \psi\right)\end{aligned} \tag{9.108}$$

and

$$\begin{aligned}\mathbf{H} &= \mathrm{Re}\left\{\widetilde{\mathbf{H}}e^{j\omega t}\right\}\\ &= \hat{\mathbf{y}}\frac{|E_0|}{\eta}\cos\left(\omega t - \beta z + \psi\right)\end{aligned} \tag{9.109}$$

Now applying Equation 9.106,

$$\mathbf{S} = \hat{\mathbf{z}}\frac{|E_0|^2}{\eta}\cos^2\left(\omega t - \beta z + \psi\right) \tag{9.110}$$

As noted earlier, $|\mathbf{S}|$ is only the instantaneous power density, which is still not quite what we are looking for. What we are actually looking for is the *time-average* power density S_{ave} – that is, the average value of $|\mathbf{S}|$ over one period T of the wave. This may be calculated as follows:

$$\begin{aligned}S_{ave} &= \frac{1}{T}\int_{t=t_0}^{t_0+T}|\mathbf{S}|dt\\ &= \frac{|E_0|^2}{\eta}\,\frac{1}{T}\int_{t=t_0}^{t_0+T}\cos^2\left(\omega t - ks + \psi\right)dt\end{aligned} \tag{9.111}$$

Since $\omega = 2\pi f = 2\pi/T$, the definite integral equals $T/2$. We obtain

$$\boxed{S_{ave} = \frac{|E_0|^2}{2\eta}} \tag{9.112}$$

It is useful to check units again at this point. Note (V/m)^2 divided by Ω is W/m^2, as expected.

Equation 9.112 is the time-average power density (units of W/m^2) associated with a sinusoidally-varying uniform plane wave in lossless media.

Note that Equation 9.112 is analogous to a well-known result from electric circuit theory. Recall the time-average power P_{ave} (units of W) associated with a voltage phasor $\widetilde{V}$ across a resistance R is

$$P_{ave} = \frac{|\widetilde{V}|^2}{2R} \tag{9.113}$$

which closely resembles Equation 9.112. The result is also analogous to the result for a voltage wave on a transmission line (Section 3.20), for which:

$$P_{ave} = \frac{|V_0^+|^2}{2Z_0} \tag{9.114}$$

where V_0^+ is a complex-valued constant representing the magnitude and phase of the voltage wave, and Z_0 is the characteristic impedance of the transmission line.

Here is a good point at which to identify a common pitfall. $|E_0|$ and $|\widetilde{V}|$ are the *peak magnitudes* of the associated real-valued physical quantities. However, these quantities are also routinely given as *root mean square* ("rms") quantities. Peak magnitudes are greater by a factor of $\sqrt{2}$, so Equation 9.112 expressed in terms of the rms quantity lacks the factor of $1/2$.

Example

Example 9.3

Power density of a typical radio wave.

A radio wave transmitted from a distant location may be perceived locally as a uniform plane wave if there is no nearby structure to scatter the wave; a good example of this is the wave arriving at the user of a cellular telephone in a rural area with no significant terrain scattering. The range of possible signal strengths varies widely, but a typical value of the electric field intensity arriving at the user's location is 10 μV/m rms. What is the corresponding power density?

Solution: From the problem statement, $|E_0| = 10\ \mu$V/m rms. We assume propagation occurs in air, which is indistinguishable from free space at cellular frequencies. If we use Equation 9.112, then we must first convert $|E_0|$ from rms to peak magnitude, which is done by multiplying by $\sqrt{2}$. Thus:

$$\begin{aligned} S_{ave} = \frac{|E_0|^2}{2\eta} &\cong \frac{\left(\sqrt{2}\cdot 10\times 10^{-6}\ \mathrm{V/m}\right)^2}{2\cdot 377\ \Omega} \\ &\cong 2.65\times 10^{-13}\ \mathrm{W/m^2} \end{aligned}$$

Alternatively, we can just use a version of Equation 9.112, which is appropriate for rms units:

$$\begin{aligned} S_{ave} = \frac{|E_{0,rms}|^2}{\eta} &\cong \frac{\left(10\times 10^{-6}\ \mathrm{V/m}\right)^2}{377\ \Omega} \\ &\cong 2.65\times 10^{-13}\ \mathrm{W/m^2} \end{aligned}$$

Either way, we obtain the correct answer, 0.265 pW/m^2 (that's *picowatts* per square meter).

Considering the prevalence of phasor representation, it is useful to have an alternative form of the Poynting vector which yields time-average power by operating directly on field quantities in phasor form. This is $\mathbf{S}_{ave}$, defined as:

$$\mathbf{S}_{ave} \triangleq \frac{1}{2}\mathrm{Re}\left\{\widetilde{\mathbf{E}}\times\widetilde{\mathbf{H}}^*\right\} \tag{9.115}$$

(Note that the magnetic field intensity phasor is conjugated.) The above expression gives the expected result for a uniform plane wave. Using Equations 9.105 and 9.107, we find

$$\mathbf{S}_{ave} = \frac{1}{2}\text{Re}\left\{\left(\hat{\mathbf{x}}E_0 e^{-j\beta z}\right) \times \left(\hat{\mathbf{y}}\frac{E_0}{\eta}e^{-j\beta z}\right)^*\right\} \tag{9.116}$$

which yields

$$\mathbf{S}_{ave} = \hat{\mathbf{z}}\frac{|E_0|^2}{2\eta} \tag{9.117}$$

as expected.

Additional Reading:

- "Poynting vector (https://en.wikipedia.org/wiki/Poynting_vector)" on Wikipedia.
- "Poynting's Theorem (https://en.wikipedia.org/wiki/Poynting's_theorem)" on Wikipedia.

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1. Be careful: The quantities *power spectral density* (W/Hz) and *power flux density* (W/(m^2·Hz)) are also sometimes referred to as "power density." In this section, we will limit the scope to *spatial* power density (W/m^2). ↩

End Matter

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