6 Steady Current and Conductivity

6 Steady Current and Conductivity

6.1 Convection and Conduction Currents

In practice, we deal with two physical mechanisms for current: convection and conduction. The distinction between these types of current is important in electromagnetic analysis.

Convection current consists of charged particles moving in response to mechanical forces, as opposed to being guided by the electric field (Sections 2.2 and/or 5.1). An example of a convection current is a cloud bearing free electrons that moves through the atmosphere driven by wind.

Conduction current consists of charged particles moving in response to the electric field and not merely being carried by motion of the surrounding material. In some materials, the electric field is also able to dislodge weakly-bound electrons from atoms, which then subsequently travel some distance before reassociating with other atoms. For this reason, the individual electrons in a conduction current do not necessarily travel the full distance over which the current is perceived to exist.

The distinction between convection and conduction is important because *Ohm's Law* (Section 6.3) – which specifies the relationship between electric field intensity and current – applies only to conduction current.

Additional Reading:

- "Electric current (https://en.wikipedia.org/wiki/Electric_current)" on Wikipedia.

6.2 Current Distributions

In elementary electric circuit theory, current is the rate at which electric charge passes a particular point in a circuit. For example, 1 A is 1 C per second. In this view current is a scalar quantity, and there are only two possible directions because charge is constrained to flow along defined channels. Direction is identified by the sign of the current with respect to a reference direction, which is in turn defined with respect to a reference voltage polarity. The convention in electrical engineering defines positive current as the flow of positive charge into the positive voltage terminal of a passive device such as a resistor, capacitor, or inductor. For an active device such as a battery, positive current corresponds to the flow of positive charge *out of* the positive voltage terminal.

However, this kind of thinking only holds up when the systems being considered are well-described as "lumped" devices connected by infinitesimally thin, filament-like connections representing wires and circuit board traces. Many important problems in electrical engineering concern situations in which the flow of current is not limited in this way. Examples include wire- and pin-type interconnects at radio frequencies, circuit board and enclosure grounding, and physical phenomena such as lightning. To accommodate the more general class of problems, we must define current as a vector quantity. Furthermore, current in these problems can spread out over surfaces and within volumes, so we must also consider *spatial distributions* of current.

Line Current. As noted above, if a current I is constrained to follow a particular path, then the only other consideration is direction. Thus, a line current is specified mathematically as $\hat{\imath}I$, where the direction $\hat{\imath}$ may vary with position along the path. For example, in a straight wire $\hat{\imath}$ is constant, whereas in a coil $\hat{\imath}$ varies with position along the coil.

Surface Current Distribution. In some cases, current may be distributed over a surface. For example, the radio-frequency current on a wire of radius a made from a metal with sufficiently high conductivity can be modeled as a uniform surface current existing on the wire surface. In this case, the current is best described as a surface current density $\mathbf{J}_s$, which is the total current I on the wire divided by the circumference $2\pi a$ of the wire:

$$\mathbf{J}_s = \hat{\mathbf{u}}\frac{I}{2\pi a} \quad \text{(units of A/m)} \tag{6.1}$$

where $\hat{\mathbf{u}}$ is the direction of current flow.

Volume Current Distribution. Imagine that current is distributed within a volume. Let $\hat{\mathbf{u}}\Delta i$ be the current passing through a small open planar surface defined within this volume, and let Δs be the area of this surface. The *volume current density* $\mathbf{J}$ at any point in the volume is defined as

$$\mathbf{J} \triangleq \lim_{\Delta s \to 0} \frac{\hat{\mathbf{u}}\Delta i}{\Delta s} = \hat{\mathbf{u}}\frac{di}{ds} \quad \text{(units of A/m}^2\text{)} \tag{6.2}$$

In general, $\mathbf{J}$ is a function of position within this volume. Subsequently the total current passing through a surface $\mathcal{S}$ is

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot \, d\mathbf{s} \quad \text{(units of A)} \tag{6.3}$$

In other words, volume current density integrated over a surface yields total current through that surface. You might recognize this as a calculation of flux, and it is.[1]

Example

Example 6.1

Current and current density in a wire of circular cross-section.

Figure 6.1 shows a straight wire having cross-sectional radius $a = 3$ mm. A battery is connected across the two ends of the wire resulting in a volume current density $\mathbf{J} = \hat{\mathbf{z}}8$ A/m^2, which is uniform throughout the wire. Find the net current I through the wire.

Solution. The net current is

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{s}$$

We choose $\mathcal{S}$ to be the cross-section perpendicular to the axis of the wire. Also, we choose $d\mathbf{s}$ to point such that I is positive with respect to the sign convention shown in Figure 6.1 , which is the usual choice in electric circuit analysis. With these choices, we have

$$\begin{aligned} I &= \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \left(\hat{\mathbf{z}}8 \text{ A/m}^2\right) \cdot \left(\hat{\mathbf{z}}\rho \; d\rho \; d\phi\right) \\ &= \left(8 \text{ A/m}^2\right) \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \rho \; d\rho \; d\phi \\ &= \left(8 \text{ A/m}^2\right) \cdot \left(\pi a^2\right) \\ &= 226 \; \mu\text{A} \end{aligned}$$

This answer is independent of the cross-sectional surface $\mathcal{S}$ used to do the calculation. For example, you could use an alternative surface that is tilted 45° away from the axis of the wire. In that case, the cross-sectional area would increase, but the dot product of $\mathbf{J}$ and $d\mathbf{s}$ would be proportionally less and the outcome would be the same. Since the choice of $\mathcal{S}$ is arbitrary – any surface with edges at the perimeter of the wire will do – you should make the choice that makes the problem as simple as possible, as we have done above.

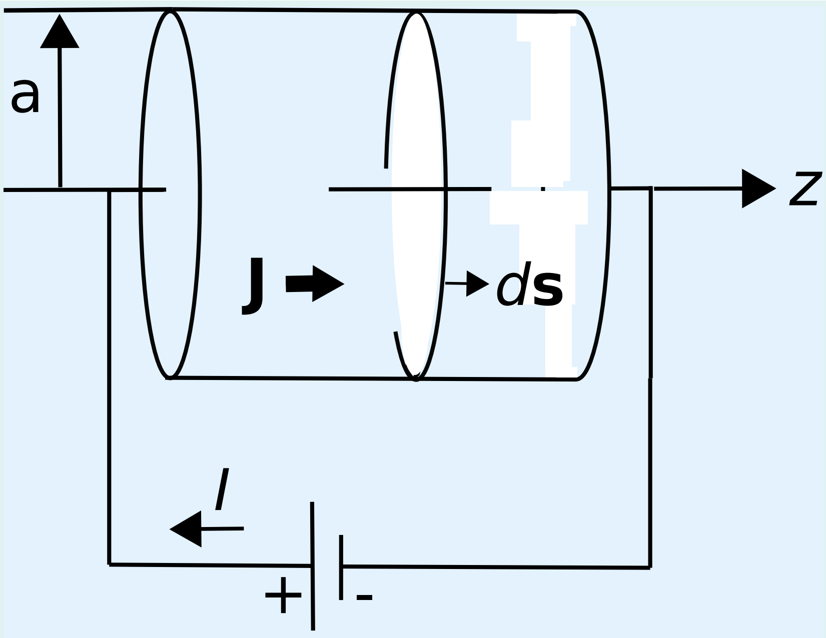

Figure 6.1. *Net current I and current density $\mathbf{J}$ in a wire of circular cross-section.*

Additional Reading:

- "Electric current (https://en.wikipedia.org/wiki/Electric_current)" on Wikipedia.
- "Current density (https://en.wikipedia.org/wiki/Current_density)" on Wikipedia.

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1. However, it is not common to refer to net current as a flux; go figure! ↩

6.3 Conductivity

Conductivity is one of the three primary "constitutive parameters" that is commonly used to characterize the electromagnetic properties of materials (Section 2.8). The key idea is this:

> Conductivity is a property of materials that determines conduction current density in response to an applied electric field.

Recall that conduction current is the flow of charge in response to an electric field (Section 6.1). Although the associated force is straightforward to calculate (e.g., Section 5.1), the result is merely the force applied, not the speed at which charge moves in response. The latter is determined by the *mobility of charge*, which is in turn determined by the atomic and molecular structure of the material. Conductivity relates current density to the applied field directly, without requiring one to grapple separately with the issues of applied force and charge mobility.

In the absence of material – that is, in a true, perfect vacuum – conductivity is zero because there is no charge available to form current, and therefore the current is zero no matter what electric field is applied. At the other extreme, a *good conductor* is a material that contains a supply of charge that is able to move freely within the material. When an electric field is applied to a good conductor, charge-bearing material constituents move in the direction determined by the electric field, creating current flow in that direction. This relationship is summarized by *Ohm's Law for Electromagnetics*:

$$\boxed{\mathbf{J} = \sigma\mathbf{E}} \tag{6.4}$$

where $\mathbf{E}$ is electric field intensity (V/m); $\mathbf{J}$ is the volume current density, a vector describing the current flow, having units of A/m^2 (see Section 6.2); and σ is conductivity. Since $\mathbf{E}$ has units of V/m, we see σ has units of $\Omega^{-1}\mathrm{m}^{-1}$, which is more commonly expressed as S/m, where 1 S ("siemens") is defined as $1\ \Omega^{-1}$. Section A.3 provides values of conductivity for a representative set of materials.

> Conductivity σ is expressed in units of S/m, where 1 S $=$ 1 Ω^{-1}.

It is important to note that the current being addressed here is *conduction current*, and not convection current, displacement current, or some other form of current – see Section 6.2 for elaboration. Summarizing:

> Ohm's Law for Electromagnetics (Equation 6.4) states that volume density of conduction current (A/m^2) equals conductivity (S/m) times electric field intensity (V/m).

The reader is likely aware that there is also an "Ohm's Law" in electric circuit theory, which states that current I (units of A) is voltage V (units of V) divided by resistance R (units of Ω); i.e., $I = V/R$. This is in fact a special case of Equation 6.4; see Section 6.4 for more about this.

As mentioned above, σ depends on both the *availability* and *mobility* of charge within the material. At the two extremes, we have *perfect insulators*, for which $\sigma = 0$, and *perfect conductors*, for which $\sigma \to \infty$. Some materials approach these extremes, whereas others fall midway between these conditions. Here are a few classes of materials that are frequently encountered:

- A perfect vacuum – “free space” – contains no charge and therefore is a perfect insulator with $\sigma = 0$.
- *Good insulators* typically have conductivities $\ll 10^{-10}$ S/m, which is sufficiently low that the resulting currents can usually be ignored. The most important example is air, which has a conductivity only slightly greater than that of free space. An important class of good insulators is *lossless dielectrics*,[1] which are well-characterized in terms of permittivity (ϵ) alone, and for which $\mu = \mu_0$ and $\sigma = 0$ may usually be assumed.
- *Poor insulators* have conductivities that are low, but nevertheless sufficiently high that the resulting currents cannot be ignored. For example, the dielectric material that is used to separate the conductors in a transmission line must be considered a poor insulator as opposed to a good (effectively lossless) insulator in order to characterize loss per length along the transmission line, which can be significant.[2] These *lossy dielectrics* are well-characterized in terms of ϵ and σ, and typically $\mu = \mu_0$ can be assumed.
- *Semiconductors* such as those materials used in integrated circuits have intermediate conductivities, typically in the range 10^{-4} to 10^{+1} S/m.
- *Good conductors* are materials with very high conductivities, typically greater than 10^5 S/m. An important category of good conductors includes metals, with certain metals including alloys of aluminum, copper, and gold reaching conductivities on the order of 10^8 S/m. In such materials, minuscule electric fields give rise to large currents. There is no significant storage of energy in such materials, and so the concept of permittivity is not relevant for good conductors.

The reader should take care to note that terms such as *good conductor* and *poor insulator* are qualitative and subject to context. What may be considered a “good insulator” in one application may be considered to be a “poor insulator” in another.

One relevant category of material was not included in the above list – namely, *perfect conductors*. A perfect conductor is a material in which $\sigma \to \infty$. It is tempting to interpret this as meaning that any electric field gives rise to infinite current density; however, this is not plausible even in the ideal limit. Instead, this condition is interpreted as meaning that $\mathbf{E} = \mathbf{J}/\sigma \to 0$ throughout the material; i.e., $\mathbf{E}$ is zero independently of any current flow in the material. An important consequence is that the potential field V is equal to a constant value throughout the material. (Recall $\mathbf{E} = -\nabla V$ (Section 5.14), so constant V means $\mathbf{E} = 0$.) We refer to the volume occupied by such a material as an *equipotential volume*. This concept is useful as an approximation of the behavior of good conductors; for example, metals are often modeled as perfectly-conducting equipotential volumes in order to simplify analysis.

> A perfect conductor is a material for which $\sigma \to \infty$, $\mathbf{E} \to 0$, and subsequently V (the electric potential) is constant.

One final note: It is important to remain aware of the assumptions we have made about materials in this book, which are summarized in Section 2.8. In particular, we continue to assume that materials are linear unless otherwise indicated. For example, whereas air is normally considered to be a good insulator and therefore a poor conductor, anyone who has ever witnessed lightning has seen a demonstration that under the right conditions – i.e., sufficiently large potential difference between earth and sky – current will readily flow through air. This particular situation is known as *dielectric breakdown* (see Section 5.21). The non-linearity of materials can become evident before reaching the point of dielectric breakdown, so one must be careful to consider this possibility when dealing with strong electric fields.

Additional Reading:

- "Electrical Resistivity and Conductivity (https://en.wikipedia.org/wiki/Electrical_resistivity_and_conductivity)" on Wikipedia.
- "Ohm's Law (https://en.wikipedia.org/wiki/Ohm's_law)" on Wikipedia.

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1. See Section 5.20 for a discussion of dielectric materials. ↩
2. Review Sections 3.4, 3.9, and associated sections for a refresher on this issue. ↩

6.4 Resistance

The concept of resistance is most likely familiar to readers via *Ohm's Law for Devices*; i.e., $V = IR$ where V is the potential difference associated with a current I. This is correct, but it is not the whole story. Let's begin with a statement of intent:

> Resistance R (Ω) is a characterization of the conductivity of a *device* (as opposed to a *material*) in terms of Ohms Law for Devices; i.e., $V = IR$.

Resistance is a property of devices such as resistors, which are intended to provide resistance, as well as being a property of most practical electronic devices, whether it is desired or not.

Resistance is a manifestation of the conductivity of the materials comprising the device, which subsequently leads to the "$V = IR$" relationship. This brings us to a very important point and a common source of confusion. *Resistance is not necessarily the real part of impedance.* Let's take a moment to elaborate. Impedance (Z) is defined as the ratio of voltage to current; i.e., V/I; or equivalently in the phasor domain as $\widetilde{V}/\tilde{I}$. Most devices – not just devices exhibiting resistance – can be characterized in terms of this ratio. Consider for example the input impedance of a terminated transmission line (Section 3.15). This impedance may have a non-zero real-valued component even when the transmission line and the terminating load are comprised of perfect conductors. Summarizing:

> Resistance results in a real-valued impedance. However, not all devices exhibiting a real-valued impedance exhibit resistance. Furthermore, the real component of a complex-valued impedance does not necessarily represent resistance.

Restating the main point in yet other words: *Resistance pertains to limited conductivity, not simply to voltage-current ratio.*

Also important to realize is that whereas conductivity σ (units of S/m) is a property of materials, resistance depends on *both* conductivity and the geometry of the device. In this section, we address the question of how the resistance of a device can be is determined. The following example serves this purpose.

Figure 6.2 shows a straight wire of length l centered on the z axis, forming a cylinder of material having conductivity σ and cross-sectional radius a. The ends of the cylinder are covered by perfectly-conducting plates to which the terminals are attached. A battery is connected to the terminals, resulting in a uniform internal electric field $\mathbf{E}$ between the plates.

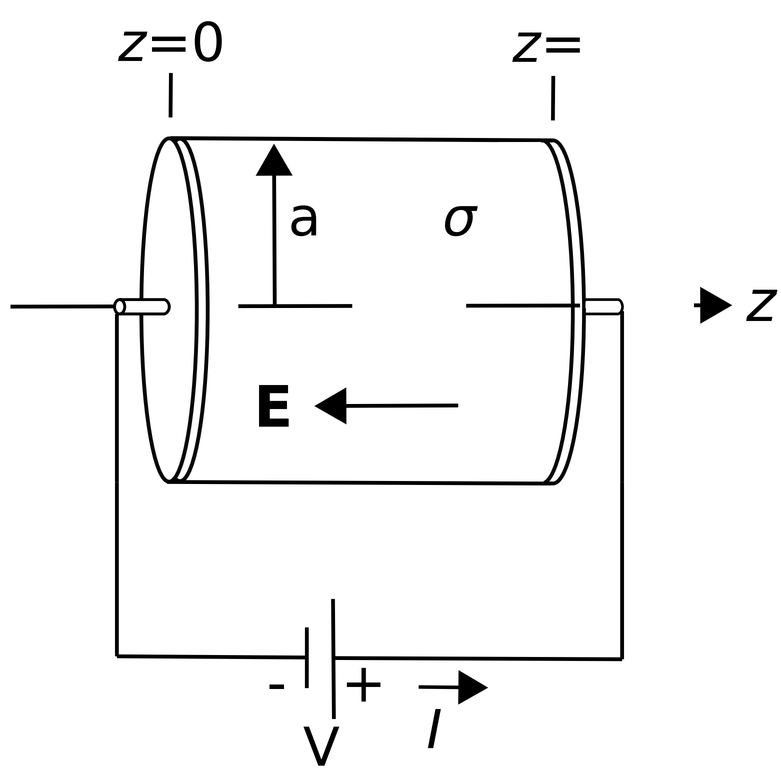

Figure 6.2. *Analysis of the resistance of straight wire of circular cross-section.*

To calculate R for this device, let us first calculate V, then I, and finally $R = V/I$. First, we compute the potential difference using the following result from Section 5.8:

$$V = -\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} \tag{6.5}$$

Here we can view V as a “given” (being the voltage of the battery), but wish to evaluate the right hand side so as to learn something about the effect of the conductivity and geometry of the wire. The appropriate choice of $\mathcal{C}$ begins at $z = 0$ and ends at $z = l$, following the axis of the cylinder. (Remember: $\mathcal{C}$ defines the reference direction for increasing potential, so the resulting potential difference will be the node voltage at the end point minus the node voltage at the start point.) Thus, $d\mathbf{l} = \hat{\mathbf{z}}dz$ and we have

$$V = -\int_{z=0}^{l} \mathbf{E} \cdot (\hat{\mathbf{z}}dz)$$

We do not yet know $\mathbf{E}$; however, we know it is constant throughout the device and points in the $-\hat{\mathbf{z}}$ direction since this is the direction of current flow and Ohm’s Law for Electromagnetics (Section 6.3) requires the electric field to point in the same direction. Thus, we may write $\mathbf{E} = -\hat{\mathbf{z}}E_z$ where E_z is a constant. We now find:

$$\begin{aligned} V &= -\int_{z=0}^{l} (-\hat{\mathbf{z}}E_z) \cdot (\hat{\mathbf{z}}dz) \\ &= +E_z \int_{z=0}^{l} dz \\ &= +E_z l \end{aligned}$$

The current I is given by (Section 6.2)

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{s}$$

We choose the surface $\mathcal{S}$ to be the cross-section perpendicular to the axis of the wire. Also, we choose $d\mathbf{s}$ to point such that I is positive with respect to the sign convention shown in Figure 6.2. With these choices, we have

$$I = \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \mathbf{J} \cdot (-\hat{\mathbf{z}}\ \rho\ d\rho\ d\phi)$$

From Ohm’s Law for Electromagnetics, we have

$$\mathbf{J} = \sigma\mathbf{E} = -\hat{\mathbf{z}}\sigma E_z \tag{6.6}$$

So now

$$\begin{aligned} I &= \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \left(-\hat{\mathbf{z}}\sigma E_z\right) \cdot \left(-\hat{\mathbf{z}}\rho\ d\rho\ d\phi\right) \\ &= \sigma E_z \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \rho\ d\rho\ d\phi \\ &= \sigma E_z \left(\pi a^2\right) \end{aligned}$$

Finally:

$$R = \frac{V}{I} = \frac{E_z l}{\sigma E_z \left(\pi a^2\right)} = \frac{l}{\sigma \left(\pi a^2\right)}$$

This is a good-enough answer for the problem posed, but it is easily generalized a bit further. Noting that πa^2 in the denominator is the cross-sectional area A of the wire, so we find:

$$\boxed{R = \frac{l}{\sigma A}} \tag{6.7}$$

i.e., the resistance of a wire having cross-sectional area A – regardless of the shape of the cross-section, is given by the above equation.

The resistance of a right cylinder of material, given by Equation 6.7, is proportional to length and inversely proportional to cross-sectional area and conductivity.

It is important to remember that Equation 6.7 presumes that the volume current density $\mathbf{J}$ is uniform over the cross-section of the wire. This is an excellent approximation for thin wires at "low" frequencies including, of course, DC. At higher frequencies it may not be a good assumption that $\mathbf{J}$ is uniformly-distributed over the cross-section of the wire, and at sufficiently high frequencies one finds instead that the current is effectively limited to the exterior surface of the wire. In the "high frequency" case, A in Equation 6.7 is reduced from the physical area to a smaller value corresponding to the reduced area through which most of the current flows. Therefore, R increases with increasing frequency. To quantify the high frequency behavior of R (including determination of what constitutes "high frequency" in this context) one requires concepts beyond the theory of electrostatics, so this is addressed elsewhere.

Example

Example 6.2

Resistance of 22 AWG hookup wire.

A common type of wire found in DC applications is 22AWG ("American Wire Gauge"; see "Additional Resources" at the end of this section) copper solid-conductor hookup wire. This type of wire has circular cross-section with diameter 0.644 mm. What is the resistance of 3 m of this wire? Assume copper conductivity of 58 MS/m.

Solution. From the problem statement, the diameter $2a = 0.644$ mm, $\sigma = 58 \times 10^6$ S/m, and $l = 3$ m. The cross-sectional area is $A = \pi a^2 \cong 3.26 \times 10^{-7}$ m^2. Using Equation 6.7 we obtain $R = 159$ mΩ.

Example

Example 6.3

Resistance of steel pipe.

A pipe is 3 m long and has inner and outer radii of 5 mm and 7 mm respectively. It is made from steel having conductivity 4 MS/m. What is the DC resistance of this pipe?

Solution. We can use Equation 6.7 if we can determine the cross-sectional area A through which the current flows. This area is simply the area defined by the outer radius, πb^2, minus the area defined by the inner radius πa^2. Thus, $A = \pi b^2 - \pi a^2 \cong 7.54 \times 10^{-5}$ m^2. From the problem statement, we also determine that $\sigma = 4 \times 10^6$ S/m and $l = 3$ m. Using Equation 6.7 we obtain $R \cong 9.95$ mΩ.

Additional Reading:

- "Resistor (https://en.wikipedia.org/wiki/Resistor)" on Wikipedia.
- "Ohm's Law (https://en.wikipedia.org/wiki/Ohm's_law)" on Wikipedia.
- "American wire gauge (https://en.wikipedia.org/wiki/American_wire_gauge)" on Wikipedia.

6.5 Conductance

Conductance, like resistance (Section 6.4), is a property of devices. Specifically:

Conductance G (Ω^{-1} or S) is the reciprocal of resistance R.

Therefore, conductance depends on both the conductivity of the materials used in the device, as well as the geometry of the device.

A natural question to ask is, why do we require the concept of conductance, if it simply the reciprocal of resistance? The short answer is that the concept of conductance is *not* required; or, rather, we need only resistance *or* conductance and not both. Nevertheless, the concept appears in engineering analysis for two reasons:

- Conductance is sometimes considered to be a more intuitive description of the underlying physics in cases where the applied voltage is considered to be the independent “stimulus” and current is considered to be the response. This is why conductance appears in the lumped element model for transmission lines (Section 3.4), for example.
- Characterization in terms of conductance may be preferred when considering the behavior of devices in parallel, since the conductance of a parallel combination is simply the sum of the conductances of the devices.

Example

Example 6.4

Conductance of a coaxial structure.

Let us now determine the conductance of a structure consisting of coaxially-arranged conductors separated by a lossy dielectric, as shown in Figure 6.3. The conductance per unit length G' (i.e., S/m) of this structure is of interest in determining the characteristic impedance of coaxial transmission line, as addressed in Sections 3.4 and 3.10.

For our present purposes, we may model the structure as two concentric perfectly-conducting cylinders of radii a and b, separated by a lossy dielectric having conductivity σ_s. We place the $+z$ axis along the common axis of the concentric cylinders so that the cylinders may be described as constant-coordinate surfaces $\rho = a$ and $\rho = b$.

There are at least 2 ways to solve this problem. One method is to follow the procedure that was used to find the capacitance of this structure in Section 5.24. Adapting that approach to the present problem, one would assume a potential difference V between the conductors, from that determine the resulting electric field intensity $\mathbf{E}$, and then using Ohm's Law for Electromagnetics (Section 6.3) determine the density $\mathbf{J} = \sigma_s \mathbf{E}$ of the current that leaks directly between conductors. From this, one is able to determine the total leakage current I, and subsequently the conductance $G \triangleq I/V$. Although highly recommended as an exercise for the student, in this section we take an alternative approach so as to demonstrate that there are a variety of approaches available for such problems.

The method we shall use below is as follows: (1) Assume a leakage current I between the conductors; (2) Determine the associated current density $\mathbf{J}$, which is possible using only geometrical considerations; (3) Determine the associated electric field intensity $\mathbf{E}$ using $\mathbf{J}/\sigma_s$; (4) Integrate $\mathbf{E}$ over a path between the conductors to get V. Then, as before, conductance $G \triangleq I/V$.

The current I is defined as shown in Figure 6.3, with reference direction according to the engineering convention that positive current flows out of the positive terminal of a source. The associated current density must flow in the same direction, and the circular symmetry of the problem therefore constrains $\mathbf{J}$ to have the form

$$\mathbf{J} = \hat{\rho}\frac{I}{A} \tag{6.8}$$

where A is the area through which I flows. In other words, current flows radially outward from the inner conductor to the outer conductor, with density that diminishes inversely with the area through which the total current flows. (It may be helpful to view $\mathbf{J}$ as a flux density and I

as a flux, as noted in Section 6.2.) This area is simply circumference $2\pi\rho$ times length l, so

$$\mathbf{J} = \hat{\rho}\frac{I}{2\pi\rho l} \tag{6.9}$$

which exhibits the correct units of A/m^2.

Now from Ohm's Law for Electromagnetics we find the electric field within the structure is

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma_s} = \hat{\rho}\frac{I}{2\pi\rho l\sigma_s} \tag{6.10}$$

Next we get V using (Section 5.8)

$$V = -\int_{\mathcal{C}} \mathbf{E}\cdot d\mathbf{l} \tag{6.11}$$

where $\mathcal{C}$ is any path from the negatively-charged outer conductor to the positively-charged inner conductor. Since this can be *any* such path (Section 5.9), we should choose the simplest one. The simplest path is the one that traverses a radial of constant ϕ and z. Thus:

$$\begin{aligned} V &= -\int_{\rho=b}^{a}\left(\hat{\rho}\frac{I}{2\pi\rho l\sigma_s}\right)\cdot(\hat{\rho}d\rho) \\ &= -\frac{I}{2\pi l\sigma_s}\int_{\rho=b}^{a}\frac{d\rho}{\rho} \\ &= +\frac{I}{2\pi l\sigma_s}\int_{\rho=a}^{b}\frac{d\rho}{\rho} \\ &= +\frac{I}{2\pi l\sigma_s}\ln\left(\frac{b}{a}\right) \end{aligned} \tag{6.12}$$

Wrapping up:

$$G \triangleq \frac{I}{V} = \frac{I}{(I/2\pi l\sigma_s)\ln(b/a)} \tag{6.13}$$

Note that factors of I in the numerator and denominator cancel out, leaving:

$$\boxed{G = \frac{2\pi l\sigma_s}{\ln(b/a)}} \tag{6.14}$$

Note that Equation 6.14 is dimensionally correct, having units of S $= \Omega^{-1}$. Also note that this is expression depends only on materials (through σ_s) and geometry (through l, a, and b). Notably it does *not* depend on current or voltage, which would imply non-linear behavior.

To make the connection back to lumped-element transmission line model parameters (Sections 3.4 and 3.10), we simply divide by l to get the per-length parameter:

$$\boxed{G' = \frac{2\pi\sigma_s}{\ln\left(b/a\right)}} \tag{6.15}$$

Figure 6.3. *Determining the conductance of a structure consisting of coaxially-arranged conductors separated by a lossy dielectric.*

Example

Example 6.5

Conductance of RG-59 coaxial cable.

RG-59 coaxial cable consists of an inner conductor having radius 0.292 mm, an outer conductor having radius 1.855 mm, and a polyethylene spacing material exhibiting conductivity of about 5.9×10^{-5} S/m. Estimate the conductance per length of RG-59.

Solution. From the problem statement, $a = 0.292$ mm, $b = 1.855$ mm, and $\sigma_s \cong 5.9 \times 10^{-5}$ S/m. Using Equation 6.15, we find $G' \cong 200$ μS/m.

6.6 Power Dissipation in Conducting Media

The displacement of charge in response to the force exerted by an electric field constitutes a reduction in the potential energy of the system (Section 5.8). If the charge is part of a steady current, there must be an associated loss of energy that occurs at a steady rate. Since power is energy per unit time, the loss of energy associated with current is expressible as power dissipation. In this section, we address two questions: (1) How much power is dissipated in this manner, and (2) What happens to the lost energy?

First, recall that *work* is force times distance traversed in response to that force (Section 5.8). Stated mathematically:

$$\Delta W = +\mathbf{F} \cdot \Delta \mathbf{l} \tag{6.16}$$

where the vector $\mathbf{F}$ is the force (units of N) exerted by the electric field, the vector $\Delta\mathbf{l}$ is the direction and distance (units of m) traversed, and ΔW is the work done (units of J) as a result. Note that a “$+$” has been explicitly indicated; this is to emphasize the distinction from the work being considered in Section 5.8. In that section, the work “ΔW” represented energy from an external source that was being used to increase the potential energy of the system by moving charge “upstream” relative to the electric field. Now, ΔW represents this internal energy as it is escaping from the system in the form of kinetic energy; therefore, positive ΔW now means a reduction in potential energy, hence the sign change.[1]

The associated power ΔP (units of W) is ΔW divided by the time Δt (units of s) required for the distance $\Delta\mathbf{l}$ to be traversed:

$$\Delta P = \frac{\Delta W}{\Delta t} = \mathbf{F} \cdot \frac{\Delta \mathbf{l}}{\Delta t} \tag{6.17}$$

Now we’d like to express force in terms of the electric field exerting this force. Recall that the force exerted by an electric field intensity $\mathbf{E}$ (units of V/m) on a particle bearing charge q (units of C) is $q\mathbf{E}$ (Section 2.2). However, we’d like to express this force in terms of a current, as opposed to a charge. An expression in terms of current can be constructed as follows. First, note that the total charge in a small volume “cell” is the volume charge density ρ_v (units of C/m^3) times the volume Δv of the cell; i.e., $q = \rho_v \Delta v$ (Section 5.3). Therefore:

$$\mathbf{F} = q\mathbf{E} = \rho_v \ \Delta v \ \mathbf{E} \tag{6.18}$$

and subsequently

$$\Delta P = \rho_v \ \Delta v \ \mathbf{E} \cdot \frac{\Delta \mathbf{l}}{\Delta t} = \mathbf{E} \cdot \left(\rho_v \frac{\Delta \mathbf{l}}{\Delta t} \right) \Delta v \tag{6.19}$$

The quantity in parentheses has units of C/m^3 $\cdot$ m $\cdot$ s^{-1}, which is A/m^2. Apparently this quantity is the volume current density $\mathbf{J}$, so we have

$$\Delta P = \mathbf{E} \cdot \mathbf{J} \ \Delta v \tag{6.20}$$

In the limit as $\Delta v \to 0$ we have

$$dP = \mathbf{E} \cdot \mathbf{J} \ dv \tag{6.21}$$

and integrating over the volume $\mathcal{V}$ of interest we obtain

$$P = \int_{\mathcal{V}} dP = \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J} \; dv \tag{6.22}$$

The above expression is commonly known as *Joule's Law*. In our situation, it is convenient to use Ohm's Law for Electromagnetics ($\mathbf{J} = \sigma\mathbf{E}$; Section 6.3) to get everything in terms of materials properties (σ), geometry ($\mathcal{V}$), and the electric field:

$$P = \int_{\mathcal{V}} \mathbf{E} \cdot (\sigma\mathbf{E}) \;\; dv \tag{6.23}$$

which is simply

$$\boxed{P = \int_{\mathcal{V}} \sigma \left|\mathbf{E}\right|^2 \;\; dv} \tag{6.24}$$

Thus:

The power dissipation associated with current is given by Equation 6.24. This power is proportional to conductivity and proportional to the electric field magnitude squared.

This result facilitates the analysis of power dissipation in materials exhibiting loss; i.e., having finite conductivity. But what is the power dissipation in a perfectly conducting material? For such a material, $\sigma \rightarrow \infty$ and $\mathbf{E} \rightarrow 0$ no matter how much current is applied (Section 6.3). In this case, Equation 6.24 is not very helpful. However, as we just noted, being a perfect conductor means $\mathbf{E} \rightarrow 0$ no matter how much current is applied, so from Equation 6.22 we have found that:

No power is dissipated in a perfect conductor.

When conductivity is finite, Equation 6.24 serves as a more-general version of a concept from elementary circuit theory, as we shall now demonstrate. Let $\mathbf{E} = \hat{\mathbf{z}}E_z$, so $\left|\mathbf{E}\right|^2 = E_z^2$. Then Equation 6.24 becomes:

$$P = \int_{\mathcal{V}} \sigma E_z^2 \; dv = \sigma E_z^2 \; \int_{\mathcal{V}} \; dv \tag{6.25}$$

The second integral in Equation 6.25 is a calculation of volume. Let's assume $\mathcal{V}$ is a cylinder aligned along the z axis. The volume of this cylinder is the cross-sectional area A times the length l. Then the above equation becomes:

$$P = \sigma E_z^2 \ A \ l \tag{6.26}$$

For reasons that will become apparent very shortly, let's reorganize the above expression as follows:

$$P = \left(\sigma E_z A\right)\left(E_z l\right) \tag{6.27}$$

Note that σE_z is the current density in A/m^2, which when multiplied by A gives the total current. Therefore, the quantity in the first set of parentheses is simply the current I. Also note that $E_z l$ is the potential difference over the length l, which is simply the node-to-node voltage V (Section 5.8). Therefore, we have found:

$$P = IV \tag{6.28}$$

as expected from elementary circuit theory.

Now, what happens to the energy associated with this dissipation of power? The displacement of charge carriers in the material is limited by the conductivity, which itself is finite because, simply put, other constituents of the material get in the way. If charge is being displaced as described in this section, then energy is being used to displace the charge-bearing particles. The motion of constituent particles is observed as heat – in fact, this is essentially the definition of heat. Therefore:

> The power dissipation associated with the flow of current in any material that is not a perfect conductor manifests as heat.

This phenomenon is known as *joule heating*, *ohmic heating*, and by other names. This conversion of electrical energy to heat is the method of operation for toasters, electric space heaters, and many other devices that generate heat. It is of course also the reason that all practical electronic devices generate heat.

Additional Reading:

- "Joule Heating (https://en.wikipedia.org/wiki/Joule_heating)" on Wikipedia.

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1. It could be argued that it is bad form to use the same variable to represent both tallies; nevertheless, it is common practice and so we simply remind the reader that it is important to be aware of the definitions of variables each time they are (re)introduced. ↩

End Matter

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