8 Time-Varying Fields

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8.1 Comparison of Static and Time-Varying Electromagnetics

Students encountering time-varying electromagnetic fields for the first time have usually been exposed to electrostatics and magnetostatics already. These disciplines exhibit many similarities as summarized in Table 8.1. The principles of time-varying electromagnetics presented in this table are all formally introduced in other sections; the sole purpose of this table is to point out the differences. We can summarize the differences as follows:

> Maxwell's Equations in the general (time-varying) case include extra terms that do not appear in the equations describing electrostatics and magnetostatics. These terms involve time derivatives of fields and describe coupling between electric and magnetic fields.

Table 8.1. *Comparison of principles governing static and time-varying electromagnetic fields. Differences in the time-varying case relative to the static case are highlighted in blue.*

	Electrostatics /	**Time-Varying**
	Magnetostatics	**(Dynamic)**
Electric & magnetic	independent	possibly coupled
fields are...		
Maxwell's Eqns.	$\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s} = Q_{encl}$	$\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s} = Q_{encl}$
(integral)	$\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} = 0$	Extension "color" failed to load
	$\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0$	$\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0$
	$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{s} = I_{encl}$	$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl} \backslash\mathrm{color}blue + \int_{\mathcal{S}} \frac{\partial}{\partial t}\mathbf{D} \cdot d\mathbf{s}$
Maxwell's Eqns	$\nabla \cdot \mathbf{D} = \rho_v$	$\nabla \cdot \mathbf{D} = \rho_v$
(differential)	$\nabla \times \mathbf{E} = 0$	$\nabla \times \mathbf{E} = \backslash\mathrm{color}blue - \frac{\partial}{\partial t}\mathbf{B}$
	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
	$\nabla \times \mathbf{H} = \mathbf{J}$	$\nabla \times \mathbf{H} = \mathbf{J} \backslash\mathrm{color}blue + \frac{\partial}{\partial t}\mathbf{D}$

The coupling between electric and magnetic fields in the time-varying case has one profound consequence in particular. It becomes possible for fields to continue to exist even after their sources – i.e., charges and currents – are turned off. What kind of field can continue to exist in the absence of a source? Such a field is commonly called a *wave*. Examples of waves include signals in transmission lines and signals propagating away from an antenna.

Additional Reading:

- "Maxwell's Equations (https://en.wikipedia.org/wiki/Maxwell's_equations)" on Wikipedia.

8.2 Electromagnetic Induction

When an electrically-conducting structure is exposed to a time-varying magnetic field, an electrical potential difference is induced across the structure. This phenomenon is known as *electromagnetic induction*. A convenient introduction to electromagnetic induction is provided by Lenz's Law. This section explains electromagnetic induction in the context of Lenz's Law and provides two examples.

Let us begin with the example depicted in Figure 8.1, involving a cylindrical coil. Attached to the terminals of the coil is a resistor for which we may identify an electric potential difference V and current I. In this particular case, the sign conventions indicated for V and I are arbitrary, but it is important to be consistent once they are established.

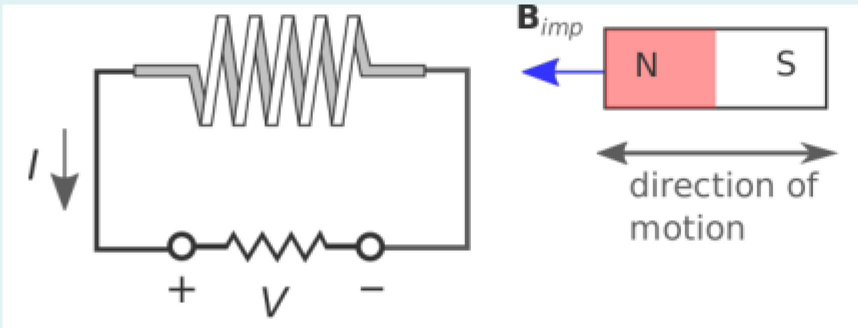

Figure 8.1. *An experiment demonstrating electromagnetic induction and Lenz's Law.*

Now let us introduce a bar magnet as shown in Figure 8.1. The magnet is centered along the axis of coil, to the right of the coil, and with its north pole facing toward the coil. The magnet is responsible for the magnetic flux density $\mathbf{B}_{imp}$. We refer to $\mathbf{B}_{imp}$ as an *impressed* magnetic field because this field exists independently of any response that may be induced by interaction with the coil. Note that $\mathbf{B}_{imp}$ points to the left inside the coil.

The experiment consists of three tests. We will find in two of these tests that current flows (i.e., $|I| > 0$), and subsequently, there is an *induced* magnetic field $\mathbf{B}_{ind}$ due to this current. It is the direction of the current and subsequently the direction of $\mathbf{B}_{ind}$ inside the coil that we wish to observe. The findings are summarized below and in Table 8.2.

- When the magnet is motionless, we have the unsurprising result that there is no current in the coil. Therefore, no magnetic field is induced, and the total magnetic field is simply equal to $\mathbf{B}_{imp}$.
- When the magnet moves *toward* the coil, we observe current that is positive with respect to the reference direction indicated in Figure 8.1. This current creates an induced magnetic field $\mathbf{B}_{ind}$ that points to the *right*, as predicted by magnetostatic considerations from the right-hand rule. Since $\mathbf{B}_{imp}$ points to the left, it appears that the induced current is opposing the increase in the magnitude of the total magnetic field.
- When the magnet moves *away* from the coil, we observe current that is negative with respect to the reference direction indicated in Figure 8.1. This current yields $\mathbf{B}_{ind}$ that points to the *left*. Since $\mathbf{B}_{imp}$ points to the left, it appears that the induced current is opposing the decrease in the magnitude of the total magnetic field.

Table 8.2. *Results of the experiment associated with Figure 8.1.*

Magnet is ...	$\vert\mathbf{B}_{imp}\vert$ in coil is ...	Circuit Response	$\mathbf{B}_{ind}$ inside coil
Motionless	constant	$V = 0$, $I = 0$	none
Moving toward coil	increasing	$V > 0$, $I > 0$	Pointing right
Moving away from coil	decreasing	$V < 0$, $I < 0$	Pointing left

The first conclusion one may draw from this experiment is that changes in the magnetic field can induce current. This was the claim made in the first paragraph of this section and is a consequence of Faraday's Law, which is tackled in detail in Section 8.3. The second conclusion – also associated with Faraday's Law – is the point of this section: The induced magnetic field – that is, the one due to the current induced in the coil – always opposes the change in the impressed magnetic field. Generalizing:

> Lenz's Law states that the current that is induced by a change in an impressed magnetic field creates an induced magnetic field that opposes (acts to reduce the effect of) the change in the total magnetic field.

When the magnet moves, three things happen: (1) A current is induced, (2) A magnetic field is induced (which adds to the impressed magnetic field), and (3) the value of V becomes non-zero. Lenz's Law does not address which of these are responding directly to the change in the impressed magnetic field, and which of these are simply responding to changes in the other quantities. Lenz's Law may leave you with the incorrect impression that it is I that is induced, and that $\mathbf{B}_{ind}$ and V are simply responding to this current. In truth, the quantity that is induced is actually V. This can be verified in the above experiment by replacing the resistor with a high-impedance voltmeter, which will indicate that V is changing even though there is negligible current flow. Current flow is simply a response to the induced potential. Nevertheless, it is common to say informally that "I is induced," even if it is only indirectly through V.

So, if Lenz's Law is simply an observation and not an explanation of the underlying physics, then what is it good for? Lenz's Law is often useful for quickly determining the direction of current flow in practical electromagnetic induction problems, without resorting to the mathematics associated with Faraday's Law. Here's an example:

Example

Example 8.1

Electromagnetic induction through a transformer.

Figure 8.2 shows a rudimentary circuit consisting of a battery and a switch on the left, a voltmeter on the right, and a transformer linking the two. It is not necessary to be familiar with transformers to follow this example; suffice it to say, the transformer considered here consists of two coils wound around a common toroidal core, which serves to contain magnetic flux. In this way, the flux generated by either coil is delivered to the other coil with negligible loss.

The experiment begins with the switch on the left in the open state. Thus, there is no current and no magnetic field apparent in the coil on the left. The voltmeter reads 0 V. When the switch is closed, what happens?

Solution. Closing the switch creates a current in the coil on the left. Given the indicated polarity of the battery, this current flows counter-clockwise through the circuit on the left, with current arriving in the left coil through the bottom terminal. Given the indicated direction of the winding in the left coil, the impressed magnetic field $\mathbf{B}_{imp}$ is oriented counter-clockwise through the toroidal core. The coil on the right "sees" $\mathbf{B}_{imp}$ increase from zero to some larger value. Since the voltmeter presumably has input high impedance, negligible current flows. However, if current were able to flow, Lenz's Law dictates that it would be induced to flow in a counter-clockwise direction around the circuit on the right, since the induced magnetic field $\mathbf{B}_{ind}$ would then be clockwise-directed so as to oppose the increase in $\mathbf{B}_{imp}$. Therefore, the potential measured at the bottom of the right coil would be higher than the potential at the top of the right coil. The figure indicates that the voltmeter measures the potential at its right terminal relative to its left terminal, so the needle will deflect to the right. This deflection will be temporary, since the current provided by the battery becomes constant at a new non-zero value and $\mathbf{B}_{ind}$ responds only to the *change* in $\mathbf{B}_{imp}$. The voltmeter reading will remain at zero for as long as the switch remains closed and the current remains steady.

Here are some follow up exercises to test your understanding of what is going on: (1) Now open the switch. What happens? (2) Repeat the original experiment, but before starting, swap the terminals on the battery.

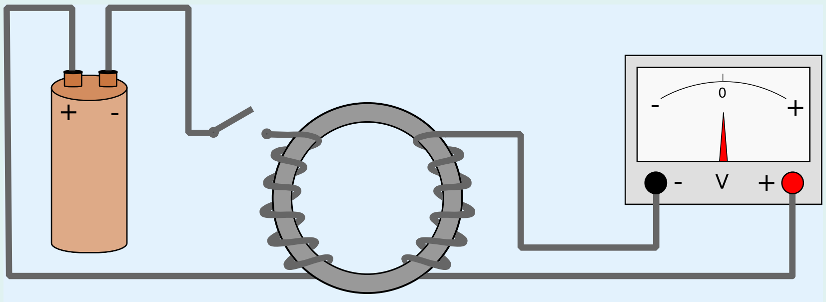

Figure 8.2. *Electromagnetic induction through a transformer.*

E. Bach CC0 1.0 (https://creativecommons.org/publicdomain/zero/1.0/deed.en) (modified)

Finally, it is worth noting that Lenz's law can also be deduced from the principle of conservation of energy. The argument is that if the induced magnetic field *reinforced* the change in the impressed magnetic field, then the sum magnetic field would increase. This would result in a further increase in the induced magnetic field, leading to a positive feedback situation. However, positive feedback cannot be supported without an external source of energy, leading to a logical contradiction. In other words, the principle of conservation of energy requires the *negative* feedback described by Lenz's law.

Additional Reading:

- "Electromagnetic Induction (https://en.wikipedia.org/wiki/Electromagnetic_induction)" on Wikipedia.
- "Lenz's Law (https://en.wikipedia.org/wiki/Lenz's_law)" on Wikipedia.

8.3 Faraday's Law

Faraday's Law describes the generation of electric potential by a time-varying magnetic flux. This is a form of *electromagnetic induction*, as discussed in Section 8.2.

To begin, consider the scenario shown in Figure 8.3. A single loop of wire in the presence of an impressed magnetic field $\mathbf{B}$. For reasons explained later, we introduce a small gap and define V_T to be the potential difference measured across the gap according to the sign convention indicated. The resistance R may be any value greater than zero, including infinity; i.e., a literal gap.

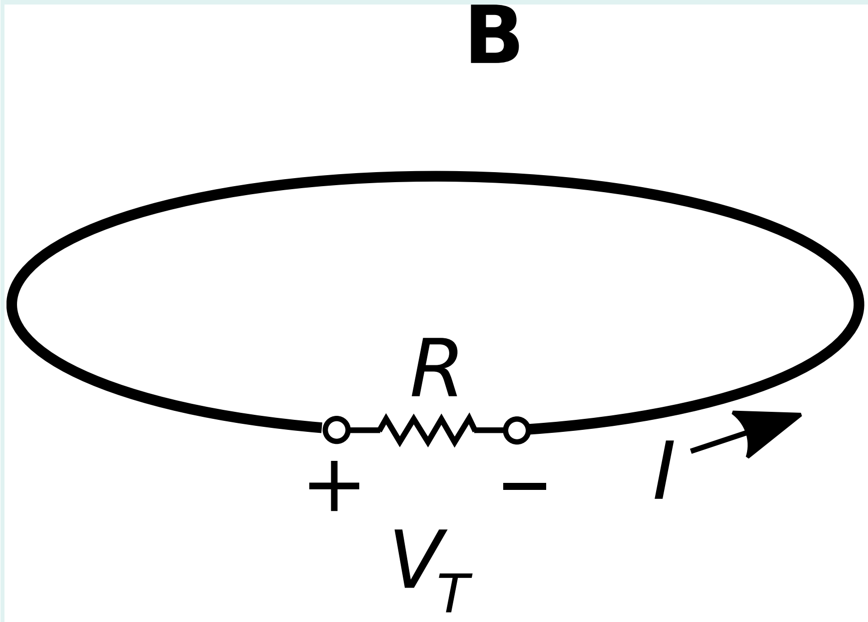

Figure 8.3. *A single loop of wire in the presence of an impressed magnetic field.*

As long as R is not infinite, we know from Lenz's Law (Section 8.2) to expect that a time-varying magnetic field will cause a current to flow in the wire. Lenz's Law also tells us the direction in which the current will flow. However, Lenz's Law does not tell us the magnitude of the current, and it sidesteps some important physics that has profound implications for the analysis and design of electrical devices, including generators and transformers.

The more complete picture is given by Faraday's Law. In terms of the scenario of Figure 8.3, Faraday's Law relates the potential V_T induced by the time variation of $\mathbf{B}$. V_T then gives rise to the current identified in Lenz's Law. The magnitude of this current is simply V_T/R. Without further ado, here's Faraday's Law for this single loop scenario:

$$V_T = -\frac{\partial}{\partial t}\Phi \quad \text{(single loop)} \tag{8.1}$$

Here Φ is the magnetic flux (units of Wb) associated with any open surface $\mathcal{S}$ bounded by the loop:

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} \tag{8.2}$$

where $\mathbf{B}$ is magnetic flux density (units of T or Wb/m^2) and $d\mathbf{s}$ is the differential surface area vector.

To make headway with Faraday's Law, one must be clear about the meanings of $\mathcal{S}$ and $d\mathbf{s}$. If the wire loop in the present scenario lies in a plane, then a good choice for $\mathcal{S}$ is the simply the planar area bounded by the loop. However, *any* surface that is bounded by the loop will work, including non-planar surfaces that extend above and/or below the plane of the loop. All that is required is that every magnetic field line that passes through the loop also passes through $\mathcal{S}$. This happens automatically if the curve $\mathcal{C}$ defining the edge of the open surface $\mathcal{S}$ corresponds to the loop. Subsequently, the magnitude of $d\mathbf{s}$ is the differential surface element ds and the direction of $d\mathbf{s}$ is the unit vector $\hat{\mathbf{n}}$ perpendicular to each point on $\mathcal{S}$, so $d\mathbf{s} = \hat{\mathbf{n}} ds$.

This leaves just one issue remaining – the orientation of $\hat{\mathbf{n}}$. This is sorted out in Figure 8.4. There are two possible ways for a vector to be perpendicular to a surface, and the direction chosen for $\hat{\mathbf{n}}$ will affect the sign of V_T. Therefore, $\hat{\mathbf{n}}$ must be somehow related to the polarity chosen for V_T. Let's consider this relationship. Let $\mathcal{C}$ begin at the "$-$" terminal of V_T and follow the entire perimeter of the loop, ending at the "$+$" terminal. Then, $\hat{\mathbf{n}}$ is determined by the following "right-hand rule:" $\hat{\mathbf{n}}$ points in the direction of the curled fingers of the right hand when the thumb of the right hand is aligned in the direction of $\mathcal{C}$. It's worth noting that this convention is precisely the convention used to relate $\hat{\mathbf{n}}$ and $\mathcal{C}$ in Stokes' Theorem (Section 4.9).

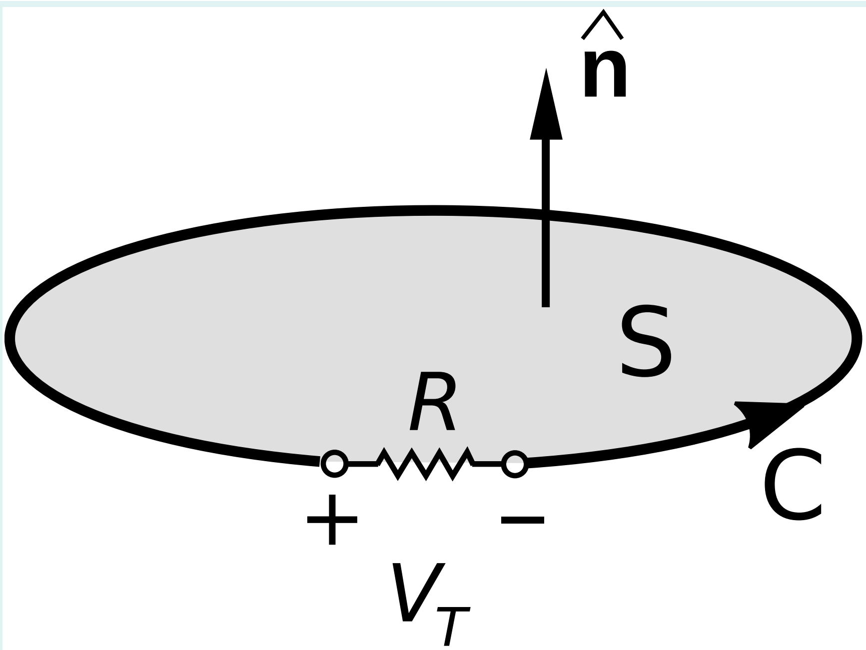

Figure 8.4. *Relationship between the polarity of V_T and orientations of $\mathcal{C}$ and $\hat{\mathbf{n}}$ in the planar single-loop scenario.*

Now let us recap how Faraday's Law is applied to the single loop scenario of Figure 8.3:

1. Assign "+" and "−" terminals to the gap voltage V_T.
2. The orientation of $\hat{\mathbf{n}}$ is determined by the right hand rule, taking the direction of $\mathcal{C}$ to be the perimeter of the loop beginning at "−" and ending at "+"
3. $\mathbf{B}$ yields a magnetic flux Φ associated with the loop according to Equation 8.2. $\mathcal{S}$ is any open surface that intersects all the magnetic field lines that pass through the loop (so you might as well choose $\mathcal{S}$ in a way that results in the simplest possible integration).
4. By Faraday's Law (Equation 8.1), V_T is the time derivative of Φ with a change of sign.
5. The current I flowing in the loop is V_T/R, with the reference direction (i.e., direction of positive current) being from "+" to "−" through the resistor. Think of the loop as a voltage source, and you'll get the correct reference direction for I.

At this point, let us reiterate that the electromagnetic induction described by Faraday's Law induces *potential* (in this case, V_T) and not *current*. The current in the loop is simply the induced voltage divided by the resistance of the loop. This point is easily lost, especially in light of Lenz's Law, which seems to imply that current, as opposed to potential, is the thing being induced.

Wondering about the significance of the minus sign in Equation 8.1? That is specifically Lenz's Law: The current I that ends up circulating in the loop generates its own magnetic field ("$\mathbf{B}_{ind}$" in Section 8.2), which is distinct from the impressed magnetic field $\mathbf{B}$ and which tends to oppose change in $\mathbf{B}$. Thus, we see that Faraday's Law subsumes Lenz's Law.

Frequently one is interested in a structure that consists of multiple identical loops. We have in mind here something like a coil with $N \geq 1$ windings tightly packed together. In this case, Faraday's Law is

$$\boxed{V_T = -N\frac{\partial}{\partial t}\Phi} \tag{8.3}$$

Note that the difference is simply that the gap potential V_T is greater by N. In other words, each winding of the coil contributes a potential given by Equation 8.1, and these potentials add in series.

> Faraday's Law, given in general by Equation 8.3, states that the potential induced in a coil is proportional to the time derivative of the magnetic flux through the coil.

The induced potential V_T is often referred to as "emf," which is a contraction of the term *electromotive force* – a misnomer to be sure, since no actual force is implied, only potential. The term "emf" is nevertheless frequently used in the context of Faraday's Law applications for historical reasons.

Previously in this section, we considered the generation of emf by time variation of $\mathbf{B}$. However, Equation 8.1 indicates that what actually happens is that the emf is the result of time variation of the magnetic flux, Φ. Magnetic flux is magnetic flux density integrated over area, so it appears that emf can also be generated simply by varying $\mathcal{S}$, independently of any time variation of $\mathbf{B}$. In other words, emf may be generated even when $\mathbf{B}$ is constant, by instead varying the shape or orientation of the coil. So, we have a variety of schemes by which we can generate emf. Here they are:

1. Time-varying $\mathbf{B}$ (as we considered previously in this section). For example, $\mathbf{B}$ might be due to a permanent magnet that is moved (e.g., translated or rotated) in the vicinity of the coil, as described in Section 8.2. Or, more commonly, $\mathbf{B}$ might be due to a different coil that bears a time-varying current. These mechanisms are collectively referred to as *transformer emf*. Transformer emf is the underlying principle of operation of transformers; for more on this see Section 8.5.
2. The perimeter $\mathcal{C}$ – and thus the surface $\mathcal{S}$ over which Φ is determined – can be time-varying. For example, a wire loop might be rotated or changed in shape in the presence of a constant magnetic field. This mechanism is referred to as *motional emf* and is the underlying principle of operation of generators (Section 8.7).
3. Transformer and motional emf can exist in various combinations. From the perspective of Faraday's Law, transformer and motional emf are the same in the sense in either case Φ is time-varying, which is all that is required to generate emf.

Finally, a comment on the generality of Faraday's Law. Above we have introduced Faraday's Law as if it were specific to loops and coils of wire. However, the truth of the matter is that Faraday's Law is fundamental physics. If you can define a closed path – current-bearing or not – then you can compute the potential difference achieved by traversing that path using Faraday's Law. The value you compute is the potential associated with electromagnetic induction, and exists independently and in addition to the potential difference associated with the static electric field (e.g., Section 5.12). In other words:

> Faraday's Law indicates the contribution of electromagnetic induction (the generation of emf by a time-varying magnetic flux) to the potential difference achieved by traversing a closed path.

In Section 8.8, this insight is used to transform the static form of Kirchoff's Voltage Law (Section 5.10) – which gives the potential difference associated with electric field only – into the *Maxwell-Faraday Equation* (Section 8.8), which is a general statement about the relationship between the instantaneous value of the electric field and the time derivative of the magnetic field.

Additional Reading:

- "Faraday's law of induction (https://en.wikipedia.org/wiki/Faraday's_law_of_induction)" on Wikipedia.

8.4 Induction in a Motionless Loop

In this section, we consider the problem depicted in Figure 8.5, which is a single motionless loop of wire in the presence of a spatially-uniform but time-varying magnetic field. A small gap is introduced in the loop, allowing us to measure the induced potential V_T. Additionally, a resistance R is connected across

V_T in order to allow a current to flow. This problem was considered in Section 8.3 as an introduction to Faraday's Law; in this section, we shall actually work the problem and calculate some values. This is intended to serve as an example of the application of Faraday's Law, a demonstration of transformer emf, and will serve as a first step toward an understanding of transformers as devices.

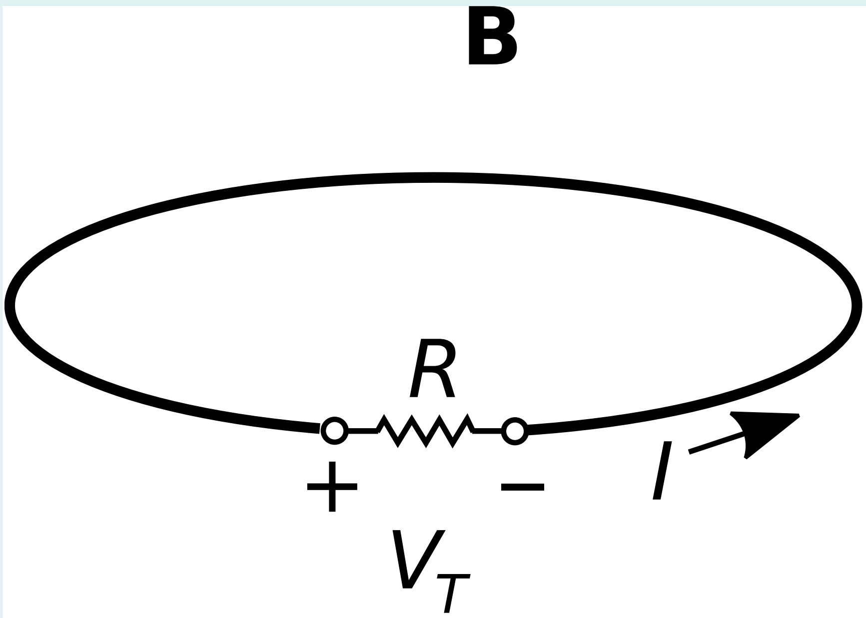

Figure 8.5. *A single loop of wire in the presence of an impressed spatially-uniform but time-varying magnetic field.*

In the present problem, the loop is centered in the $z = 0$ plane. The magnetic flux density is $\mathbf{B} = \hat{\mathbf{b}}B(t)$; i.e., time-varying magnitude $B(t)$ and a constant direction $\hat{\mathbf{b}}$. Because this magnetic field is spatially uniform (i.e., the same everywhere), we will find that only the area of the loop is important, and not it's specific shape. For this reason, it will not be necessary to specify the radius of the loop or even require that it be a circular loop. Our task is to find expressions for V_T and I.

To begin, remember that Faraday's Law is a calculation of electric potential and not current. So, the approach is to first find V_T, and then find the current I that flows through the gap resistance in response.

The sign convention for V_T is arbitrary; here, we have selected "$+$" and "$-$" terminals as indicated in Figure 8.5.[1] Following the standard convention for the reference direction of current through a passive device, I should be directed as shown in Figure 8.5. It is worth repeating that these conventions for the signs of V_T and I are merely *references*; for example, we may well find that I is negative, which means that current flows in a clockwise direction in the loop.

We now invoke Faraday's Law:

$$V_T = -N \frac{\partial}{\partial t} \Phi \tag{8.4}$$

The number of windings N in the loop is 1, and Φ is the magnetic flux through the loop. Thus:

$$V_T = -\frac{\partial}{\partial t}\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{s} \tag{8.5}$$

where $\mathcal{S}$ is any open surface that intersects all of the magnetic field lines that pass through the loop. The simplest such surface is simply the planar surface defined by the perimeter of the loop. Then $d\mathbf{s} = \hat{\mathbf{n}}ds$, where ds is the differential surface element and $\hat{\mathbf{n}}$ is the normal to the plane of the loop. Which of the two possible normals to the loop? This is determined by the right-hand rule of Stokes' Theorem. From the "$-$" terminal, we point the thumb of the right hand in the direction that leads to the "$+$" terminal by traversing the perimeter of the loop. When we do this, the curled fingers of the right hand intersect $\mathcal{S}$ in the same direction as $\hat{\mathbf{n}}$. To maintain the generality of results derived below, we shall not make the substitution $\hat{\mathbf{n}} = +\hat{\mathbf{z}}$; nevertheless we see this is the case for a loop parallel to the $z = 0$ plane with the polarity of V_T indicated in Figure 8.5.

Taking this all into account, we have

$$\begin{aligned} V_T &= -\frac{\partial}{\partial t}\int_{\mathcal{S}} \left(\hat{\mathbf{b}}B(t)\right)\cdot\left(\hat{\mathbf{n}}ds\right) \\ &= -\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}\right)\frac{\partial}{\partial t}\int_{\mathcal{S}} B(t)\ ds \end{aligned} \tag{8.6}$$

Since the magnetic field is uniform, $B(t)$ may be extracted from the integral. Furthermore, the shape and the orientation of the loop are time-invariant, so the remaining integral may be extracted from the time derivative operation. This leaves:

$$V_T = -\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}\right)\left(\frac{\partial}{\partial t}B(t)\right)\int_{\mathcal{S}} ds \tag{8.7}$$

The integral in this expression is simply the area of the loop, which is a constant; let the symbol A represent this area. We obtain

$$\boxed{V_T = -\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}A\right)\frac{\partial}{\partial t}B(t)} \tag{8.8}$$

which is the expression we seek. Note that the quantity $\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}A$ is the *projected area* of the loop. The projected area is equal to A when the the magnetic field lines are perpendicular to the loop (i.e., $\hat{\mathbf{b}} = \hat{\mathbf{n}}$), and decreases to zero as $\hat{\mathbf{b}}\cdot\hat{\mathbf{n}} \to 0$. Summarizing:

The magnitude of the transformer emf induced by a spatially-uniform magnetic field is equal to the projected area times the time rate of change of the magnetic flux density, with a change of sign. (Equation 8.8).

A few observations about this result:

- As promised earlier, we have found that the shape of the loop is irrelevant; i.e., a *square* loop having the same area and planar orientation would result in the same V_T. This is because the magnetic field is spatially uniform, and because it is the magnetic flux (Φ) and not the magnetic field or shape of the loop alone that determines the induced potential.
- The induced potential is proportional to A; i.e., V_T can be increased by increasing the area of the loop.
- The peak magnitude of the induced potential is maximized when the plane of the loop is perpendicular to the magnetic field lines.
- The induced potential goes to zero when the plane of the loop is parallel to the magnetic field lines. Said another way, there is no induction unless magnetic field lines pass through the loop.
- The induced potential is proportional to the rate of change of $\mathbf{B}$. If $\mathbf{B}$ is constant in time, then there is no induction.

Finally, the current in the loop is simply

$$I = \frac{V_T}{R} \tag{8.9}$$

Again, electromagnetic induction induces potential, and the current flows only in response to the induced potential as determined by Ohm's Law. In particular, if the resistor is removed, then $R \to \infty$ and $I \to 0$, but V_T is unchanged.

One final comment is that even though the current I is not a *direct* result of electromagnetic induction, we can use I as a check of the result using Lenz's Law (Section 8.2). We'll demonstrate this in the example below.

Example

Example 8.2

Induction in a motionless circular loop by a *linearly*-increasing magnetic field

Let the loop be planar in the $z = 0$ plane and circular with radius $a = 10$ cm. Let the magnetic field be $\hat{\mathbf{z}}B(t)$ where

$$\begin{aligned} B(t) &= 0 &&, \ t < 0 \\ &= B_0 t/t_0 &&, \ 0 \leq t \leq t_0 \\ &= B_0 &&, \ t > t_0 \end{aligned} \tag{8.10}$$

i.e., $B(t)$ begins at zero and increases linearly to B_0 at time t_0, after which it remains constant at B_0. Let $B_0 = 0.2$ T, $t_0 = 3$ s, and let the loop be closed by a resistor $R = 1\ \mathrm{k}\Omega$. What current I flows in the loop?

Solution. Adopting the sign conventions of Figure 8.5 we first note that $\hat{\mathbf{n}} = +\hat{\mathbf{z}}$; this is determined by the right-hand rule with respect to the indicated polarity of V_T. Thus, Equation 8.8 becomes

$$V_T = -\left(\hat{\mathbf{b}} \cdot \hat{\mathbf{z}}A\right) \frac{\partial}{\partial t} B(t)$$

Note $\hat{\mathbf{b}} \cdot \hat{\mathbf{z}}A = A$ since $\hat{\mathbf{b}} = \hat{\mathbf{z}}$; i.e., because the plane of the loop is perpendicular to the magnetic field lines. Since the loop is circular, $A = \pi a^2$. Also

$$\begin{aligned} \frac{\partial}{\partial t} B(t) &= 0 &&, \ t < 0 \\ &= B_0/t_0 &&, \ 0 \leq t \leq t_0 \\ &= 0 &&, \ t > t_0 \end{aligned}$$

Putting this all together:

$$V_T = -\pi a^2 \frac{B_0}{t_0} = -2.09\ \mathrm{mV}\ ,\ 0 \leq t \leq t_0$$

and $V_T = 0$ before and after this time interval, since $\mathbf{B}$ is constant during those times. Subsequently the induced current is

$$I = \frac{V_T}{R} = -2.09\ \mu\mathrm{A}\ ,\ 0 \leq t \leq t_0$$

and $I = 0$ before and after this time interval. We have found that the induced current is a constant clockwise flow that exists only while $\mathbf{B}$ is increasing.

Finally, let's see if the result is consistent with Lenz's Law. The current induced while $\mathbf{B}$ is changing gives rise to an induced magnetic field $\mathbf{B}_{ind}$. From the right-hand rule that relates the direction of I to the direction of $\mathbf{B}_{ind}$ (Section 7.5), the direction of $\mathbf{B}_{ind}$ is generally $-\hat{\mathbf{z}}$ inside the loop. In other words, the magnetic field associated with the induced current opposes the increasing impressed magnetic field that induced the current, in accordance with Lenz's Law.

Example

Example 8.3

Induction in a motionless circular loop by a *sinusoidally*-varying magnetic field.

Let us repeat the previous example, but now with

$$B(t) = B_0 \sin 2\pi f_0 t$$

with $f_0 = 1$ kHz.

Solution. Now

$$\frac{\partial}{\partial t} B(t) = 2\pi f_0 B_0 \cos 2\pi f_0 t$$

So

$$V_T = -2\pi^2 f_0 a^2 B_0 \cos 2\pi f_0 t$$

Subsequently

$$I = \frac{V_T}{R} = -\frac{2\pi^2 f_0 a^2 B_0}{R} \cos 2\pi f_0 t$$

Substituting values, we have:

$$I = -(39.5 \text{ mA}) \cos\left[(6.28 \text{ krad/s})t\right]$$

It should be no surprise that V_T and I vary sinusoidally, since the source ($\mathbf{B}$) varies sinusoidally. A bit of useful trivia here is that V_T and I are 90° out of phase with the source. It is also worth noting what happens when $\mathbf{B}(t) = 0$. This occurs twice per period, at $t = n/2f_0$ where n is any integer, including $t = 0$. At these times $\mathbf{B}(t)$ is zero, but V_T and hence I_R are decidedly non-zero; in fact, they are at their maximum magnitude. Again, it is the *change* in $\mathbf{B}$ that induces voltage and subsequently current, not $\mathbf{B}$ itself.

1. A good exercise for the student is to repeat this problem with the terminal polarity reversed; one should obtain the same answer. ↩

8.5 Transformers: Principle of Operation

A transformer is a device that connects two electrical circuits through a shared magnetic field. Transformers are used in impedance transformation, voltage level conversion, circuit isolation, conversion between single-ended and differential signal modes, and other applications.[1] The underlying electromagnetic principle is Faraday's Law (Section 8.3) – in particular, transformer emf.

The essential features of a transformer can be derived from the simple experiment shown in Figures 8.6 and 8.7. In this experiment, two coils are arranged along a common axis. The winding pitch is small, so that all magnetic field lines pass through the length of the coil, and no lines pass between the windings. To further contain the magnetic field, we assume both coils are wound on the same core, consisting of some material exhibiting high permeability. The upper coil has N_1 turns and the lower coil has N_2 turns.

In Part I of this experiment (Figure 8.6), the upper coil is connected to a sinusoidally-varying voltage source $V_1^{(1)}$ in which the subscript refers to the coil and the superscript refers to "Part I" of this experiment. The voltage source creates a current in the coil, which in turn creates a time-varying magnetic field $\mathbf{B}$ in the core.

Figure 8.6. *Part I of an experiment demonstrating the linking of electric circuits using a transformer.*

Figure 8.7. *Part II of an experiment demonstrating the linking of electric circuits using a transformer.*

The lower coil has N_2 turns wound in the *opposite direction* and is open-circuited. Given the closely-spaced windings and use of a high-permeability core, we assume that the magnetic field within the lower coil is equal to $\mathbf{B}$ generated in the upper coil. The potential induced in the lower coil is $V_2^{(1)}$ with reference polarity indicated in the figure. From Faraday's Law we have

$$V_2^{(1)} = -N_2 \frac{\partial}{\partial t} \Phi_2 \tag{8.11}$$

where Φ_2 is the flux through a single turn of the lower coil. Thus:

$$V_2^{(1)} = -N_2 \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{B} \cdot (-\hat{\mathbf{z}} ds) \tag{8.12}$$

Note that the direction of $d\mathbf{s} = -\hat{\mathbf{z}} ds$ is determined by the polarity we have chosen for $V_2^{(1)}$.

In Part II of the experiment (Figure 8.7), we make changes as follows. We apply a voltage $V_2^{(2)}$ to the lower coil and open-circuit the upper coil. Further, we adjust $V_2^{(2)}$ such that the induced magnetic flux density is again $\mathbf{B}$ – that is, equal to the field in Part I of the experiment. Now we have

$$V_1^{(2)} = -N_1 \frac{\partial}{\partial t} \Phi_1 \tag{8.13}$$

which is

$$V_1^{(2)} = -N_1 \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{B} \cdot (+\hat{\mathbf{z}} ds) \tag{8.14}$$

For reasons that will become apparent in a moment, let's shift the leading minus sign into the integral. We then have

$$V_1^{(2)} = +N_1 \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{B} \cdot (-\hat{\mathbf{z}} ds) \tag{8.15}$$

Comparing this to Equations 8.11 and 8.12, we see that we may rewrite this in terms of the flux in the lower coil in Part I of the experiment:

$$V_1^{(2)} = +N_1 \frac{\partial}{\partial t} \Phi_2 \tag{8.16}$$

In fact, we can express this in terms of the *potential* in Part I of the experiment:

$$\begin{aligned} V_1^{(2)} &= \left(-\frac{N_1}{N_2}\right) \left(-N_2 \frac{\partial}{\partial t} \Phi_2\right) \\ &= \left(-\frac{N_1}{N_2}\right) V_2^{(1)} \end{aligned} \tag{8.17}$$

We have found that the potential in the upper coil in Part II is related in a simple way to the potential in the lower coil in Part I of the experiment. If we had done Part II first, we would have obtained the same result but with the superscripts swapped. Therefore, it must be generally true – regardless of the arrangement of terminations – that

$$V_1 = -\frac{N_1}{N_2} V_2 \tag{8.18}$$

This expression should be familiar from elementary circuit theory – except possibly for the minus sign. The minus sign is a consequence of the fact that the coils are wound in opposite directions. We can make the above expression a little more general as follows:

$$\boxed{\frac{V_1}{V_2} = p\frac{N_1}{N_2}} \tag{8.19}$$

where p is defined to $+1$ when the coils are wound in the same direction and -1 when coils are wound in opposite directions. (It is an excellent exercise to confirm that this is true by repeating the above analysis with winding direction changed for either the upper or lower coil, for which p will then turn out to be $+1$.) This is the "transformer law" of basic electric circuit theory, from which all other characteristics of transformers as two-port circuit devices can be obtained (See Section 8.6 for follow-up on that). Summarizing:

The ratio of coil voltages in an ideal transformer is equal to the ratio of turns with sign determined by the relative directions of the windings, per Equation 8.19.

A more familiar transformer design is shown in Figure 8.8 – coils wound on a toroidal core as opposed to a cylindrical core. Why do this? This arrangement confines the magnetic field linking the two coils to the core, as opposed to allowing field lines to extend beyond the device. This confinement is important in order to prevent fields originating outside the transformer from interfering with the magnetic field linking the coils, which would lead to electromagnetic interference (EMI) and electromagnetic compatibility (EMC) problems. The principle of operation is in all other respects the same.

Figure 8.8. *Transformer implemented as coils sharing a toroidal core. Here $p = +1$.*

Additional Reading:

- "Transformer (https://en.wikipedia.org/wiki/Transformer)" on Wikipedia.
- "Balun (https://en.wikipedia.org/wiki/Balun)" on Wikipedia.

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1. See "Additional Reading" at the end of this section for more on these applications. ↩

8.6 Transformers as Two-Port Devices

Section 8.5 explains the principle of operation of the ideal transformer. The relationship governing the terminal voltages V_1 and V_2 was found to be

$$\frac{V_1}{V_2} = p\frac{N_1}{N_2} \tag{8.20}$$

where N_1 and N_2 are the number of turns in the associated coils and p is either $+1$ or -1 depending on the relative orientation of the windings; i.e., whether the reference direction of the associated fluxes is the same or opposite, respectively.

We shall now consider ratios of current and impedance in ideal transformers, using the two-port model shown in Figure 8.9. By virtue of the reference current directions and polarities chosen in this figure, the power delivered by the source V_1 is $V_1 I_1$, and the power absorbed by the load Z_2 is $-V_2 I_2$. Assuming the transformer windings have no resistance, and assuming the magnetic flux is perfectly contained within the core, the power absorbed by the load must equal the power provided by the source; i.e., $V_1 I_1 = -V_2 I_2$. Thus, we have[1]

$$\boxed{\frac{I_1}{I_2} = -\frac{V_2}{V_1} = -p\frac{N_2}{N_1}} \tag{8.21}$$

Figure 8.9. *The transformer as a two-port circuit device.*

We can develop an impedance relationship for ideal transformers as follows. Let Z_1 be the input impedance of the transformer; that is, the impedance looking into port 1 from the source. Thus, we have

$$\begin{aligned} Z_1 &\triangleq \frac{V_1}{I_1} \\ &= \frac{+p\left(N_1/N_2\right)V_2}{-p\left(N_2/N_1\right)I_2} \\ &= -\left(\frac{N_1}{N_2}\right)^2\left(\frac{V_2}{I_2}\right) \end{aligned} \tag{8.22}$$

In Figure 8.9, Z_2 is the the output impedance of port 2; that is, the impedance looking out port 2 into the load. Therefore, $Z_2 = -V_2/I_2$ (once again the minus sign is a result of our choice of sign conventions in Figure 8.9). Substitution of this result into the above equation yields

$$Z_1 = \left(\frac{N_1}{N_2}\right)^2 Z_2 \tag{8.23}$$

and therefore

$$\boxed{\frac{Z_1}{Z_2} = \left(\frac{N_1}{N_2}\right)^2} \tag{8.24}$$

Thus, we have demonstrated that a transformer scales impedance in proportion to the square of the turns ratio N_1/N_2. Remarkably, the impedance transformation depends *only* on the turns ratio, and is independent of the relative direction of the windings (p).

The relationships developed above should be viewed as AC expressions, and are not normally valid at DC. This is because transformers exhibit a fundamental limitation in their low-frequency performance. To see this, first recall Faraday's Law:

$$V = -N\frac{\partial}{\partial t}\Phi \tag{8.25}$$

If the magnetic flux Φ isn't time-varying, then there is no induced electric potential, and subsequently no linking of the signals associated with the coils. At very low but non-zero frequencies, we encounter another problem that gets in the way – *magnetic saturation*. To see this, note we can obtain an expression for Φ from Faraday's Law by integrating with respect to time, yielding

$$\Phi(t) = -\frac{1}{N}\int_{t_0}^{t} V(\tau)d\tau + \Phi(t_0) \tag{8.26}$$

where t_0 is some earlier time at which we know the value of $\Phi(t_0)$. Let us assume that $V(t)$ is sinusoidally-varying. Then the peak value of Φ after $t = t_0$ depends on the frequency of $V(t)$. If the frequency of $V(t)$ is very low, then Φ can become very large. Since the associated cross-sectional areas of the coils are presumably constant, this means that the magnetic field becomes very large. The problem with that is that most practical high-permeability materials suitable for use as transformer cores exhibit magnetic saturation; that is, the rate at which the magnetic field is able to increase decreases with increasing magnetic field magnitude (see Section 7.16). The result of all this is that a transformer may work fine at (say) 1 MHz, but at (say) 1 Hz the transformer may exhibit an apparent loss associated with this saturation. Thus, practical transformers exhibit highpass frequency response.

It should be noted that the highpass behavior of practical transistors can be useful. For example, a transformer can be used to isolate an undesired DC offset and/or low-frequency noise in the circuit attached to one coil from the circuit attached to the other coil.

The DC-isolating behavior of a transformer also allows the transformer to be used as a *balun*, as illustrated in Figure 8.10. A balun is a two-port device that transforms a single-ended ("unbalanced") signal – that is, one having an explicit connection to a datum (e.g., ground) – into a differential ("balanced") signal, for which there is no explicit connection to a datum. Differential signals have many benefits in circuit design, whereas inputs and outputs to devices must often be in single-ended form. Thus, a common use of transformers is to effect the conversion between single-ended and differential circuits. Although a transformer is certainly not the only device that can be used as a balun, it has one very big advantage, namely bandwidth.

Figure 8.10. *Transformers used to convert a single-ended ("unbalanced") signal to a differential (balanced) signal, and back.*

Additional Reading:

- "Transformer (https://en.wikipedia.org/wiki/Transformer)" on Wikipedia.
- "Two-port network (https://en.wikipedia.org/wiki/Two-port_network)" on Wikipedia.
- "Saturation (magnetic) (https://en.wikipedia.org/wiki/Saturation_(magnetic))" on Wikipedia.
- "Balun (https://en.wikipedia.org/wiki/Balun)" on Wikipedia.
- S.W. Ellingson, "Differential Circuits" (Sec. 8.7) in *Radio Systems Engineering*, Cambridge Univ. Press, 2016.

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1. The minus signs in this equation are a result of the reference polarity and directions indicated in Figure 8.9. These are more-or-less standard in electrical two-port theory (see "Additional Reading" at the end of this section), but are certainly not the only reasonable choice. If you see these expressions without the minus signs, it probably means that a different combination of reference directions/polarities is in effect. ↩

8.7 The Electric Generator

A generator is a device that transforms mechanical energy into electrical energy, typically by electromagnetic induction via Faraday's Law (Section 8.3). For example, a generator might consist of a gasoline engine that turns a crankshaft to which is attached a system of coils and/or magnets. This rotation changes the relative orientations of the coils with respect to the magnetic field in a time-varying manner, resulting in a time-varying magnetic flux and subsequently induced electric potential. In this case, the induced potential is said to be a form of *motional emf*, as it is due entirely to changes in geometry as opposed to changes in the magnitude of the magnetic field. Coal- and steam-fired generators, hydroelectric generators, wind turbines, and other energy generation devices operate using essentially this principle.

Figure 8.11 shows a rudimentary generator, which serves as to illustrate the relevant points. This generator consists of a planar loop that rotates around the z axis; therefore, the rotation can be parameterized in ϕ. In this case, the direction of rotation is specified to be in the $+\phi$ direction. The frequency of rotation is f_0; that is, the time required for the loop to make one complete revolution is $1/f_0$. We assume a time-invariant and spatially-uniform magnetic flux density $\mathbf{B} = \hat{\mathbf{b}}B_0$ where $\hat{\mathbf{b}}$ and B_0 are both constants. For illustration purposes, the loop is indicated to be circular. However, because the magnetic field is time-invariant and spatially-uniform, the specific shape of the loop is not important, as we shall see in a moment. Only the area of the loop will matter.

Figure 8.11. *A rudimentary single-loop generator, shown at time $t = 0$.*

The induced potential is indicated as V_T with reference polarity as indicated in the figure. This potential is given by Faraday's Law:

$$V_T = -\frac{\partial}{\partial t}\Phi \tag{8.27}$$

Here Φ is the magnetic flux associated with an open surface $\mathcal{S}$ bounded by the loop:

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} \tag{8.28}$$

As usual, $\mathcal{S}$ can be any surface that intersects all magnetic field lines passing through the loop, and also as usual, the simplest choice is simply the planar area bounded by the loop. The differential surface element $d\mathbf{s}$ is $\hat{\mathbf{n}}ds$, where $\hat{\mathbf{n}}$ is determined by the reference polarity of V_T according to the "right hand rule" convention from Stokes' Theorem. Making substitutions, we have

$$\begin{aligned}\Phi &= \int_{\mathcal{S}} \left(\hat{\mathbf{b}}B_0\right) \cdot \left(\hat{\mathbf{n}}ds\right) \\ &= \left[\hat{\mathbf{b}} \cdot \hat{\mathbf{n}}\right] B_0 \int_{\mathcal{S}} ds \\ &= \left[\hat{\mathbf{b}} \cdot \hat{\mathbf{n}}A\right] B_0\end{aligned} \tag{8.29}$$

where A is the area of the loop.

To make headway, an expression for $\hat{\mathbf{n}}$ is needed. The principal difficulty here is that $\hat{\mathbf{n}}$ is rotating with the loop, so it is time-varying. To sort this out, first consider the situation at time $t = 0$, which is the moment illustrated in Figure 8.11. Beginning at the "$-$" terminal, we point the thumb of our right hand in the direction that leads to the "$+$" terminal by traversing the loop; $\hat{\mathbf{n}}$ is then the direction perpendicular to the plane of the loop in which the fingers of our right hand pass through the loop. We see that at $t = 0$, $\hat{\mathbf{n}} = +\hat{\mathbf{y}}$. A bit later, the loop will have rotated by one-fourth of a complete rotation, and at that time $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$. This occurs at $t = 1/(4f_0)$. Later still, the loop will have rotated by one-half of a compete rotation, and then $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$. This occurs at $t = 1/(2f_0)$. It is apparent that

$$\hat{\mathbf{n}}(t) = -\hat{\mathbf{x}}\sin 2\pi f_0 t + \hat{\mathbf{y}}\cos 2\pi f_0 t \tag{8.30}$$

as can be confirmed by checking the three special cases identified above. Now applying Faraday's Law, we find

$$\begin{aligned} V_T &= -\frac{\partial}{\partial t}\Phi \\ &= -\frac{\partial}{\partial t}\left[\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}(t)\right]AB_0 \\ &= -\left[\hat{\mathbf{b}}\cdot\frac{\partial}{\partial t}\hat{\mathbf{n}}(t)\right]AB_0 \end{aligned} \tag{8.31}$$

For notational convenience we make the following definition:

$$\hat{\mathbf{n}}'(t) \triangleq -\frac{1}{2\pi f_0}\frac{\partial}{\partial t}\hat{\mathbf{n}}(t) \tag{8.32}$$

which is simply the time derivative of $\hat{\mathbf{n}}$ divided by $2\pi f_0$ so as to retain a unit vector. The reason for including a change of sign will become apparent in a moment. Applying this definition, we find

$$\hat{\mathbf{n}}'(t) = +\hat{\mathbf{x}}\cos 2\pi f_0 t + \hat{\mathbf{y}}\sin 2\pi f_0 t \tag{8.33}$$

Note that this is essentially the definition of the radial basis vector $\hat{\rho}$ from the cylindrical coordinate system (which is why we applied the minus sign in Equation 8.32). Recognizing this, we write

$$\hat{\rho}(t) = +\hat{\mathbf{x}}\cos 2\pi f_0 t + \hat{\mathbf{y}}\sin 2\pi f_0 t \tag{8.34}$$

and finally

$$\boxed{V_T = +2\pi f_0 AB_0 \hat{\mathbf{b}}\cdot\hat{\rho}(t)} \tag{8.35}$$

If the purpose of this device is to generate power, then presumably we would choose the magnetic field to be in a direction that maximizes the maximum value of $\hat{\mathbf{b}}\cdot\hat{\rho}(t)$. Therefore, power is optimized for $\mathbf{B}$ polarized entirely in some combination of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, and with $\mathbf{B}\cdot\hat{\mathbf{z}} = 0$. Under that constraint, we see that V_T varies sinusoidally with frequency f_0 and exhibits peak magnitude

$$\boxed{\max\ |V_T(t)| = 2\pi f_0 AB_0} \tag{8.36}$$

It’s worth noting that the maximum voltage magnitude is achieved when the plane of the loop is parallel to $\mathbf{B}$; i.e., when $\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}(t) = 0$ so that $\Phi(t) = 0$. Why is that? Because this is when $\Phi(t)$ is most rapidly increasing or decreasing. Conversely, when the plane of the loop is perpendicular to $\mathbf{B}$ (i.e., $\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}(t) = 1$), $|\Phi(t)|$ is maximum but its time-derivative is zero, so $V_T(t) = 0$ at this instant.

Example

Example 8.4

Rudimentary electric generator.

The generator in Figure 8.11 consists of a circular loop of radius $a = 1$ cm rotating at 1000 revolutions per second in a static and spatially-uniform magnetic flux density of 1 mT in the $+\hat{\mathbf{x}}$ direction. What is the induced potential?

Solution. From the problem statement, $f_0 = 1$ kHz, $B_0 = 1$ mT, and $\hat{\mathbf{b}} = +\hat{\mathbf{x}}$. Therefore $\hat{\mathbf{b}} \cdot \hat{\rho}(t) = \hat{\mathbf{x}} \cdot \hat{\rho}(t) = \cos 2\pi f_0 t$. The area of the loop is $A = \pi a^2$. From Equation 8.35 we obtain

$$V_T(t) \cong (1.97 \text{ mV}) \cos\left[(6.28 \text{ krad/s})\, t\right]$$

Finally, we note that it is not really necessary for the loop to rotate in the presence of a magnetic field with constant $\hat{\mathbf{b}}$; it works equally well for the loop to be stationary and for $\hat{\mathbf{b}}$ to rotate – in fact, this is essentially the same problem. In some practical generators, both the potential-generating coils and fields (generated by some combination of magnets and coils) rotate.

Additional Reading:

- “Electric Generator (https://en.wikipedia.org/wiki/Electric_generator)” on Wikipedia.

8.8 The Maxwell-Faraday Equation

In this section, we generalize Kirchoff’s Voltage Law (KVL), previously encountered as a principle of electrostatics in Sections 5.10 and 5.11. KVL states that in the absence of a time-varying magnetic flux, the electric potential accumulated by traversing a closed path $\mathcal{C}$ is zero. Here is that idea in mathematical form:

$$V = \oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} = 0 \tag{8.37}$$

Now recall Faraday’s Law (Section 8.3):

$$V = -\frac{\partial}{\partial t}\Phi = -\frac{\partial}{\partial t}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} \tag{8.38}$$

Here, $\mathcal{S}$ is any open surface that intersects all magnetic field lines passing through $\mathcal{C}$, with the relative orientations of $\mathcal{C}$ and $d\mathbf{s}$ determined in the usual way by the Stokes' Theorem convention (Section 4.9). Note that Faraday's Law agrees with KVL in the magnetostatic case. If magnetic flux is constant, then Faraday's Law says $V = 0$. However, Faraday's Law is very clearly *not* consistent with KVL if magnetic flux is time-varying. The correction is simple enough; we can simply set these expressions to be equal. Here we go:

$$\boxed{\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s}} \tag{8.39}$$

This general form is known by a variety of names; here we refer to it as the *Maxwell-Faraday Equation* (MFE).

The integral form of the Maxwell-Faraday Equation (Equation 8.39) states that the electric potential associated with a closed path $\mathcal{C}$ is due entirely to electromagnetic induction, via Faraday's Law.

Despite the great significance of this expression as one of Maxwell's Equations, one might argue that all we have done is simply to write Faraday's Law in a slightly more verbose way. This is true. The *real* power of the MFE is unleashed when it is expressed in differential, as opposed to integral form. Let us now do this.

We can transform the left-hand side of Equation 8.39 into a integral over $\mathcal{S}$ using Stokes' Theorem. Applying Stokes' theorem on the left, we obtain

$$\int_{\mathcal{S}} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} \tag{8.40}$$

Now exchanging the order of integration and differentiation on the right hand side:

$$\int_{\mathcal{S}} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \int_{\mathcal{S}} \left(-\frac{\partial}{\partial t}\mathbf{B}\right) \cdot d\mathbf{s} \tag{8.41}$$

The surface $\mathcal{S}$ on both sides is the same, and we have not constrained $\mathcal{S}$ in any way. $\mathcal{S}$ can be any mathematically-valid open surface anywhere in space, having any size and any orientation. The only way the above expression can be universally true under these conditions is if the integrands on each side are equal at every point in space. Therefore,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}} \tag{8.42}$$

which is the MFE in differential form.

What does this mean? Recall that the curl of $\mathbf{E}$ is a way to take a directive of $\mathbf{E}$ with respect to position (Section 4.8). Therefore the MFE constrains spatial derivatives of $\mathbf{E}$ to be simply related to the rate of change of $\mathbf{B}$. Said plainly:

> The differential form of the Maxwell-Faraday Equation (Equation 8.42) relates the change in the electric field with position to the change in the magnetic field with time.

Now that *is* arguably new and useful information. We now see that electric and magnetic fields are coupled not only for line integrals and fluxes, but also at each point in space.

Additional Reading:

- "Faraday's Law of Induction (https://en.wikipedia.org/wiki/Faraday's_law_of_induction)" on Wikipedia.
- "Maxwell's Equations (https://en.wikipedia.org/wiki/Maxwell's_equations)" on Wikipedia.

8.9 Displacement Current and Ampere's Law

In this section, we generalize Ampere's Law, previously encountered as a principle of magnetostatics in Sections 7.4 and 7.9. Ampere's Law states that the current I_{encl} flowing through closed path $\mathcal{C}$ is equal to the line integral of the magnetic field intensity $\mathbf{H}$ along $\mathcal{C}$. That is:

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl} \tag{8.43}$$

We shall now demonstrate that this equation is unreliable if the current is not steady; i.e., not DC.

First, consider the situation shown in Figure 8.12. Here, a current I flows in the wire, subsequently generating a magnetic field $\mathbf{H}$ that circulates around the wire (Section 7.5). When we perform the integration in Ampere's Law along any path $\mathcal{C}$ enclosing the wire, the result is I, as expected. In this case, Ampere's Law is working even when I is time-varying.

Figure 8.12. *Ampere's Law applied to a continuous line of current.*

C. Burks (modified)

Now consider the situation shown in Figure 8.13, in which we have introduced a parallel-plate capacitor. In the DC case, this situation is simple. No current flows, so there is no magnetic field and Ampere's Law is trivially true. In the AC case, the current I *can* be non-zero, but we must be clear about the physical origin of this current. What is happening is that for one half of a period, a source elsewhere in the circuit is moving positive charge to one side of the capacitor and negative charge to the other side. For the other half-period, the source is exchanging the charge, so that negative charge appears on the previously positively-charged side and vice-versa. Note that at no point is current flowing directly from one side of the capacitor to the other; instead, all current must flow through the circuit in order to arrive at the other plate. Even though there is no current between the plates, there is current in the wire, and therefore there is also a magnetic field associated with that current.

Figure 8.13. *Ampere's Law applied to a parallel plate capacitor.*

C. Burks (modified)

Now we are ready to shine a light on the problem. Recall that from Stokes' Theorem (Section 4.9), the line integral over $\mathcal{C}$ is mathematically equivalent to an integral over any open surface $\mathcal{S}$ that is bounded by $\mathcal{C}$. Two such surfaces are shown in Figure 8.12 and Figure 8.13, indicated as $\mathcal{S}_1$ and $\mathcal{S}_2$. In the wire-only scenario of Figure 8.12, the choice of $\mathcal{S}$ clearly doesn't matter; any valid surface intersects current equal to I. Similarly in the scenario of Figure 8.13, everything seems fine if we choose $\mathcal{S} = \mathcal{S}_1$. If, on the other hand, we select $\mathcal{S}_2$ in the parallel-plate capacitor case, then we have a problem. There is no current flowing through $\mathcal{S}_2$, so the right side of Equation 8.43 is zero even though the left side is potentially non-zero. So, it appears that something necessary for the time-varying case is missing from Equation 8.43.

To resolve the problem, we postulate an additional term in Ampere's Law that is non-zero in the above scenario. Specifically, we propose:

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_c + I_d \tag{8.44}$$

where I_c is the enclosed current (formerly identified as I_{encl}) and I_d is the proposed new term. If we are to accept this postulate, then here is a list of things we know about I_d:

- I_d has units of current (A).
- $I_d = 0$ in the DC case and is potentially non-zero in the AC case. This implies that I_d is the time derivative of some other quantity.
- I_d must be somehow related to the electric field.

How do we know I_d must be related to the electric field? This is because the Maxwell-Faraday Equation (Section 8.8) tells us that spatial derivatives of $\mathbf{E}$ are related to time derivatives of $\mathbf{H}$; i.e., $\mathbf{E}$ and $\mathbf{H}$ are coupled in the time-varying (here, AC) case. This coupling between $\mathbf{E}$ and $\mathbf{H}$ must also be at work here, but we have not yet seen $\mathbf{E}$ play a role. This is pretty strong evidence that I_d depends on the electric field.

Without further ado, here's I_d:

$$I_d = \int_{\mathcal{S}} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \tag{8.45}$$

where $\mathbf{D}$ is the electric flux density (units of C/m^2) and is equal to $\epsilon\mathbf{E}$ as usual, and $\mathcal{S}$ is the same open surface associated with $\mathcal{C}$ in Ampere's Law. Note that this expression meets our expectations: It is determined by the electric field, it is zero when the electric field is constant (i.e., not time varying), and has units of current.

The quantity I_d is commonly known as *displacement current*. It should be noted that this name is a bit misleading, since I_d is not a current in the conventional sense. Certainly, it is not a *conduction* current – conduction current is represented by I_c, and there is no current conducted through an ideal capacitor. It is not unreasonable to think of I_d as current in a more general sense, for the following reason. At one instant, charge is distributed one way and at another, it is distributed in another way. If you define current as a time variation in the charge distribution relative to $\mathcal{S}$ – regardless of the path taken by the charge – then I_d is a current. However, this distinction is a bit philosophical, so it may be less confusing to interpret "displacement current" instead as a separate electromagnetic quantity that just happens to have units of current.

Now we are able to write the general form of Ampere's Law that applies even when sources are time-varying. Here it is:

$$\boxed{\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_c + \int_{\mathcal{S}} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}} \tag{8.46}$$

As is the case in the Maxwell-Faraday Equation, most of the utility of Ampere's Law is unleashed when expressed in differential form. To obtain this form the first step is to write I_c as an integral of over $\mathcal{S}$; this is simply (see Section 6.2):

$$I_c = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{s} \tag{8.47}$$

where $\mathbf{J}$ is the volume current density (units of A/m^2). So now we have

$$\begin{aligned} \oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} &= \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{s} + \int_{\mathcal{S}} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \\ &= \int_{\mathcal{S}} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} \end{aligned} \tag{8.48}$$

We can transform the left side of the above equation into a integral over $\mathcal{S}$ using Stokes' Theorem (Section 4.9). We obtain

$$\int_{\mathcal{S}} (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_{\mathcal{S}} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} \tag{8.49}$$

The surface $\mathcal{S}$ on both sides is the same, and we have not constrained $\mathcal{S}$ in any way. $\mathcal{S}$ can be any mathematically-valid open surface anywhere in space, having any size and any orientation. The only way the above expression can be universally true under these conditions is if the integrands on each side are equal at every point in space. Therefore:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}} \tag{8.50}$$

which is Ampere's Law in differential form.

What does Equation 8.50 mean? Recall that the curl of $\mathbf{H}$ is a way to describe the direction and rate of change of $\mathbf{H}$ with position. Therefore, this equation constrains spatial derivatives of $\mathbf{H}$ to be simply related to $\mathbf{J}$ and the time derivative of $\mathbf{D}$ (displacement current). Said plainly:

> The differential form of the general (time-varying) form of Ampere's Law (Equation 8.50) relates the change in the magnetic field with position to the change in the electric field with time, plus current.

As is the case in the Maxwell-Faraday Equation (Section 8.8), we see that electric and magnetic fields become coupled at each point in space when sources are time-varying.

Additional Reading:

- "Displacement Current (https://en.wikipedia.org/wiki/Displacement_current)" on Wikipedia.
- "Ampere's Circuital Law (https://en.wikipedia.org/wiki/Ampere's_circuital_law)" on Wikipedia.
- "Maxwell's Equations (https://en.wikipedia.org/wiki/Maxwell's_equations)" on Wikipedia.

End Matter

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