

5 Electrostatics

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Electrostatics is the theory of the electric field in conditions in which its behavior is independent of magnetic fields, including

- The electric field associated with fixed distributions of electric charge
- *Capacitance* (the ability of a structure to store energy in an electric field)
- The *energy* associated with the electrostatic field
- *Steady current* induced in a conducting material in the presence of an electrostatic field (essentially, Ohm's Law)

The term “static” refers to the fact that these aspects of electromagnetic theory can be developed by assuming sources are time-invariant; we might say that electrostatics is the study of the electric field at DC. However, many aspects of electrostatics are relevant to AC, radio frequency, and higher-frequency applications as well.

Additional Reading:

- “[Electrostatics](https://en.wikipedia.org/wiki/Electrostatics) (<https://en.wikipedia.org/wiki/Electrostatics>)” on Wikipedia.

5.1 Coulomb's Law

Consider two charge-bearing particles in free space, identified as “particle 1” and “particle 2” in Figure 5.1. Let the charges borne by these particles be q_1 and q_2 , and let R be the distance between them. If the particles bear charges of the same sign (i.e., if $q_1 q_2$ is positive), then the particles repel; otherwise, they attract. This repulsion or attraction can be quantified as a force experienced by each particle. Physical observations reveal that the magnitude of the force is proportional to $q_1 q_2$, and inversely proportional to R^2 . For particle 2 we find:

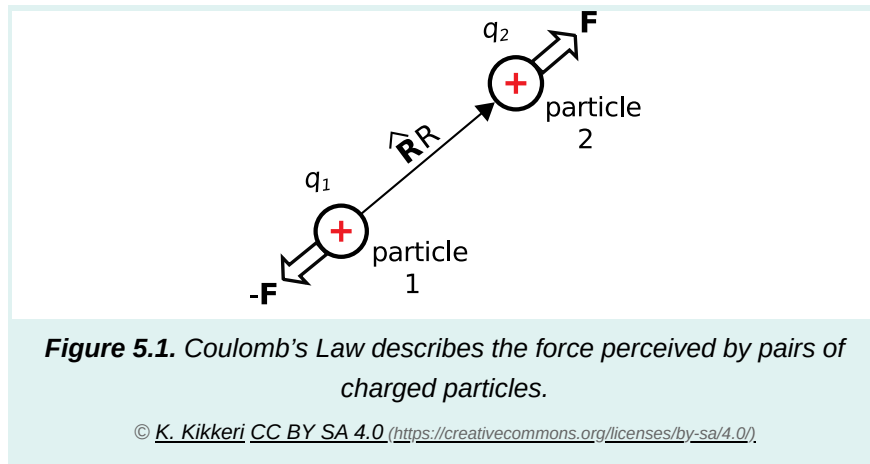
$$\mathbf{F} = \hat{\mathbf{R}} F_0 \frac{q_1 q_2}{R^2} \quad (5.1)$$

where $\hat{\mathbf{R}}$ is the unit vector pointing from the particle 1 to the particle 2, and F_0 is a constant. The value of F_0 must have units of inverse permittivity; i.e., $(\text{F/m})^{-1}$. This is most easily seen by dimensional analysis of the above relationship, including the suspected factor:

$$\frac{C \cdot C}{F/m \cdot m^2} = \frac{C \cdot C}{F \cdot m} = \frac{C \cdot C}{C/V \cdot m} = \frac{C \cdot V}{m} = \frac{J}{m} = N$$

where we have used the facts that $1 F = 1 C/V$, $1 V = 1 J/C$, and $1 N = 1 J/m$. This finding suggests that $F_0 \propto \epsilon^{-1}$, where ϵ is the permittivity of the medium in which the particles exist. Observations confirm that the force is in fact inversely proportional to the permittivity, with an additional factor of $1/4\pi$ (unitless). Putting this all together we obtain what is commonly known as *Coulomb's Law*:

$$\mathbf{F} = \hat{\mathbf{R}} \frac{q_1 q_2}{4\pi\epsilon R^2} \quad (5.2)$$



Subsequently, the force perceived by particle 2 is equal and opposite; i.e., equal to $-\mathbf{F}$.

Separately, it is known that $\mathbf{F}$ can be described in terms of the electric field intensity $\mathbf{E}_1$ associated with particle 1:

$$\mathbf{F} = q_2 \mathbf{E}_1 \quad (5.3)$$

This is essentially the *definition* of $\mathbf{E}_1$, as explained in Section 2.2. Combining this result with Coulomb's Law, we obtain a means to directly calculate the field associated with the first particle in the absence of the second particle:

$$\mathbf{E}_1 = \hat{\mathbf{R}} \frac{q_1}{4\pi\epsilon R^2} \quad (5.4)$$

where now $\hat{\mathbf{R}}$ is the vector beginning at the particle 1 and ending at the point to be evaluated.

The electric field intensity associated with a point charge (Equation 5.4) is (1) directed away from positive charge, (2) proportional to the magnitude of the charge, (3) inversely proportional to the permittivity of the medium, and (3) inversely proportional to distance squared.

We have described this result as originating from Coulomb's Law, which is based on physical observations. However, the same result may be obtained directly from Maxwell's Equations using Gauss' Law (Section 5.5).

Example

Example 5.1

Electric field of a point charge at the origin.

A common starting point in electrostatic analysis is the field associated with a particle bearing charge q at the origin of the coordinate system. Because the electric field is directed radially away from a positively-charged source particle in all directions, this field is most conveniently described in the spherical coordinate system. Thus, $\hat{\mathbf{R}}$ becomes $\hat{\mathbf{r}}$, R becomes r , and we have

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{r}} \frac{q}{4\pi\epsilon r^2} \quad (5.5)$$

Here's a numerical example. What is the electric field intensity at a distance $1 \mu\text{m}$ from a single electron located at the origin, in free space? In this case, $q \cong -1.60 \times 10^{-19} \text{ C}$ (don't forget that minus sign!), $\epsilon = \epsilon_0$, $r = 1 \mu\text{m}$, and we find:

$$\mathbf{E}(\mathbf{r}) = -\hat{\mathbf{r}} (1.44 \text{ kV/m}) \quad (5.6)$$

This is large relative to electric field strengths commonly encountered in engineering applications. The strong electric field of the electron is not readily apparent because electrons in common materials tend to be accompanied by roughly equal amounts of positive charge, such as the protons of atoms. Sometimes, however, the effect of individual electrons *does* become significant in practical electronics through a phenomenon known as *shot noise*.

Additional Reading:

- “Coulomb’s Law (https://en.wikipedia.org/wiki/Coulomb's_law)” on Wikipedia.
- “Shot Noise (https://en.wikipedia.org/wiki/Shot_noise)” on Wikipedia.

5.2 Electric Field Due to Point Charges

The electric field intensity associated with a single particle bearing charge q_1 , located at the origin, is (Section [5.1](#))

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{r}} \frac{q_1}{4\pi\epsilon r^2} \quad (5.7)$$

If this particle is instead located at some position $\mathbf{r}_1$, then the above expression may be written as follows:

$$\mathbf{E}(\mathbf{r}; \mathbf{r}_1) = \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|} \frac{q_1}{4\pi\epsilon |\mathbf{r} - \mathbf{r}_1|^2} \quad (5.8)$$

or, combining like terms in the denominator:

$$\mathbf{E}(\mathbf{r}; \mathbf{r}_1) = \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3} \frac{q_1}{4\pi\epsilon} \quad (5.9)$$

Now let us consider the field due to multiple such particles. Under the usual assumptions about the permittivity of the medium (reminder: Section [2.8](#)), the property of superposition applies. Using this principle, we conclude:

The electric field resulting from a set of charged particles is equal to the sum of the fields associated with the individual particles.

Stated mathematically:

$$\mathbf{E}(\mathbf{r}) = \sum_{n=1}^N \mathbf{E}(\mathbf{r}; \mathbf{r}_n) \quad (5.10)$$

where N is the number of particles. Thus, we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} q_n \quad (5.11)$$

5.3 Charge Distributions

In principle, the smallest unit of electric charge that can be isolated is the charge of a single electron, which is $\cong -1.60 \times 10^{-19}$ C. This is very small, and we rarely deal with electrons one at a time, so it is usually more convenient to describe charge as a quantity that is continuous over some region of space. In particular, it is convenient to describe charge as being distributed in one of three ways: along a curve, over a surface, or within a volume.

Line Charge Distribution. Imagine that charge is distributed along a curve $\mathcal{C}$ through space. Let Δq be the total charge along a short segment of the curve, and let Δl be the length of this segment. The *line charge density* ρ_l at any point along the curve is defined as

$$\rho_l \triangleq \lim_{\Delta l \rightarrow 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad (5.12)$$

which has units of C/m. We may then define ρ_l to be a function of position along the curve, parameterized by l ; e.g., $\rho_l(l)$. Then, the total charge Q along the curve is

$$Q = \int_{\mathcal{C}} \rho_l(l) dl \quad (5.13)$$

which has units of C. In other words, line charge density integrated over length yields total charge.

Surface Charge Distribution. Imagine that charge is distributed over a surface. Let Δq be the total charge on a small patch on this surface, and let Δs be the area of this patch. The *surface charge density* ρ_s at any point on the surface is defined as

$$\rho_s \triangleq \lim_{\Delta s \rightarrow 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad (5.14)$$

which has units of C/m². Let us define ρ_s to be a function of position on this surface. Then the total charge over a surface $\mathcal{S}$ is

$$Q = \int_{\mathcal{S}} \rho_s ds \quad (5.15)$$

In other words, surface charge density integrated over a surface yields total charge.

Volume Charge Distribution. Imagine that charge is distributed over a volume. Let Δq be the total charge in a small cell within this volume, and let Δv be the volume of this cell. The *volume charge density* ρ_v at any point in the volume is defined as

$$\rho_v \triangleq \lim_{\Delta v \rightarrow 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \quad (5.16)$$

which has units of C/m³. Since ρ_v is a function of position within this volume, the total charge within a volume $\mathcal{V}$ is

$$Q = \int_{\mathcal{V}} \rho_v dv \quad (5.17)$$

In other words, volume charge density integrated over a volume yields total charge.

5.4 Electric Field Due to a Continuous Distribution of Charge

The electric field intensity associated with N charged particles is (Section 5.2):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} q_n \quad (5.18)$$

where q_n and $\mathbf{r}_n$ are the charge and position of the n^{th} particle. However, it is common to have a continuous distribution of charge as opposed to a countable number of charged particles. In this section, we extend Equation 5.18 using the concept of continuous distribution of charge (Section 5.3) so that we may address this more general class of problems.

Distribution of Charge Along a Curve. Consider a continuous distribution of charge along a curve $\mathcal{C}$. The curve can be divided into short segments of length Δl . Then, the charge associated with the n^{th} segment, located at $\mathbf{r}_n$, is

$$q_n = \rho_l(\mathbf{r}_n) \Delta l \quad (5.19)$$

where ρ_l is charge density (units of C/m) at $\mathbf{r}_n$. Substituting this expression into Equation 5.18, we obtain

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} \rho_l(\mathbf{r}_n) \Delta l \quad (5.20)$$

Taking the limit as $\Delta l \rightarrow 0$ yields:

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{\mathcal{C}} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho_l(\mathbf{r}') dl} \quad (5.21)$$

where $\mathbf{r}'$ represents the varying position along $\mathcal{C}$ with integration.

The simplest example of a curve is a straight line. It is straightforward to use Equation [5.21](#) to determine the electric field due to a distribution of charge along a straight line. However, it is much easier to analyze that particular distribution using Gauss' Law, as shown in Section [5.6](#). The following example addresses a charge distribution for which Equation [5.21](#) is more appropriate.

Example

Example 5.2

Electric field along the axis of a ring of uniformly-distributed charge.

Consider a ring of radius a in the $z = 0$ plane, centered on the origin, as shown in Figure 5.2. Let the charge density along this ring be uniform and equal to ρ_l (C/m). Find the electric field along the z axis.

Solution. The source charge position is given in cylindrical coordinates as

$$\mathbf{r}' = \hat{\rho}a \quad (5.22)$$

The position of a field point along the z axis is simply

$$\mathbf{r} = \hat{\mathbf{z}}z \quad (5.23)$$

Thus,

$$\mathbf{r} - \mathbf{r}' = -\hat{\rho}a + \hat{\mathbf{z}}z \quad (5.24)$$

and

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{a^2 + z^2} \quad (5.25)$$

Equation 5.21 becomes:

$$\mathbf{E}(z) = \frac{1}{4\pi\epsilon} \int_0^{2\pi} \frac{-\hat{\rho}a + \hat{\mathbf{z}}z}{[a^2 + z^2]^{3/2}} \rho_l (a d\phi) \quad (5.26)$$

Pulling factors that do not vary with ϕ out of the integral and factoring into separate integrals for the $\hat{\phi}$ and $\hat{\mathbf{z}}$ components, we obtain:

$$\frac{\rho_l a}{4\pi\epsilon [a^2 + z^2]^{3/2}} \left[-a \int_0^{2\pi} \hat{\rho} d\phi + \hat{\mathbf{z}}z \int_0^{2\pi} d\phi \right] \quad (5.27)$$

The second integral is equal to 2π . The first integral is equal to zero. To see this, note that the integral is simply summing values of $\hat{\rho}$ for all possible values of ϕ . Since $\hat{\rho}(\phi + \pi) = -\hat{\rho}(\phi)$, the integrand for any given value of ϕ is equal and opposite the integrand π radians later. (This is one example of a *symmetry* argument.) Thus, we obtain

$$\mathbf{E}(z) = \hat{\mathbf{z}} \frac{\rho_l a}{2\epsilon} \frac{z}{[a^2 + z^2]^{3/2}} \quad (5.28)$$

It is a good exercise to confirm that this result is dimensionally correct. It is also recommended to confirm that when $z \gg a$, the result is approximately the same as that expected from a particle having the same total charge as the ring.

Figure 5.2. Calculating the electric field along the axis of a ring of charge.

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Distribution of Charge Over a Surface. Consider a continuous distribution of charge over a surface $\mathcal{S}$. The surface can be divided into small patches having area Δs . Then, the charge associated with the n^{th} patch, located at $\mathbf{r}_n$, is

$$q_n = \rho_s(\mathbf{r}_n) \Delta s \quad (5.29)$$

where ρ_s is the surface charge density (units of C/m²) at $\mathbf{r}_n$. Substituting this expression into Equation 5.18, we obtain

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} \rho_s(\mathbf{r}_n) \Delta s \quad (5.30)$$

Taking the limit as $\Delta s \rightarrow 0$ yields:

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{\mathcal{S}} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho_s(\mathbf{r}') ds} \quad (5.31)$$

where $\mathbf{r}'$ represents the varying position over $\mathcal{S}$ with integration.

Example

Example 5.3

Electric field along the axis of a disk of uniformly-distributed charge.

Consider a circular disk of radius a in the $z = 0$ plane, centered on the origin, as shown in Figure 5.3. Let the charge density over this disk be uniform and equal to ρ_s (C/m²). Find the electric field along the z axis.

Solution. The source charge position is given in cylindrical coordinates as

$$\mathbf{r}' = \hat{\rho}\rho \quad (5.32)$$

The position of a field point along the z axis is simply

$$\mathbf{r} = \hat{\mathbf{z}}z \quad (5.33)$$

Thus,

$$\mathbf{r} - \mathbf{r}' = -\hat{\rho}\rho + \hat{\mathbf{z}}z \quad (5.34)$$

and

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{\rho^2 + z^2} \quad (5.35)$$

Equation 5.31 becomes:

$$\mathbf{E}(z) = \frac{1}{4\pi\epsilon} \int_{\rho=0}^a \int_{\phi=0}^{2\pi} \frac{-\hat{\rho}\rho + \hat{\mathbf{z}}z}{[\rho^2 + z^2]^{3/2}} \rho_s (\rho d\rho d\phi) \quad (5.36)$$

To solve this integral, first rearrange the double integral into a single integral over ϕ followed by integration over ρ :

$$\frac{\rho_s}{4\pi\epsilon} \int_{\rho=0}^a \frac{\rho}{[\rho^2 + z^2]^{3/2}} \left[\int_{\phi=0}^{2\pi} (-\hat{\rho}\rho + \hat{\mathbf{z}}z) d\phi \right] d\rho \quad (5.37)$$

Now we address the integration over ϕ shown in the square brackets in the above expression:

$$\int_{\phi=0}^{2\pi} (-\hat{\rho}\rho + \hat{z}z) d\phi = -\rho \int_{\phi=0}^{2\pi} \hat{\rho} d\phi + \hat{z}z \int_{\phi=0}^{2\pi} d\phi \quad (5.38)$$

The first integral on the right is zero for the following reason. As the integral progresses in ϕ , the vector $\hat{\rho}$ rotates. Because the integration is over a complete revolution (i.e., ϕ from 0 to 2π), the contribution from each pointing of $\hat{\rho}$ is canceled out by another pointing of $\hat{\rho}$ that is in the opposite direction. Since there is an equal number of these canceling pairs of pointings, the result is zero. Thus:

$$\begin{aligned} \int_{\phi=0}^{2\pi} (-\hat{\rho}\rho + \hat{z}z) d\phi &= 0 + \hat{z}z \int_{\phi=0}^{2\pi} d\phi \\ &= \hat{z}2\pi z \end{aligned} \quad (5.39)$$

Substituting this into Expression [5.37](#) we obtain:

$$\begin{aligned} &\frac{\rho_s}{4\pi\epsilon} \int_{\rho=0}^a \frac{\rho}{[\rho^2 + z^2]^{3/2}} [\hat{z}2\pi z] d\rho \\ &= \hat{z} \frac{\rho_s z}{2\epsilon} \int_{\rho=0}^a \frac{\rho d\rho}{[\rho^2 + z^2]^{3/2}} \end{aligned} \quad (5.40)$$

This integral can be solved using integration by parts and trigonometric substitution. Since the solution is tedious and there is no particular principle of electromagnetics demonstrated by this solution, we shall simply state the result:

$$\begin{aligned} \int_{\rho=0}^a \frac{\rho d\rho}{[\rho^2 + z^2]^{3/2}} &= \left. \frac{-1}{\sqrt{\rho^2 + z^2}} \right|_{\rho=0}^a \\ &= \frac{-1}{\sqrt{a^2 + z^2}} + \frac{1}{|z|} \end{aligned} \quad (5.41)$$

Substituting this result:

$$\begin{aligned} \mathbf{E}(z) &= \hat{z} \frac{\rho_s z}{2\epsilon} \left(\frac{-1}{\sqrt{a^2 + z^2}} + \frac{1}{|z|} \right) \\ &= \hat{z} \frac{\rho_s}{2\epsilon} \left(\frac{-z}{\sqrt{a^2 + z^2}} + \frac{z}{|z|} \right) \\ &= \hat{z} \frac{\rho_s}{2\epsilon} \left(\frac{-z}{\sqrt{a^2 + z^2}} + \text{sgn } z \right) \end{aligned} \quad (5.42)$$

where “sgn” is the “signum” function; i.e., $\text{sgn } z = +1$ for $z > 0$ and $\text{sgn } z = -1$ for $z < 0$.
Summarizing:

$$\mathbf{E}(z) = \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon} \left(\operatorname{sgn} z - \frac{z}{\sqrt{a^2 + z^2}} \right) \quad (5.43)$$

It is a good exercise to confirm that this result is dimensionally correct and yields an electric field vector that points in the expected direction and with the expected dependence on a and z .

Figure 5.3. Calculating the electric field along the axis of a disk of charge.

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A special case of the “disk of charge” scenario considered in the preceding example is an *infinite sheet* of charge. The electric field from an infinite sheet of charge is a useful theoretical result. We get the field in this case simply by letting $a \rightarrow \infty$ in Equation 5.43, yielding:

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon} \operatorname{sgn} z \quad (5.44)$$

Again, it is useful to confirm that this is dimensionally correct: C/m² divided by F/m yields V/m. Also, note that Equation 5.44 is the electric field at *any* point above or below the charge sheet – not just on z axis. This follows from symmetry. From the perspective of any point in space, the edges of the sheet are the same distance (i.e., infinitely far) away.

Distribution of Charge In a Volume. Consider a continuous distribution of charge within a volume $\mathcal{V}$. The volume can be divided into small cells (volume elements) having volume Δv . Then, the charge associated with the n^{th} cell, located at $\mathbf{r}_n$, is

$$q_n = \rho_v(\mathbf{r}_n) \Delta v \quad (5.45)$$

where ρ_v is volume charge density (units of C/m³) at $\mathbf{r}_n$. Substituting this expression into Equation 5.18, we obtain

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} \rho_v(\mathbf{r}_n) \Delta v \quad (5.46)$$

Taking the limit as $\Delta v \rightarrow 0$ yields:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{\mathcal{V}} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho_v(\mathbf{r}') dv \quad (5.47)$$

where $\mathbf{r}'$ represents the varying position over $\mathcal{V}$ with integration.

5.5 Gauss' Law: Integral Form

Gauss' Law is one of the four fundamental laws of classical electromagnetics, collectively known as *Maxwell's Equations*. Before diving in, the reader is strongly encouraged to review Section [2.4](#). In that section, Gauss' Law emerges from the interpretation of the electric field as a flux density. Section [2.4](#) does not actually identify Gauss' Law, but here it is:

Gauss' Law (Equation [5.48](#)) states that the flux of the electric field through a closed surface is equal to the enclosed charge.

Gauss' Law is expressed mathematically as follows:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{encl} \quad (5.48)$$

where $\mathbf{D}$ is the electric flux density $\epsilon\mathbf{E}$, S is a closed surface with differential surface normal $d\mathbf{s}$, and Q_{encl} is the enclosed charge. We can see the law is dimensionally correct; $\mathbf{D}$ has units of C/m^2 , thus integrating $\mathbf{D}$ over a surface gives a quantity with units of $\text{C/m}^2 \cdot \text{m}^2 = \text{C}$, which are the units of charge.

Gauss' Law has a number of applications in electromagnetic theory. One of them, as explored below, is as a method to compute the electric field in response to a distribution of electric charge. Note that a method to do this, based on Coulomb's Law, is described in Sections [5.1](#), [5.2](#), and [5.4](#). Gauss' Law provides an alternative method that is easier or more useful in certain applications.

Example

Example 5.4

Electric field associated with a charged particle, using Gauss' Law.

In this example, we demonstrate the ability of Gauss' Law to predict the field associated with a charge distribution. Let us do this for the simplest possible charge distribution. A particle of charge q located at the origin, for which we already have the answer (Section 5.1).

Solution. Gauss' Law applies to *any* surface that encloses the charge, so for simplicity we chose a sphere of radius r centered at the origin. Note that Q_{encl} on the right hand side is just q for any surface having $r > 0$. Gauss' Law in this case becomes

$$\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \mathbf{D} \cdot (\hat{\mathbf{r}} r^2 \sin \theta \, d\theta \, d\phi) = q \quad (5.49)$$

If we can solve for $\mathbf{D}$, we can get $\mathbf{E}$ using $\mathbf{D} = \epsilon \mathbf{E}$. The simplest way to solve for $\mathbf{D}$ is to use a symmetry argument, which proceeds as follows. In this problem, the *magnitude* of $\mathbf{D}$ can depend only on r , and not θ or ϕ . This is because the charge has no particular orientation, and the sphere is centered on the charge. Similarly, it is clear that $\mathbf{D}$ must point either directly toward or directly away from the charge. In other words, $\mathbf{D} = \hat{\mathbf{r}} D(r)$. Substituting this in the above equation, we encounter the dot product $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}$, which is simply 1. Since $D(r)$ and r^2 are constants with respect to the integration, we obtain:

$$r^2 D(r) \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin \theta \, d\theta \, d\phi = q \quad (5.50)$$

The remaining integral is simply 4π , thus we obtain:

$$D(r) = \frac{q}{4\pi r^2} \quad (5.51)$$

Bringing the known vector orientation of $\mathbf{D}$ back into the equation, we obtain

$$\mathbf{D} = \hat{\mathbf{r}} \frac{q}{4\pi r^2} \quad (5.52)$$

and finally using $\mathbf{D} = \epsilon \mathbf{E}$ we obtain the expected result

$$\mathbf{E} = \hat{\mathbf{r}} \frac{q}{4\pi\epsilon r^2} \quad (5.53)$$

Here's the point you should take away from the above example:

Gauss' Law combined with a symmetry argument may be sufficient to determine the electric field due to a charge distribution. Thus, Gauss' Law may be an easier alternative to Coulomb's Law in some applications.

Additional Reading:

- “Gauss' Law (https://en.wikipedia.org/wiki/Gauss'_law),” on Wikipedia.

5.6 Electric Field Due to an Infinite Line Charge using Gauss' Law

Section [5.5](#) explains one application of Gauss' Law, which is to find the electric field due to a charged particle. In this section, we present another application – the electric field due to an infinite line of charge. The result serves as a useful “building block” in a number of other problems, including determination of the capacitance of coaxial cable (Section [5.24](#)). Although this problem can be solved using the “direct” approach described in Section [5.4](#) (and it is an excellent exercise to do so), the Gauss' Law approach demonstrated here turns out to be relatively simple.

Example

Example 5.5

Electric field associated with an infinite line charge, using Gauss' Law.

Use Gauss' Law to determine the electric field intensity due to an infinite line of charge along the z axis, having charge density ρ_l (units of C/m), as shown in Figure 5.4.

Solution. Gauss' Law requires integration over a surface that encloses the charge. So, our first problem is to determine a suitable surface. A cylinder of radius a that is concentric with the z axis, as shown in Figure 5.4, is maximally symmetric with the charge distribution and so is likely to yield the simplest possible analysis. At first glance, it seems that we may have a problem since the charge extends to infinity in the $+z$ and $-z$ directions, so it's not clear how to enclose all of the charge. Let's suppress that concern for a moment and simply choose a cylinder of finite length l . In principle, we can solve the problem first for this cylinder of finite size, which contains only a fraction of the charge, and then later let $l \rightarrow \infty$ to capture the rest of the charge. (In fact, we'll find when the time comes it will not be necessary to do that, but we shall prepare for it anyway.)

Here's Gauss' Law:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{encl} \quad (5.54)$$

where $\mathbf{D}$ is the electric flux density $\epsilon\mathbf{E}$, S is a closed surface with outward-facing differential surface normal $d\mathbf{s}$, and Q_{encl} is the enclosed charge.

The first order of business is to constrain the form of $\mathbf{D}$ using a symmetry argument, as follows. Consider the field of a point charge q at the origin (Section 5.5):

$$\mathbf{D} = \hat{\mathbf{r}} \frac{q}{4\pi r^2} \quad (5.55)$$

We can "assemble" an infinite line of charge by adding particles in pairs. One pair is added at a time, with one particle on the $+z$ axis and the other on the $-z$ axis, with each located an equal distance from the origin. We continue to add particle pairs in this manner until the resulting charge extends continuously to infinity in both directions. The principle of superposition indicates that the resulting field will be the sum of the fields of the particles (Section 5.2). Thus, we see that $\mathbf{D}$ cannot have any component in the $\hat{\phi}$ direction because none of the fields of the constituent particles have a component in that direction. Similarly, we see that the magnitude of $\mathbf{D}$ cannot depend on ϕ because none of the fields of the constituent particles depends on ϕ and because the charge distribution is identical

(“invariant”) with rotation in ϕ . Also, note that for any choice of z the distribution of charge above and below that plane of constant z is identical; therefore, $\mathbf{D}$ cannot be a function of z and $\mathbf{D}$ cannot have any component in the $\hat{z}$ direction. Therefore, the direction of $\mathbf{D}$ must be radially outward; i.e., in the $\hat{\rho}$ direction, as follows:

$$\mathbf{D} = \hat{\rho} D_{\rho}(\rho) \quad (5.56)$$

Next, we observe that Q_{encl} on the right hand side of Equation 5.54 is equal to $\rho_l l$. Thus, we obtain

$$\oint_{\mathcal{S}} [\hat{\rho} D_{\rho}(\rho)] \cdot d\mathbf{s} = \rho_l l \quad (5.57)$$

The cylinder $\mathcal{S}$ consists of a flat top, curved side, and flat bottom. Expanding the above equation to reflect this, we obtain

$$\begin{aligned} \rho_l l = & \int_{top} [\hat{\rho} D_{\rho}(\rho)] \cdot (+\hat{z} d\mathbf{s}) \\ & + \int_{side} [\hat{\rho} D_{\rho}(\rho)] \cdot (+\hat{\rho} d\mathbf{s}) \\ & + \int_{bottom} [\hat{\rho} D_{\rho}(\rho)] \cdot (-\hat{z} d\mathbf{s}) \end{aligned} \quad (5.58)$$

Examination of the dot products indicates that the integrals associated with the top and bottom surfaces must be zero. In other words, the flux through the top and bottom is zero because $\mathbf{D}$ is perpendicular to these surfaces. We are left with

$$\rho_l l = \int_{side} [D_{\rho}(\rho)] d\mathbf{s} \quad (5.59)$$

The side surface is an open cylinder of radius $\rho = a$, so $D_{\rho}(\rho) = D_{\rho}(a)$, a constant over this surface. Thus:

$$\rho_l l = \int_{side} [D_{\rho}(a)] d\mathbf{s} = [D_{\rho}(a)] \int_{side} d\mathbf{s} \quad (5.60)$$

The remaining integral is simply the area of the side surface, which is $2\pi a \cdot l$. Solving for $D_{\rho}(a)$ we obtain

$$D_{\rho}(a) = \frac{\rho_l l}{2\pi a l} = \frac{\rho_l}{2\pi a} \quad (5.61)$$

Remarkably, we see $D_\rho(a)$ is independent of l , So the concern raised in the beginning of this solution – that we wouldn't be able to enclose all of the charge – doesn't matter.

Completing the solution, we note the result must be the same for any value of ρ (not just $\rho = a$), so

$$\mathbf{D} = \hat{\rho} D_\rho(\rho) = \hat{\rho} \frac{\rho l}{2\pi\rho} \quad (5.62)$$

and since $\mathbf{D} = \epsilon\mathbf{E}$:

$$\boxed{\mathbf{E} = \hat{\rho} \frac{\rho l}{2\pi\epsilon\rho}} \quad (5.63)$$

This completes the solution. We have found that the electric field is directed radially away from the line charge, and decreases in magnitude in inverse proportion to distance from the line charge. Suggestion: Check to ensure that this solution is dimensionally correct.

Figure 5.4. Finding the electric field of an infinite line of charge using Gauss' Law.

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5.7 Gauss' Law: Differential Form

The integral form of Gauss' Law (Section 5.5) is a calculation of enclosed charge Q_{encl} using the surrounding density of electric flux:

$$\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s} = Q_{encl} \quad (5.64)$$

where $\mathbf{D}$ is electric flux density and $\mathcal{S}$ is the enclosing surface. It is also sometimes necessary to do the inverse calculation (i.e., determine electric field associated with a charge distribution). This is sometimes possible using Equation 5.64 if the symmetry of the problem permits; see examples in Section 5.5 and 5.6. If the problem does not exhibit the necessary symmetry, then it seems that one must fall back to the family of techniques presented in Section 5.4 requiring direct integration over the charge, which is derived from Coulomb's Law.

However, even the Coulomb's Law / direct integration approach has a limitation that is very important to recognize: It does not account for the presence of structures that may influence the electric field. For example, the electric field due to a charge in free space is different from the electric field due to the same charge located near a perfectly-conducting surface. In fact, these approaches do not account for the possibility of *any* spatial variation in material composition, which rules out their use in many engineering applications.

To address this broader scope of problems, we require an alternative form of Gauss' Law that applies at individual points in space. That is, we require Gauss' Law expressed in the form of a differential equation, as opposed to an integral equation. This facilitates the use of Gauss' Law even in problems that do not exhibit sufficient symmetry and that involve material boundaries and spatial variations in material constitutive parameters. Given this differential equation and the boundary conditions imposed by structure and materials, we may then solve for the electric field in these more complicated scenarios. In this section, we derive the desired differential form of Gauss' Law. Elsewhere (in particular, in Section 5.15) we use this equation as a tool to find electric fields in problems involving material boundaries.

There are in fact two methods to develop the desired differential equation. One method is via the definition of divergence, whereas the other is via the divergence theorem. Both methods are presented below because each provides a different bit of insight. Let's explore the first method:

Derivation via the definition of divergence. Let the geometrical volume enclosed by $\mathcal{S}$ be $\mathcal{V}$, which has volume V (units of m^3). Dividing both sides of Equation 5.64 by V and taking the limit as $V \rightarrow 0$:

$$\lim_{V \rightarrow 0} \frac{\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s}}{V} = \lim_{V \rightarrow 0} \frac{Q_{\text{encl}}}{V} \quad (5.65)$$

The quantity on the right hand side is the volume charge density ρ_v (units of C/m^3) at the point at which we converge after letting the volume go to zero. The left hand side is, by definition, the *divergence* of $\mathbf{D}$, indicated in mathematical notation as " $\nabla \cdot \mathbf{D}$ " (see Section 4.6). Thus, we have *Gauss' Law in differential form*:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \quad (5.66)$$

To interpret this equation, recall that divergence is simply the flux (in this case, *electric flux*) per unit volume.

Gauss' Law in differential form (Equation 5.66) says that the electric flux per unit volume originating from a point in space is equal to the volume charge density at that point.

Derivation via the divergence theorem. Equation [5.66](#) may also be obtained from Equation [5.64](#) using the Divergence Theorem (Section [4.7](#)), which in the present case may be written:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{D}) dv = \oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s} \quad (5.67)$$

From Equation [5.64](#), we see that the right hand side of the equation may be replaced with the enclosed charge:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{D}) dv = Q_{encl} \quad (5.68)$$

Furthermore, the enclosed charge can be expressed as an integration of the volume charge density ρ_v over $\mathcal{V}$:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{D}) dv = \int_{\mathcal{V}} \rho_v dv \quad (5.69)$$

The above relationship must hold regardless of the specific location or shape of $\mathcal{V}$. The only way this is possible is if the integrands are equal. Thus, $\nabla \cdot \mathbf{D} = \rho_v$, and we have obtained Equation [5.66](#).

Example

Example 5.6

Determining the charge density at a point, given the associated electric field.

The electric field intensity in free space is

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{x}}Ax^2 + \hat{\mathbf{y}}Bz + \hat{\mathbf{z}}Cx^2z$$

where $\hat{\mathbf{x}}$ is only supported in math mode, $\hat{\mathbf{y}}$ is only supported in math mode, and $\hat{\mathbf{z}}$ is only supported in math mode. What is the charge density at $\mathbf{r} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 \text{ m}$?

Solution. First, we use $\mathbf{D} = \epsilon \mathbf{E}$ to get $\mathbf{D}$. Since the problem is in free space, $\epsilon = \epsilon_0$. Thus we have that the volume charge density is

$$\begin{aligned}\rho_v &= \nabla \cdot \mathbf{D} \\ &= \nabla \cdot (\epsilon_0 \mathbf{E}) = \epsilon_0 \nabla \cdot \mathbf{E} \\ &= \epsilon_0 \left[\frac{\partial}{\partial x} (Ax^2) + \frac{\partial}{\partial y} (Bz) + \frac{\partial}{\partial z} (Cx^2z) \right] \\ &= \epsilon_0 [2Ax + 0 + Cx^2]\end{aligned}$$

Now calculating the charge density at the specified location $\mathbf{r}$:

$$\begin{aligned}&\epsilon_0 [2(3 \text{ V/m}^3)(2 \text{ m}) + 0 + (1 \text{ V/m}^4)(2 \text{ m})^2] \\ &= \epsilon_0 (16 \text{ V/m}) \\ &= 142 \text{ pC/m}^3\end{aligned}$$

To obtain the electric field from the charge distribution in the presence of boundary conditions imposed by materials and structure, we must enforce the relevant boundary conditions. These boundary conditions are presented in Sections [5.17](#) and [5.18](#). Frequently, a simpler approach requiring only the boundary conditions on the electric potential ($V(\mathbf{r})$) is possible; this is presented in Section [5.15](#).

Furthermore, the reader should note the following. Gauss' Law does not always necessarily *fully* constrain possible solutions for the electric field. For that, we might also need Kirchoff's Voltage Law; see Section [5.11](#).

Before moving on, it is worth noting that Equation [5.66](#) can be solved in the special case in which there are no boundary conditions to satisfy; i.e., for charge only, in a uniform and unbounded medium. In fact, no additional electromagnetic theory is required to do this. Here's the solution:

$$\mathbf{D}(\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho_v(\mathbf{r}') dv \quad (5.70)$$

which we recognize as one of the results obtained in Section 5.4 (after dividing both sides by ϵ to get $\mathbf{E}$). It is reasonable to conclude that Gauss' Law (in either integral or differential form) is fundamental, whereas Coulomb's Law is merely a consequence of Gauss' Law.

Additional Reading:

- “Gauss' Law (https://en.wikipedia.org/wiki/Gauss'_law)” on Wikipedia.
- “Partial differential equation (https://en.wikipedia.org/wiki/Partial_differential_equation)” on Wikipedia.
- “Boundary value problem (https://en.wikipedia.org/wiki/Boundary_value_problem)” on Wikipedia.

5.8 Force, Energy, and Potential Difference

The force $\mathbf{F}_e$ experienced by a particle at location $\mathbf{r}$ bearing charge q in an electric field intensity $\mathbf{E}$ is (see Sections 2.2 and/or Section 5.1)

$$\mathbf{F}_e = q\mathbf{E}(\mathbf{r}) \quad (5.71)$$

If left alone in free space, this particle would immediately begin to move. The resulting displacement represents a loss of potential energy. This loss can be quantified using the concept of *work*, W . The incremental work ΔW done by moving the particle a short distance Δl , over which we assume the change in $\mathbf{F}_e$ is negligible, is

$$\Delta W \approx -\mathbf{F}_e \cdot \hat{\mathbf{l}} \Delta l \quad (5.72)$$

where in this case $\hat{\mathbf{l}}$ is the unit vector in the direction of the motion; i.e., the direction of $\mathbf{F}_e$. The minus sign indicates that potential energy of the system consisting of the electric field and the particle is being reduced. Like a spring that was previously compressed and is now released, the system is “relaxing.”

To confirm that work defined in this way is an expression of energy, consider the units. The product of force (units of N) and distance (units of m) has units of N·m, and 1 N·m is 1 J of energy.

Now, what if the motion of the particle is due to factors other than the force associated with the electric field? For example, we might consider “resetting” the system to its original condition by applying an external force to overcome $\mathbf{F}_e$. Equation 5.72 still represents the change in potential energy of the system, but now $\hat{\mathbf{l}}$ changes sign. The same magnitude of work is done, but now this work is positive. In other words, positive work requires the application of an *external* force that opposes and overcomes the

force associated with the electric field, thereby increasing the potential energy of the system. With respect to the analogy of a mechanical spring used above, positive work is achieved by compressing the spring.

It is also worth noting that the purpose of the dot product in Equation 5.72 is to ensure that only the component of motion parallel to the direction of the electric field is included in the energy tally. This is simply because motion in any other direction cannot be due to $\mathbf{E}$, and therefore does not increase or decrease the associated potential energy.

We can make the relationship between work and the electric field explicit by substituting Equation 5.71 into Equation 5.72, yielding

$$\Delta W \approx -q\mathbf{E}(\mathbf{r}) \cdot \hat{\mathbf{l}}\Delta l \quad (5.73)$$

Equation 5.73 gives the work only for a short distance around $\mathbf{r}$. Now let us try to generalize this result. If we wish to know the work done over a larger distance, then we must account for the possibility that $\mathbf{E}$ varies along the path taken. To do this, we may sum contributions from points along the path traced out by the particle, i.e.,

$$W \approx \sum_{n=1}^N \Delta W(\mathbf{r}_n) \quad (5.74)$$

where $\mathbf{r}_n$ are positions defining the path. Substituting Equation 5.73, we have

$$W \approx -q \sum_{n=1}^N \mathbf{E}(\mathbf{r}_n) \cdot \hat{\mathbf{l}}(\mathbf{r}_n) \Delta l \quad (5.75)$$

Taking the limit as $\Delta l \rightarrow 0$ we obtain

$$W = -q \int_{\mathcal{C}} \mathbf{E}(\mathbf{r}) \cdot \hat{\mathbf{l}}(\mathbf{r}) dl \quad (5.76)$$

where $\mathcal{C}$ is the path (previously, the sequence of $\mathbf{r}_n$'s) followed. Now omitting the explicit dependence on $\mathbf{r}$ in the integrand for clarity:

$$W = -q \int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} \quad (5.77)$$

where $d\mathbf{l} = \hat{\mathbf{l}}dl$ as usual. Now, we are able to determine the change in potential energy for a charged particle moving along any path in space, given the electric field.

At this point, it is convenient to formally define the electric *potential difference* V_{21} between the start point (1) and end point (2) of $\mathcal{C}$. V_{21} is defined as the work done by traversing $\mathcal{C}$, per unit of charge:

$$V_{21} \triangleq \frac{W}{q} \quad (5.78)$$

This has units of J/C, which is volts (V). Substituting Equation [5.77](#), we obtain:

$$\boxed{V_{21} = - \int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l}} \quad (5.79)$$

An advantage of analysis in terms of electrical potential as opposed to energy is that we will no longer have to explicitly state the value of the charge involved.

The potential difference V_{21} between two points in space, given by Equation [5.79](#), is the change in potential energy of a charged particle divided by the charge of the particle. Potential energy is also commonly known as “voltage” and has units of V.

Example**Example 5.7**

Potential difference in a uniform electric field.

Consider an electric field $\mathbf{E}(\mathbf{r}) = \hat{\mathbf{z}}E_0$, which is constant in both magnitude and direction throughout the domain of the problem. The path of interest is a line beginning at $\hat{\mathbf{z}}z_1$ and ending at $\hat{\mathbf{z}}z_2$. What is V_{21} ? (It's worth noting that the answer to this problem is a building block for a vast number of problems in electromagnetic analysis.)

Solution. From Equation 5.79 we have

$$V_{21} = - \int_{z_1}^{z_2} (\hat{\mathbf{z}}E_0) \cdot \hat{\mathbf{z}}dz = -E_0(z_2 - z_1) \quad (5.80)$$

Note V_{21} is simply the electric field intensity times the distance between the points. This may seem familiar. For example, compare this to the findings of the battery-charged capacitor experiment described in Section 2.2. There too we find that potential difference equals electric field intensity times distance, and the signs agree.

The solution to the preceding example is simple because the direct path between the two points is parallel to the electric field. If the path between the points had been *perpendicular* to $\mathbf{E}$, then the solution is even easier – V_{21} is simply zero. In all other cases, V_{21} is proportional to the component of the direct path between the start and end points that is parallel to $\mathbf{E}$, as determined by the dot product.

5.9 Independence of Path

In Section 5.8, we found that the potential difference (“voltage”) associated with a path $\mathcal{C}$ in an electric field intensity $\mathbf{E}$ is

$$V_{21} = - \int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} \quad (5.81)$$

where the curve begins at point 1 and ends at point 2. Let these points be identified using the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.^[1] Then:

$$V_{21} = - \int_{\mathbf{r}_1, \text{ along } \mathcal{C}}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{l} \quad (5.82)$$

The associated work done by a particle bearing charge q is

$$W_{21} = qV_{21} \quad (5.83)$$

This work represents the change in potential energy of the system consisting of the electric field and the charged particle. So, it must also be true that

$$W_{21} = W_2 - W_1 \quad (5.84)$$

where W_2 and W_1 are the potential energies when the particle is at $\mathbf{r}_2$ and $\mathbf{r}_1$, respectively. It is clear from the above equation that W_{21} does not depend on $\mathcal{C}$; it depends only on the positions of the start and end points and not on any of the intermediate points along $\mathcal{C}$. That is,

$$V_{21} = - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{l} \quad , \text{ independent of } \mathcal{C} \quad (5.85)$$

Since the result of the integration in Equation 5.85 is independent of the path of integration, *any* path that begins at $\mathbf{r}_1$ and ends at $\mathbf{r}_2$ yields the same value of W_{21} and V_{21} . We refer to this concept as *independence of path*.

The integral of the electric field over a path between two points depends only on the locations of the start and end points and is independent of the path taken between those points.

A practical application of this concept is that some paths may be easier to use than others, so there may be an advantage in computing the integral in Equation 5.85 using some path other than the path actually traversed.

1. See Section 4.1 for a refresher on this concept. ←

5.10 Kirchoff's Voltage Law for Electrostatics: Integral Form

As explained in Section 5.9, the electrical potential at point $\mathbf{r}_2$ relative to $\mathbf{r}_1$ in an electric field $\mathbf{E}$ (V/m) is

$$V_{21} = - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{l} \quad (5.86)$$

where the path of integration may be any path that begins and ends at the specified points. Consider what happens if the selected path through space begins and ends at the *same* point; i.e., $\mathbf{r}_2 = \mathbf{r}_1$. In this case, the path of integration is a closed loop. Since V_{21} depends only on the positions of the start and end points and because the potential energy at those points is the same, we conclude:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad (5.87)$$

This principle is known as *Kirchoff's Voltage Law for Electrostatics*.

Kirchoff's Voltage Law for Electrostatics (Equation [5.87](#)) states that the integral of the electric field over a closed path is zero.

It is worth noting that this law is a generalization of a principle of which the reader is likely already aware. In electric circuit theory, the sum of voltages over any closed loop in a circuit is zero. This is also known as Kirchoff's Voltage Law because it is precisely the same principle. To obtain Equation [5.87](#) for an electric circuit, simply partition the closed path into branches, with each branch representing one component. Then, the integral of $\mathbf{E}$ over each branch is the branch voltage; i.e., units of V/m times units of m yields units of V. Then, the sum of these branch voltages over any closed loop is zero, as dictated by Equation [5.87](#).

Finally, be advised that Equation [5.87](#) is specific to electrostatics. In electrostatics, it is assumed that the electric field is independent of the magnetic field. This is true if the magnetic field is either zero or not time-varying. If the magnetic field is time-varying, then Equation [5.87](#) must be modified to account for the effect of the magnetic field, which is to make the right hand side potentially different from zero. The generalized version of this expression that correctly accounts for that effect is known as the *Maxwell-Faraday Equation* (Section [8.8](#)).

Additional Reading:

- "[Maxwell's Equations](https://en.wikipedia.org/wiki/Maxwell's_equations) (https://en.wikipedia.org/wiki/Maxwell's_equations)" on Wikipedia.
- "[Kirchoff's Circuit Laws](https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws) (https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws)" on Wikipedia.

5.11 Kirchoff's Voltage Law for Electrostatics: Differential Form

The integral form of Kirchoff's Voltage Law for electrostatics (KVL; Section [5.10](#)) states that an integral of the electric field along a closed path is equal to zero:

$$\oint_c \mathbf{E} \cdot d\mathbf{l} = 0 \quad (5.88)$$

where $\mathbf{E}$ is electric field intensity and $\mathcal{C}$ is the closed curve. In this section, we derive the differential form of this equation. In some applications, this differential equation, combined with boundary conditions imposed by structure and materials (Sections [5.17](#) and [5.18](#)), can be used to solve for the electric field in arbitrarily complicated scenarios. A more immediate reason for considering this differential equation is that we gain a little more insight into the behavior of the electric field, disclosed at the end of this section.

The equation we seek may be obtained using Stokes' Theorem (Section [4.9](#)), which in the present case may be written:

$$\int_{\mathcal{S}} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} \quad (5.89)$$

where $\mathcal{S}$ is any surface bounded by $\mathcal{C}$, and $d\mathbf{s}$ is the normal to that surface with direction determined by right-hand rule. The integral form of KVL tells us that the right hand side of the above equation is zero, so:

$$\int_{\mathcal{S}} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = 0 \quad (5.90)$$

The above relationship must hold regardless of the specific location or shape of $\mathcal{S}$. The only way this is possible for all possible surfaces is if the integrand is zero at every point in space. Thus, we obtain the desired expression:

$$\boxed{\nabla \times \mathbf{E} = 0} \quad (5.91)$$

Summarizing:

The differential form of Kirchoff's Voltage Law for electrostatics (Equation [5.91](#)) states that the curl of the electrostatic field is zero.

Equation [5.91](#) is a partial differential equation. As noted above, this equation, combined with the appropriate boundary conditions, can be solved for the electric field in arbitrarily-complicated scenarios. Interestingly, it is not the only such equation available for this purpose – Gauss' Law (Section [5.7](#)) also does this. Thus, we see a *system* of partial differential equations emerging, and one may correctly infer that the electric field is not necessarily fully constrained by either equation alone.

Additional Reading:

- “[Maxwell's Equations](https://en.wikipedia.org/wiki/Maxwell's_equations) (https://en.wikipedia.org/wiki/Maxwell's_equations)” on Wikipedia.
- “[Boundary value problem](https://en.wikipedia.org/wiki/Boundary_value_problem) (https://en.wikipedia.org/wiki/Boundary_value_problem)” on Wikipedia.

5.12 Electric Potential Field Due to Point Charges

The electric field intensity due to a point charge q at the origin is (see Section [5.1](#) or [5.5](#))

$$\mathbf{E} = \hat{\mathbf{r}} \frac{q}{4\pi\epsilon r^2} \quad (5.92)$$

In Sections [5.8](#) and [5.9](#), it was determined that the potential difference measured from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ is

$$V_{21} = - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{l} \quad (5.93)$$

This method for calculating potential difference is often a bit awkward. To see why, consider an example from circuit theory, shown in Figure [5.5](#). In this example, consisting of a single resistor and a ground node, we've identified four quantities:

- The resistance R
- The current I through the resistor
- The node voltage V_1 , which is the potential difference measured from ground to the left side of the resistor
- The node voltage V_2 , which is the potential difference measured from ground to the right side of the resistor

Let's say we wish to calculate the potential difference V_{21} across the resistor. There are two ways this can be done:

- $V_{21} = -IR$
- $V_{21} = V_2 - V_1$

The advantage of the second method is that it is not necessary to know I , R , or indeed anything about what is happening between the nodes; it is only necessary to know the node voltages. The point is that it is often convenient to have a common *datum* – in this example, ground – with respect to which the potential differences at all other locations of interest can be defined. When we have this, calculating potential differences reduced to simply subtracting predetermined node potentials.

Figure 5.5. A resistor in a larger circuit, used as an example to demonstrate the concept of node voltages.

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So, can we establish a datum in general electrostatic problems that works the same way? The answer is yes. The datum is arbitrarily chosen to be a sphere that encompasses the universe; i.e., a sphere with radius $\rightarrow \infty$. Employing this choice of datum, we can use Equation 5.93 to define $V(\mathbf{r})$, the potential at point $\mathbf{r}$, as follows:

$$V(\mathbf{r}) \triangleq - \int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \quad (5.94)$$

The electrical potential at a point, given by Equation 5.94, is defined as the potential difference measured beginning at a sphere of infinite radius and ending at the point $\mathbf{r}$. The potential obtained in this manner is with respect to the potential infinitely far away.

In the particular case where $\mathbf{E}$ is due to the point charge at the origin:

$$V(\mathbf{r}) = - \int_{\infty}^{\mathbf{r}} \left[\hat{\mathbf{r}} \frac{q}{4\pi\epsilon r^2} \right] \cdot d\mathbf{l} \quad (5.95)$$

The principle of *independence of path* (Section 5.9) asserts that the path of integration doesn't matter as long as the path begins at the datum at infinity and ends at $\mathbf{r}$. So, we should choose the easiest such path. The radial symmetry of the problem indicates that the easiest path will be a line of constant θ and ϕ , so we choose $d\mathbf{l} = \hat{\mathbf{r}} dr$. Continuing:

$$V(\mathbf{r}) = - \int_{\infty}^{\mathbf{r}} \left[\hat{\mathbf{r}} \frac{q}{4\pi\epsilon r^2} \right] \cdot [\hat{\mathbf{r}} dr] \quad (5.96)$$

$$= - \frac{q}{4\pi\epsilon} \int_{\infty}^{\mathbf{r}} \frac{1}{r^2} dr \quad (5.97)$$

$$= + \frac{q}{4\pi\epsilon} \frac{1}{r} \Big|_{\infty}^{\mathbf{r}} \quad (5.98)$$

so

$$V(\mathbf{r}) = + \frac{q}{4\pi\epsilon r} \quad (5.99)$$

(Suggestion: Confirm that Equation 5.99 is dimensionally correct.) In the context of the circuit theory example above, this is the “node voltage” at $\mathbf{r}$ when the datum is defined to be the surface of a sphere at infinity. Subsequently, we may calculate the potential difference from any point $\mathbf{r}_1$ to any other point $\mathbf{r}_2$ as

$$V_{21} = V(\mathbf{r}_2) - V(\mathbf{r}_1) \quad (5.100)$$

and that will typically be a *lot* easier than using Equation 5.93.

It is not often that one deals with systems consisting of a single charged particle. So, for the above technique to be truly useful, we need a straightforward way to determine the potential field $V(\mathbf{r})$ for arbitrary distributions of charge. The first step in developing a more general expression is to determine the result for a particle located at a point $\mathbf{r}'$ somewhere other than the origin. Since Equation 5.99 depends only on charge and the distance between the field point $\mathbf{r}$ and $\mathbf{r}'$, we have

$$V(\mathbf{r}; \mathbf{r}') \triangleq + \frac{q'}{4\pi\epsilon |\mathbf{r} - \mathbf{r}'|} \quad (5.101)$$

where, for notational consistency, we use the symbol q' to indicate the charge. Now applying superposition, the potential field due to N charges is

$$V(\mathbf{r}) = \sum_{n=1}^N V(\mathbf{r}; \mathbf{r}_n) \quad (5.102)$$

Substituting Equation 5.101 we obtain:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{q_n}{|\mathbf{r} - \mathbf{r}_n|} \quad (5.103)$$

Equation 5.103 gives the electric potential at a specified location due to a finite number of charged particles.

The potential field due to continuous distributions of charge is addressed in Section 5.13.

5.13 Electric Potential Field due to a Continuous Distribution of Charge

The electrostatic potential field at $\mathbf{r}$ associated with N charged particles is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{q_n}{|\mathbf{r} - \mathbf{r}_n|} \quad (5.104)$$

where q_n and $\mathbf{r}_n$ are the charge and position of the n^{th} particle. However, it is more common to have a continuous distribution of charge as opposed to a countable number of charged particles. We now consider how to compute $V(\mathbf{r})$ three types of these commonly-encountered distributions. Before beginning, it's worth noting that the methods will be essentially the same, from a mathematical viewpoint, as those developed in Section 5.4; therefore, a review of that section may be helpful before attempting this section.

Continuous Distribution of Charge Along a Curve. Consider a continuous distribution of charge along a curve $\mathcal{C}$. The curve can be divided into short segments of length Δl . Then, the charge associated with the n^{th} segment, located at $\mathbf{r}_n$, is

$$q_n = \rho_l(\mathbf{r}_n) \Delta l \quad (5.105)$$

where ρ_l is the line charge density (units of C/m) at $\mathbf{r}_n$. Substituting this expression into Equation 5.104, we obtain

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\rho_l(\mathbf{r}_n)}{|\mathbf{r} - \mathbf{r}_n|} \Delta l \quad (5.106)$$

Taking the limit as $\Delta l \rightarrow 0$ yields:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{\mathcal{C}} \frac{\rho_l(l)}{|\mathbf{r} - \mathbf{r}'|} dl \quad (5.107)$$

where $\mathbf{r}'$ represents the varying position along $\mathcal{C}$ with integration along the length l .

Continuous Distribution of Charge Over a Surface. Consider a continuous distribution of charge over a surface $\mathcal{S}$. The surface can be divided into small patches having area Δs . Then, the charge associated with the n^{th} patch, located at $\mathbf{r}_n$, is

$$q_n = \rho_s(\mathbf{r}_n) \Delta s \quad (5.108)$$

where ρ_s is surface charge density (units of C/m²) at $\mathbf{r}_n$. Substituting this expression into Equation 5.104, we obtain

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\rho_s(\mathbf{r}_n)}{|\mathbf{r} - \mathbf{r}_n|} \Delta s \quad (5.109)$$

Taking the limit as $\Delta s \rightarrow 0$ yields:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{\mathcal{S}} \frac{\rho_s(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} ds \quad (5.110)$$

where $\mathbf{r}'$ represents the varying position over $\mathcal{S}$ with integration.

Continuous Distribution of Charge in a Volume. Consider a continuous distribution of charge within a volume $\mathcal{V}$. The volume can be divided into small cells (volume elements) having area Δv . Then, the charge associated with the n^{th} cell, located at $\mathbf{r}_n$, is

$$q_n = \rho_v(\mathbf{r}_n) \Delta v \quad (5.111)$$

where ρ_v is the volume charge density (units of C/m³) at $\mathbf{r}_n$. Substituting this expression into Equation 5.104, we obtain

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \frac{\rho_v(\mathbf{r}_n)}{|\mathbf{r} - \mathbf{r}_n|} \Delta v \quad (5.112)$$

Taking the limit as $\Delta v \rightarrow 0$ yields:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{\mathcal{V}} \frac{\rho_v(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv \quad (5.113)$$

where $\mathbf{r}'$ represents the varying position over $\mathcal{V}$ with integration.

5.14 Electric Field as the Gradient of Potential

In Section 5.8, it was determined that the electrical potential difference V_{21} measured over a path $\mathcal{C}$ is given by

$$V_{21} = - \int_{\mathcal{C}} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} \quad (5.114)$$

where $\mathbf{E}(\mathbf{r})$ is the electric field intensity at each point $\mathbf{r}$ along $\mathcal{C}$. In Section 5.12, we defined the scalar electric potential field $V(\mathbf{r})$ as the electric potential difference at $\mathbf{r}$ relative to a datum at infinity. In this section, we address the “inverse problem” – namely, how to calculate $\mathbf{E}(\mathbf{r})$ given $V(\mathbf{r})$. Specifically, we are interested in a direct “point-wise” mathematical transform from one to the other. Since Equation 5.114 is in the form of an integral, it should not come as a surprise that the desired expression will be in the form of a differential equation.

We begin by identifying the contribution of an infinitesimal length of the integral to the total integral in Equation [5.114](#). At point $\mathbf{r}$, this is

$$dV = -\mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} \quad (5.115)$$

Although we can proceed using any coordinate system, the following derivation is particularly simple in Cartesian coordinates. In Cartesian coordinates,

$$d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz \quad (5.116)$$

We also note that for any scalar function of position, including $V(\mathbf{r})$, it is true that

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \quad (5.117)$$

Note the above relationship is not specific to electromagnetics; it is simply mathematics. Also note that $dx = d\mathbf{l} \cdot \hat{\mathbf{x}}$ and so on for dy and dz . Making these substitutions into the above equation, we obtain:

$$dV = \frac{\partial V}{\partial x} (d\mathbf{l} \cdot \hat{\mathbf{x}}) + \frac{\partial V}{\partial y} (d\mathbf{l} \cdot \hat{\mathbf{y}}) + \frac{\partial V}{\partial z} (d\mathbf{l} \cdot \hat{\mathbf{z}}) \quad (5.118)$$

This equation may be rearranged as follows:

$$dV = \left(\left[\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right] V \right) \cdot d\mathbf{l} \quad (5.119)$$

Comparing the above equation to Equation [5.115](#), we find:

$$\mathbf{E}(\mathbf{r}) = - \left[\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right] V \quad (5.120)$$

Note that the quantity in square brackets is the gradient operator “ ∇ ” (Section [4.5](#)). Thus, we may write

$$\boxed{\mathbf{E} = -\nabla V} \quad (5.121)$$

which is the relationship we seek.

The electric field intensity at a point is the gradient of the electric potential at that point after a change of sign (Equation [5.121](#)).

Using Equation [5.121](#), we can immediately find the electric field at any point $\mathbf{r}$ if we can describe V as a function of $\mathbf{r}$. Furthermore, this relationship between V and $\mathbf{E}$ has a useful physical interpretation. Recall that the gradient of a scalar field is a vector that points in the direction in which that field increases most quickly. Therefore:

The electric field points in the direction in which the electric potential most rapidly decreases.

This result should not come as a complete surprise; for example, the reader should already be aware that the electric field points away from regions of net positive charge and toward regions of net negative charge (Sections [2.2](#) and/or [5.1](#)). What is new here is that both the magnitude and direction of the electric field may be determined given only the potential field, without having to consider the charge that is the physical source of the electrostatic field.

Example**Example 5.8**

Electric field of a charged particle, beginning with the potential field.

In this example, we determine the electric field of a particle bearing charge q located at the origin. This may be done in a “direct” fashion using Coulomb's Law (Section [5.1](#)). However, here we have the opportunity to find the electric field using a different method. In Section [5.12](#) we found the scalar potential for this source was:

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon r} \quad (5.122)$$

So, we may obtain the electric field using Equation [5.121](#):

$$\mathbf{E} = -\nabla V = -\nabla \left(\frac{q}{4\pi\epsilon r} \right) \quad (5.123)$$

Here $V(\mathbf{r})$ is expressed in spherical coordinates, so we have (Section [B.2](#)):

$$\mathbf{E} = - \left[\hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right] \left(\frac{q}{4\pi\epsilon r} \right) \quad (5.124)$$

In this case, $V(\mathbf{r})$ does not vary with ϕ or θ , so the second and third terms of the gradient are zero. This leaves

$$\begin{aligned} \mathbf{E} &= -\hat{\mathbf{r}} \frac{\partial}{\partial r} \left(\frac{q}{4\pi\epsilon r} \right) \\ &= -\hat{\mathbf{r}} \frac{q}{4\pi\epsilon} \frac{\partial}{\partial r} \frac{1}{r} \\ &= -\hat{\mathbf{r}} \frac{q}{4\pi\epsilon} \left(-\frac{1}{r^2} \right) \end{aligned} \quad (5.125)$$

So we find

$$\mathbf{E} = +\hat{\mathbf{r}} \frac{q}{4\pi\epsilon r^2} \quad (5.126)$$

as was determined in Section [5.1](#).

5.15 Poisson's and Laplace's Equations

The electric scalar potential field $V(\mathbf{r})$, defined in Section 5.12, is useful for a number of reasons including the ability to conveniently compute potential differences (i.e., $V_{21} = V(\mathbf{r}_2) - V(\mathbf{r}_1)$) and the ability to conveniently determine the electric field by taking the gradient (i.e., $\mathbf{E} = -\nabla V$). One way to obtain $V(\mathbf{r})$ is by integration over the source charge distribution, as described in Section 5.13. This method is awkward in the presence of material interfaces, which impose boundary conditions on the solutions that must be satisfied simultaneously. For example, the electric potential on a perfectly conducting surface is constant^[1] – a constraint which is not taken into account in any of the expressions in Section 5.13.

In this section, we develop an alternative approach to calculating $V(\mathbf{r})$ that accommodates these boundary conditions, and thereby facilitates the analysis of the scalar potential field in the vicinity of structures and spatially-varying material properties. This alternative approach is based on *Poisson's Equation*, which we now derive.

We begin with the differential form of Gauss' Law (Section 5.7):

$$\nabla \cdot \mathbf{D} = \rho_v \quad (5.127)$$

Using the relationship $\mathbf{D} = \epsilon \mathbf{E}$ (and keeping in mind our standard assumptions about material properties, summarized in Section 2.8) we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon} \quad (5.128)$$

Next, we apply the relationship (Section 5.14):

$$\mathbf{E} = -\nabla V \quad (5.129)$$

yielding

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon} \quad (5.130)$$

This is Poisson's Equation, but it is not in the form in which it is commonly employed. To obtain the alternative form, consider the operator $\nabla \cdot \nabla$ in Cartesian coordinates:

$$\begin{aligned} \nabla \cdot \nabla &= \left[\frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right] \cdot \left[\frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right] \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \nabla^2 \end{aligned} \quad (5.131)$$

i.e., the operator $\nabla \cdot \nabla$ is identically the Laplacian operator ∇^2 (Section 4.10). Furthermore, this is true regardless of the coordinate system employed. Thus, we obtain the following form of Poisson's Equation:

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \quad (5.132)$$

Poisson's Equation (Equation 5.132) states that the Laplacian of the electric potential field is equal to the volume charge density divided by the permittivity, with a change of sign.

Note that Poisson's Equation is a partial differential equation, and therefore can be solved using well-known techniques already established for such equations. In fact, Poisson's Equation is an *inhomogeneous* differential equation, with the inhomogeneous part $-\rho_v/\epsilon$ representing the source of the field. In the presence of material structure, we identify the relevant boundary conditions at the interfaces between materials, and the task of finding $V(\mathbf{r})$ is reduced to the purely mathematical task of solving the associated boundary value problem (see "Additional Reading" at the end of this section). This approach is particularly effective when one of the materials is a perfect conductor or can be modeled as such a material. This is because – as noted at the beginning of this section – the electric potential at all points on the surface of a perfect conductor must be equal, resulting in a particularly simple boundary condition.

In many other applications, the charge responsible for the electric field lies outside the domain of the problem; i.e., we have non-zero electric field (hence, potentially non-zero electric potential) in a region that is free of charge. In this case, Poisson's Equation simplifies to *Laplace's Equation*:

$$\boxed{\nabla^2 V = 0 \text{ (source-free region)}} \quad (5.133)$$

Laplace's Equation (Equation 5.133) states that the Laplacian of the electric potential field is zero in a source-free region.

Like Poisson's Equation, Laplace's Equation, combined with the relevant boundary conditions, can be used to solve for $V(\mathbf{r})$, but only in regions that contain no charge.

Additional Reading:

- "Poisson's equation (https://en.wikipedia.org/wiki/Poisson's_equation)" on Wikipedia.
- "Boundary value problem (https://en.wikipedia.org/wiki/Boundary_value_problem)" on Wikipedia.
- "Laplace's equation (https://en.wikipedia.org/wiki/Laplace's_equation)" on Wikipedia.

1. This fact is probably already known to the reader from past study of elementary circuit theory; however, this is established in the context of electromagnetics in Section [5.19](#). ←

5.16 Potential Field Within a Parallel Plate Capacitor

This section presents a simple example that demonstrates the use of Laplace's Equation (Section [5.15](#)) to determine the potential field in a source free region. The example, shown in Figure [5.6](#), pertains to an important structure in electromagnetic theory – the parallel plate capacitor. Here we are concerned only with the potential field $V(\mathbf{r})$ between the plates of the capacitor; you do not need to be familiar with capacitance or capacitors to follow this section (although you're welcome to look ahead to Section [5.22](#) for a preview, if desired). What *is* recommended before beginning is a review of the battery-charged capacitor experiment discussed in Section [2.2](#). In this section you'll see a rigorous derivation of what we figured out in an informal way in that section.

The parallel-plate capacitor in Figure [5.6](#) consists of two perfectly-conducting circular disks separated by a distance d by a spacer material having permittivity ϵ . There is no charge present in the spacer material, so Laplace's Equation applies. That equation is (Section [5.15](#)):

$$\nabla^2 V = 0 \quad (\text{source-free region}) \quad (5.134)$$

Let V_C be the potential difference between the plates, which would also be the potential difference across the terminals of the capacitor. The radius a of the plates is larger than d by enough that we may neglect what is going on at the edges of the plates – more on this will be said as we work the problem. Under this assumption, what is the electric potential field $V(\mathbf{r})$ between the plates?

Figure 5.6. A parallel plate capacitor, as a demonstration of the use of Laplace's Equation.

This problem has cylindrical symmetry, so it makes sense to continue to use cylindrical coordinates with the z axis being perpendicular to the plates. Equation [5.134](#) in cylindrical coordinates is:

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] V = 0 \quad (5.135)$$

or perhaps a little more clearly written as follows:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (5.136)$$

Since the problem has radial symmetry, $\partial V/\partial\phi = 0$. Since $d \ll a$, we expect the fields to be approximately constant with ρ until we get close to the edge of the plates. Therefore, we assume $\partial V/\partial\rho$ is negligible and can be taken to be zero. Thus, we are left with

$$\frac{\partial^2 V}{\partial z^2} \approx 0 \quad \text{for } \rho \ll a \quad (5.137)$$

The general solution to Equation 5.137 is obtained simply by integrating both sides twice, yielding

$$V(z) = c_1 z + c_2 \quad (5.138)$$

where c_1 and c_2 are constants that must be consistent with the boundary conditions. Thus, we must develop appropriate boundary conditions. Let the node voltage at the negative ($z = 0$) terminal be V_- . Then the voltage at the positive ($z = +d$) terminal is $V_- + V_C$. Therefore:

$$V(z = 0) = V_- \quad (5.139)$$

$$V(z = +d) = V_- + V_C \quad (5.140)$$

These are the relevant boundary conditions. Substituting $V(z = 0) = V_-$ into Equation 5.138 yields $c_2 = V_-$. Substituting $V(z = +d) = V_- + V_C$ into Equation 5.138 yields $c_1 = V_C/d$. Thus, the answer to the problem is

$$V(z) \approx \frac{V_C}{d} z + V_- \quad \text{for } \rho \ll a \quad (5.141)$$

Note that the above result is dimensionally correct and confirms that the potential deep inside a “thin” parallel plate capacitor changes linearly with distance between the plates.

Further, you should find that application of the equation $\mathbf{E} = -\nabla V$ (Section 5.14) to the solution above yields the expected result for the electric field intensity: $\mathbf{E} \approx -\hat{\mathbf{z}}V_C/d$. This is precisely the result that we arrived at (without the aid of Laplace’s Equation) in Section 2.2.

A reasonable question to ask at this point would be, what about the potential field close to the edge of the plates, or, for that matter, beyond the plates? The field in this region is referred to as a *fringing field*. For the fringing field, $\partial V/\partial\rho$ is no longer negligible and must be taken into account. In addition, it is necessary to modify the boundary conditions to account for the outside surfaces of the plates (that is, the sides of the plates that face away from the dielectric) and to account for the effect of the boundary between the spacer material and free space. These issues make the problem much more difficult. When an accurate calculation of a fringing field is necessary, it is common to resort to a numerical solution of Laplace’s Equation. Fortunately, accurate calculation of fringing fields is usually not required in practical engineering applications.

5.17 Boundary Conditions on the Electric Field Intensity ($\mathbf{E}$)

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At an interface between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. These discontinuities can be described mathematically as *boundary conditions* and used to constrain solutions for the associated electromagnetic quantities. In this section, we derive boundary conditions on the electric field intensity $\mathbf{E}$.

To begin, consider a region consisting of only two media that meet at an interface defined by the mathematical surface $\mathcal{S}$, as shown in Figure 5.7.

Figure 5.7. At the surface of a perfectly-conducting region, $\mathbf{E}$ may be perpendicular to the surface (two leftmost possibilities), but may not exhibit a component that is tangent to the surface (two rightmost possibilities).

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If either one of the materials is a perfect electrical conductor (PEC), then $\mathcal{S}$ is an equipotential surface; i.e., the electric potential V is constant everywhere on $\mathcal{S}$. Since $\mathbf{E}$ is proportional to the spatial rate of change of potential (recall $\mathbf{E} = -\nabla V$; Section 5.14), we find:

The component of $\mathbf{E}$ that is tangent to a perfectly-conducting surface is zero.

This is sometimes expressed informally as follows:

$$E_{tan} = 0 \quad \text{on PEC surface} \quad (5.142)$$

where “ E_{tan} ” is understood to be the component of $\mathbf{E}$ that is tangent to $\mathcal{S}$. Since the tangential component of $\mathbf{E}$ on the surface of a perfect conductor is zero, the electric field at the surface must be oriented entirely in the direction perpendicular to the surface, as shown in Figure 5.7.

The following equation expresses precisely the same idea, but includes the calculation of the tangential component as part of the statement:

$$\mathbf{E} \times \hat{\mathbf{n}} = 0 \quad (\text{on PEC surface}) \quad (5.143)$$

where $\hat{\mathbf{n}}$ is either normal (i.e., unit vector perpendicular to the surface) to each point on $\mathcal{S}$. This expression works because the cross product of any two vectors is perpendicular to either vector (Section 4.1), and any vector which is perpendicular to $\hat{\mathbf{n}}$ is tangent to $\mathcal{S}$.

We now determine a more general boundary condition that applies even when neither of the media bordering $\mathcal{S}$ is a perfect conductor. The desired boundary condition can be obtained directly from Kirchhoff's Voltage Law (KVL; Section 5.10):

$$\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} = 0 \quad (5.144)$$

Let the closed path of integration take the form of a rectangle centered on $\mathcal{S}$, as shown in Figure 5.8.

Figure 5.8. Use of KVL to determine the boundary condition on $\mathbf{E}$.

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Let the sides A , B , C , and D be perpendicular or parallel to the surface, respectively. Let the length of the perpendicular sides be w , and let the length of the parallel sides be l . From KVL we have

$$\begin{aligned} \oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} &= \int_A \mathbf{E} \cdot d\mathbf{l} \\ &+ \int_B \mathbf{E} \cdot d\mathbf{l} \\ &+ \int_C \mathbf{E} \cdot d\mathbf{l} \\ &+ \int_D \mathbf{E} \cdot d\mathbf{l} = 0 \end{aligned} \quad (5.145)$$

Now, let us reduce w and l together while (1) maintaining a constant ratio $w/l \ll 1$ and (2) keeping $\mathcal{C}$ centered on $\mathcal{S}$. In this process, the contributions from the B and D segments become equal in magnitude but opposite in sign; i.e.,

$$\int_B \mathbf{E} \cdot d\mathbf{l} + \int_D \mathbf{E} \cdot d\mathbf{l} \rightarrow 0 \quad (5.146)$$

This leaves

$$\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} \rightarrow \int_A \mathbf{E} \cdot d\mathbf{l} + \int_C \mathbf{E} \cdot d\mathbf{l} \rightarrow 0 \quad (5.147)$$

Let us define the unit vector $\hat{\mathbf{t}}$ ("tangent") as shown in Figure 5.8. When the lengths of sides A and C become sufficiently small, we can write the above expression as follows:

$$\mathbf{E}_1 \cdot \hat{\mathbf{t}} \Delta l - \mathbf{E}_2 \cdot \hat{\mathbf{t}} \Delta l \rightarrow 0 \quad (5.148)$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the fields evaluated on the two sides of the boundary and $\Delta l \rightarrow 0$ is the length of sides A and C while this is happening. Note that the only way Equation 5.148 can be true is if the tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$ are equal. In other words:

The tangential component of $\mathbf{E}$ must be continuous across an interface between dissimilar media.

Note that this is a generalization of the result we obtained earlier for the case in which one of the media was a PEC – in that case, the tangent component of $\mathbf{E}$ on the other side of the interface must be zero because it is zero in the PEC medium.

As before, we can express this idea in compact mathematical notation. Using the same idea used to obtain Equation 5.143, we have found

$$\mathbf{E}_1 \times \hat{\mathbf{n}} = \mathbf{E}_2 \times \hat{\mathbf{n}} \quad \text{on } \mathcal{S} \quad (5.149)$$

or, as it is more commonly written:

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \quad \text{on } \mathcal{S}} \quad (5.150)$$

We conclude this section with a note about the broader applicability of this boundary condition:

Equation 5.150 is the boundary condition that applies to $\mathbf{E}$ for both the electrostatic *and the general (time-varying) case*.

Although a complete explanation is not possible without the use of the Maxwell-Faraday Equation (Section 8.8), the reason why this boundary condition applies in the time-varying case can be disclosed here. In the presence of time-varying magnetic fields, the right-hand side of Equation 5.144 may become non-zero and is proportional to the area defined by the closed loop. However, the above derivation requires the area of this loop to approach zero, in which case the possible difference from Equation 5.144 also converges to zero. Therefore, the boundary condition expressed in Equation 5.150 applies generally.

5.18 Boundary Conditions on the Electric Flux Density ($\mathbf{D}$)

In this section, we derive boundary conditions on the electric flux density $\mathbf{D}$. The considerations are quite similar to those encountered in the development of boundary conditions on the electric *field intensity* ($\mathbf{E}$) in Section 5.17, so the reader may find it useful to review that section before attempting this section. This section also assumes familiarity with the concepts of electric flux, electric flux density, and Gauss' Law; for a refresher, Sections 2.4 and 5.5 are suggested.

To begin, consider a region at which two otherwise-homogeneous media meet at an interface defined by the mathematical surface $\mathcal{S}$, as shown in Figure 5.9. Let one of these regions be a perfect electrical conductor (PEC).

Figure 5.9. The component of $\mathbf{D}$ that is perpendicular to a perfectly-conducting surface is equal to the charge density on the surface.

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In Section 5.17, we established that the tangential component of the electric field must be zero, and therefore, the electric field is directed entirely in the direction perpendicular to the surface. We further know that the electric field within the conductor is identically zero. Therefore, $\mathbf{D}$ at any point on $\mathcal{S}$ is entirely in the direction perpendicular to the surface and pointing into the non-conducting medium. However, it is also possible to determine the magnitude of $\mathbf{D}$. We shall demonstrate in this section that

At the surface of a perfect conductor, the magnitude of $\mathbf{D}$ is equal to the surface charge density ρ_s (units of C/m²) at that point.

The following equation expresses precisely the same idea, but includes the calculation of the perpendicular component as part of the statement:

$$\mathbf{D} \cdot \hat{\mathbf{n}} = \rho_s \quad (\text{on PEC surface}) \quad (5.151)$$

where $\hat{\mathbf{n}}$ is the normal to $\mathcal{S}$ pointing into the non-conducting region. (Note that the orientation of $\hat{\mathbf{n}}$ is now important; we have assumed $\hat{\mathbf{n}}$ points into region 1, and we must now stick with this choice.) Before proceeding with the derivation, it may be useful to note that this result is not surprising. The very definition of electric flux (Section 2.4) indicates that $\mathbf{D}$ should correspond in the same way to a surface charge density. However, we can show this rigorously, and in the process we can generalize this result to the more-general case in which neither of the two materials are PEC.

The desired more-general boundary condition may be obtained from the integral form of Gauss' Law (Section 5.5), as illustrated in Figure 5.10.

Figure 5.10. Use of Gauss' Law to determine the boundary condition on $\mathbf{D}$.

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Let the surface of integration $\mathcal{S}'$ take the form of closed cylinder centered at a point on the interface and for which the flat ends are parallel to the surface and perpendicular to $\hat{\mathbf{n}}$. Let the radius of this cylinder be a , and let the length of the cylinder be $2h$. From Gauss' Law we have

$$\begin{aligned}
\oint_{\mathcal{S}'} \mathbf{D} \cdot d\mathbf{s} &= \int_{top} \mathbf{D} \cdot d\mathbf{s} \\
&+ \int_{side} \mathbf{D} \cdot d\mathbf{s} \\
&+ \int_{bottom} \mathbf{D} \cdot d\mathbf{s} = Q_{encl}
\end{aligned} \tag{5.152}$$

where the “top” and “bottom” are in Regions 1 and 2, respectively, and Q_{encl} is the charge enclosed by $\mathcal{S}'$. Now let us reduce h and a together while (1) maintaining a constant ratio $h/a \ll 1$ and (2) keeping $\mathcal{S}'$ centered on $\mathcal{S}$. Because $h \ll a$, the area of the side can be made negligible relative to the area of the top and bottom. Then as $h \rightarrow 0$ we are left with

$$\int_{top} \mathbf{D} \cdot d\mathbf{s} + \int_{bottom} \mathbf{D} \cdot d\mathbf{s} \rightarrow Q_{encl} \tag{5.153}$$

As the area of the top and bottom sides become infinitesimal, the variation in $\mathbf{D}$ over these areas becomes negligible. Now we have simply:

$$\mathbf{D}_1 \cdot \hat{\mathbf{n}} \Delta A + \mathbf{D}_2 \cdot (-\hat{\mathbf{n}}) \Delta A \rightarrow Q_{encl} \tag{5.154}$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are the electric flux density vectors in medium 1 and medium 2, respectively, and ΔA is the area of the top and bottom sides. The above expression can be rewritten

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \rightarrow \frac{Q_{encl}}{\Delta A} \tag{5.155}$$

Note that the left side of the equation must represent a actual, physical surface charge; this is apparent from dimensional analysis and the fact that h is now infinitesimally small. Therefore:

$$\boxed{\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s} \tag{5.156}$$

where, as noted above, $\hat{\mathbf{n}}$ points into region 1. Summarizing:

Any discontinuity in the normal component of the electric flux density across the boundary between two material regions is equal to the surface charge.

Now let us verify that this is consistent with our preliminary finding, in which Region 2 was a PEC. In that case $\mathbf{D}_2 = 0$, so we see that Equation 5.151 is satisfied, as expected. If neither Region 1 nor Region 2 is PEC *and* there is no surface charge on the interface, then we find $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$; i.e.,

In the absence of surface charge, the normal component of the electric flux density must be continuous across the boundary.

Finally, we note that since $\mathbf{D} = \epsilon\mathbf{E}$, Equation 5.156 implies the following boundary condition on $\mathbf{E}$:

$$\hat{\mathbf{n}} \cdot (\epsilon_1 \mathbf{E}_1 - \epsilon_2 \mathbf{E}_2) = \rho_s \quad (5.157)$$

where ϵ_1 and ϵ_2 are the permittivities in Regions 1 and 2, respectively. The above equation illustrates one reason why we sometimes prefer the “flux” interpretation of the electric field to the “field intensity” interpretation of the electric field.

5.19 Charge and Electric Field for a Perfectly Conducting Region

In this section, we consider the behavior of charge and the electric field in the vicinity of a perfect electrical conductor (PEC).

First, note that the electric field – both the electric field intensity $\mathbf{E}$ and electric flux density $\mathbf{D}$ – throughout a PEC region is zero. This is because the electrical potential throughout a PEC region must be constant. (This idea is explored further in Section 6.3.) Recall that the electric field is proportional to the spatial rate of change of electrical potential (i.e., $\mathbf{E} = -\nabla V$; Section 5.14). Thus, the electric field must be zero throughout a PEC region.

Second, the electric field is oriented directly away from (i.e., perpendicular to) the PEC surface, and the magnitude of $\mathbf{D}$ is equal to the surface charge density ρ_s (C/m²) (Section 5.18).

Now we address the question of charge distribution; i.e., the location and density of charge. Consider the scenario shown in Figure 5.11.

Figure 5.11. An infinite flat slab of PEC in the presence of an applied electric field.

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Here, a flat slab of PEC material is embedded in dielectric material.^[1] The thickness of the slab is finite, whereas the length and width of the slab is infinite. The region above the slab is defined as Region 1 and has permittivity ϵ_1 . The region below the slab is defined as Region 2 and has permittivity ϵ_2 . Electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ are present in Regions 1 and 2, respectively, as shown in Figure 5.11. To begin, let us assume that these fields are the result of some external stimulus that results in the direction of these fields being generally upward, as shown in Figure 5.11.

Now, what do we know about $\mathbf{E}_1$ and $\mathbf{E}_2$? First, both fields must satisfy the relevant boundary conditions. That is, the component of $\mathbf{E}_1$ that is tangent to the upper PEC surface is zero, so that $\mathbf{E}_1$ is directed entirely in a direction perpendicular to the surface. Similarly, the component of $\mathbf{E}_2$ that is tangent to the lower PEC surface is zero, so that $\mathbf{E}_2$ is directed entirely in a direction perpendicular to the surface. At this point we have not determined the magnitudes or signs of $\mathbf{E}_1$ and $\mathbf{E}_2$; we have established only that there are no non-zero components tangential to (i.e., parallel to) the PEC surfaces.

Next, recall that the electric field must be zero within the slab. This means that there must be zero net charge within the slab, since any other distribution of charge will result in a non-zero electric field, and subsequently a potential difference between locations within the slab. Therefore:

There can be no static charge *within* a PEC.

It follows that

Charge associated with a PEC lies entirely on the surface.

Outside the slab, the boundary conditions on $\mathbf{D}_1$ and $\mathbf{D}_2$ in the dielectric regions require these fields to be non-zero when the surface charge density on the PEC is non-zero. The surface charge supports the discontinuity in the normal component of the electric fields. Specifically, $\mathbf{D}_1$ and $\mathbf{D}_2$ have the same magnitude $|\rho_s|$ because the surface charge densities on both sides of the slab have equal magnitude. However, the electric field intensity $\mathbf{E}_1 = \mathbf{D}_1/\epsilon_1$, whereas $\mathbf{E}_2 = \mathbf{D}_2/\epsilon_2$; i.e., these are different. That is, the electric field intensities are unequal unless the permittivities in each dielectric region are equal.

Finally, let us consider the structure of the electric field in more general cases. Figure 5.12 shows field lines in a homogeneous dielectric material in which a point charge and PEC regions of various shapes are embedded. Note that electric field lines now bend in the dielectric so as to satisfy the requirement that the tangential component of the electric field be zero on PEC surfaces. Also note that the charge distribution arranges itself on the PEC surfaces so as to maintain zero electric field and constant potential within the cube.

Figure 5.12. Electric field lines due to a point charge in the vicinity of PEC regions (shaded) of various shapes.

by C. Burks (modified)

1. For the purposes of this section, it suffices to interpret “dielectric” as a “nonconducting and well characterized entirely in terms of its permittivity.” For more, see Section 5.20. ←

5.20 Dielectric Media

Dielectric is particular category of materials that exhibit low conductivity^[1] because their constituent molecules remain intact when exposed to an electric field, as opposed to shedding electrons as is the case in good conductors. Subsequently, dielectrics do not effectively pass current, and are therefore considered “good insulators” as well as “poor conductors.” An important application of dielectrics in electrical engineering is as a spacer material in printed circuit boards (PCBs), coaxial cables, and capacitors.

Examples of dielectrics include air, glass, teflon, and fiberglass epoxy (the material used in common “FR4” printed circuit boards). These and other dielectrics are listed along with values of their constitutive parameters in Section [A.1](#).

The electromagnetic properties of dielectric materials are quantified primarily by relative permittivity ϵ_r (Section [2.3](#)), which ranges from very close to 1 upward to roughly 50, and is less than 6 or so for most commonly-encountered materials having low moisture content. The permeability of dielectric materials is approximately equal to the free-space value (i.e., $\mu \approx \mu_0$); therefore, these materials are said to be “non-magnetic.”

Additional Reading:

- “[Dielectric](https://en.wikipedia.org/wiki/Dielectric) (<https://en.wikipedia.org/wiki/Dielectric>)” on Wikipedia.

1. See Section [2.8](#) for a refresher on parameters describing properties of materials. ↩

5.21 Dielectric Breakdown

The permittivity of an ideal dielectric is independent of the magnitude of an applied electric field; the material is said to be “linear.”^[1] However, all practical dielectrics fail in this respect with sufficiently strong electric field. Typically, the failure is abrupt and is observed as a sudden, dramatic increase in conductivity, signaling that electrons are being successfully dislodged from their host molecules. The threshold value of the electric field intensity at which this occurs is known as the *dielectric strength*, and the sudden change in behavior observed in the presence of an electric field greater than this threshold value is known as *dielectric breakdown*.

Dielectric strength varies from about 3 MV/m for air to about 200 MV/m in mica (a dielectric commonly used in capacitors).

Dielectric breakdown is typically accompanied by “arcing,” which is a sudden flow of current associated with the breakdown. A well known example of this phenomenon is lightning, which occurs when charge is exchanged between sky and ground when air (a dielectric) exhibits breakdown. Dielectric breakdown in solids typically damages the material.

Additional Reading:

- “[Electrical Breakdown](https://en.wikipedia.org/wiki/Electrical_breakdown) (https://en.wikipedia.org/wiki/Electrical_breakdown)” on Wikipedia.

1. See Section [2.8](#) for a review of this concept. ↩

5.22 Capacitance

When separate regions of positive and negative charge exist in proximity, Coulomb forces (Section [5.1](#)) will attempt to decrease the separation between the charges. As noted in Section [5.8](#), this can be interpreted as a tendency of a system to reduce its potential energy. If the charges are fixed in place, then the potential energy remains constant. This potential energy is proportional to the Coulomb force. Referring back to Section [5.1](#), the Coulomb force is:

- Proportional to quantity of positive charge squared
- Inversely proportional to the separation between the charges squared
- Inversely proportional to the permittivity of the material separating the charges

Therefore, the potential energy of the system is likewise dependent on charge, separation, and permittivity. Furthermore, we see that the ability of a system to store energy in this manner depends on the geometry of the charge distribution and the permittivity of the intervening material.

Now recall that the electric field intensity $\mathbf{E}$ is essentially defined in terms of the Coulomb force; i.e., $\mathbf{F} = q\mathbf{E}$ (Section [2.2](#)). So, rather than thinking of the potential energy of the system as being associated with the Coulomb force, it is equally valid to think of the potential energy as being stored in the electric field associated with the charge distribution. It follows from the previous paragraph that the energy stored in the electric field depends on the geometry of the charge distribution and the permittivity of the intervening media. This relationship is what we mean by *capacitance*. We summarize as follows:

Capacitance is the ability of a structure to store energy in an electric field.

and

The capacitance of a structure depends on its geometry and the permittivity of the medium separating regions of positive and negative charge.

Note that capacitance does *not* depend on charge, which we view as either a stimulus or response from this point of view. The corresponding response or stimulus, respectively, is the potential associated with this charge. This leads to the following definition:

$$C \triangleq \frac{Q_+}{V} \quad (5.158)$$

where Q_+ (units of C) is the total positive charge, V (units of V) is the potential associated with this charge (defined such that it is positive), and C (units of F) is the associated capacitance. So:

In practice, capacitance is defined as the ratio of charge present on one conductor of a two-conductor system to the potential difference between the conductors (Equation [5.158](#)).

In other words, a structure is said to have greater capacitance if it stores more charge – and therefore stores more energy – in response to a given potential difference.

Figure [5.13](#) shows the relevant features of this definition. Here, a battery imposes the potential difference V between two regions of perfectly-conducting material. Q_+ is the total charge on the surface of the PEC region attached to the positive terminal of the battery. An equal amount of negative charge appears on the surface of the PEC region attached to the negative terminal of the battery (Section [5.19](#)). This charge distribution gives rise to an electric field. Assuming the two PEC regions are fixed in place, Q_+ will increase linearly with increasing V , at a rate determined by the capacitance C of the structure.

Figure 5.13. Electrostatic interpretation of capacitance.

A *capacitor* is a device that is designed to exhibit a specified capacitance. We can now make the connection to the concept of the capacitor as it appears in elementary circuit theory. In circuit theory, the behavior of devices is characterized in terms of terminal voltage V_T in response to terminal current I_T , and vice versa. First, note that current does not normally flow through a capacitor,^[1] so when we refer to “terminal current” for a capacitor, what we *really* mean is the flow of charge arriving or departing from one of the conductors via the circuit, which is equal to the flow of charge departing or arriving (respectively) at the other conductor. This gives the appearance of current flow through the capacitor when the current is examined from outside the capacitor. With that settled, we proceed as follows. Using Equation [5.158](#), we express the voltage V_T across the terminals of a capacitor having capacitance C :

$$V_T = \frac{Q_+}{C} \quad (5.159)$$

We seek a relationship between V_T and I_T . Current is charge per unit time, so the charge on either conductor is the integral of I_T over time; i.e.:

$$Q_+(t) = \int_{t_0}^t I_T(\tau) d\tau + Q_+(t_0) \quad (5.160)$$

where t_0 is an arbitrarily-selected start time. In other words, amps integrated over time is charge. If we define I_T as being positive in the direction of the flow of positive charge as is the usual convention, then we have:

$$V_T(t) = \frac{1}{C} \int_{t_0}^t I_T(\tau) d\tau + \frac{1}{C} Q_+(t_0) \quad (5.161)$$

Again applying Equation 5.158, we see that the second term is simply $V_T(t_0)$. This is the expected relationship from elementary circuit theory.

Finally, solving for I_T we obtain the differential form of this relationship:

$$I_T(t) = C \frac{d}{dt} V_T(t) \quad (5.162)$$

Additional Reading:

- “[Capacitance](https://en.wikipedia.org/wiki/Capacitance) (<https://en.wikipedia.org/wiki/Capacitance>)” on Wikipedia.
- “[Capacitor](https://en.wikipedia.org/wiki/Capacitor) (<https://en.wikipedia.org/wiki/Capacitor>)” on Wikipedia.

1. If it does, it's probably experiencing *dielectric breakdown*; see Section 5.21. ↩

5.23 The Thin Parallel Plate Capacitor

Let us now determine the capacitance of a common type of capacitor known as the *thin parallel plate capacitor*, shown in Figure 5.14. This capacitor consists of two flat plates, each having area A , separated by distance d . To facilitate discussion, let us place the origin of the coordinate system at the center of the lower plate, with the $+z$ axis directed toward the upper plate such that the upper plate lies in the $z = +d$ plane.

Figure 5.14. Thin parallel plate capacitor.

Below we shall find the capacitance by assuming a particular charge on one plate, using the boundary condition on the electric flux density $\mathbf{D}$ to relate this charge density to the internal electric field, and then integrating over the electric field between the plates to obtain the potential difference. Then, capacitance is the ratio of the assumed charge to the resulting potential difference.

The principal difficulty in this approach is finding the electric field. To appreciate the problem, first consider that if the area of the plates was infinite, then the electric field would be very simple; it would begin at the positively-charged plate and extend in a perpendicular direction toward the negatively-charged plate (Section 5.19). Furthermore, the field would be constant everywhere between the plates. This much is apparent from symmetry alone. However, when the plate area is finite, then we expect a *fringing field* to emerge. “Fringing field” is simply a term applied to the non-uniform field that appears near the edge of the plates. The field is non-uniform in this region because the boundary conditions on the *outside* (outward-facing) surfaces of the plates have a significant effect in this region. In the central region of the capacitor, however, the field is not much different from the field that exists in the case of infinite plate area.

In any parallel plate capacitor having finite plate area, some fraction of the energy will be stored by the approximately uniform field of the central region, and the rest will be stored in the fringing field. We can make the latter negligible relative to the former by making the capacitor very “thin,” in the sense that the smallest identifiable dimension of the plate is much greater than d . Under this condition, we may obtain a good approximation of the capacitance by simply neglecting the fringing field, since an insignificant fraction of the energy is stored there.

Imposing the “thin” condition leads to three additional simplifications. First, the surface charge distribution may be assumed to be approximately uniform over the plate, which greatly simplifies the analysis. Second, the shape of the plates becomes irrelevant; they might be circular, square, triangular, etc. When computing capacitance in the “thin” case, only the plate area A is important. Third, the thickness of each of the plates becomes irrelevant.

We are now ready to determine the capacitance of the thin parallel plate capacitor. Here are the steps:

1. Assume a total positive charge Q_+ on the upper plate.
2. Invoking the “thin” condition, we assume the charge density on the plates is uniform. Thus, the surface charge density on bottom side of the upper plate is $\rho_{s,+} = Q_+/A$ (C/m²).
3. From the boundary condition on the bottom surface of the upper plate, $\mathbf{D}$ on this surface is $-\hat{\mathbf{z}}\rho_{s,+}$.
4. The total charge on the lower plate, Q_- , must be equal and opposite the total charge on the upper plate; i.e, $Q_- = -Q_+$. Similarly, the surface charge density on the upper surface of the lower plate, $\rho_{s,-}$, must be $-\rho_{s,+}$.
5. From the boundary condition on the top surface of the lower plate (Section 5.18), $\mathbf{D}$ on this surface is $+\hat{\mathbf{z}}\rho_{s,-}$. Since $+\hat{\mathbf{z}}\rho_{s,-} = -\hat{\mathbf{z}}\rho_{s,+}$, $\mathbf{D}$ on the facing sides of the plates is equal.
6. Again invoking the “thin” condition, we assume $\mathbf{D}$ between the plates has approximately the same structure as we would see if the plate area was infinite. Therefore, we are justified in assuming $\mathbf{D} \approx -\hat{\mathbf{z}}\rho_{s,+}$ *everywhere* between the plates. (You might also see that this is self-evident from the definition of $\mathbf{D}$ as the flux density of electric charge (Section 2.4).)
7. With an expression for the electric field in hand, we may now compute the potential difference V between the plates as follows (Section 5.8):

$$\begin{aligned}
 V &= - \int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} \\
 &= - \int_0^d \left(\frac{1}{\epsilon} \mathbf{D} \right) \cdot (\hat{\mathbf{z}} dz) \\
 &= - \int_0^d \left(-\hat{\mathbf{z}} \frac{\rho_{s,+}}{\epsilon} \right) \cdot (\hat{\mathbf{z}} dz) \\
 &= + \frac{\rho_{s,+} d}{\epsilon}
 \end{aligned} \tag{5.163}$$

8. Finally,

$$C = \frac{Q_+}{V} = \frac{\rho_{s,+} A}{\rho_{s,+} d/\epsilon} = \frac{\epsilon A}{d} \tag{5.164}$$

Summarizing:

$$\boxed{C \approx \frac{\epsilon A}{d}} \tag{5.165}$$

The capacitance of a parallel plate capacitor having plate separation much less than the size of the plate is given by Equation 5.165. This is an approximation because the fringing field is

neglected.

It's worth noting that this is dimensionally correct; i.e., F/m times m^2 divided by m yields F . It's also worth noting the effect of the various parameters:

Capacitance increases in proportion to permittivity and plate area and decreases in proportion to distance between the plates.

Example

Example 5.9

Printed circuit board capacitance.

Printed circuit boards commonly include a “ground plane,” which serves as the voltage datum for the board, and at least one “power plane,” which is used to distribute a DC supply voltage (See “Additional Reading” at the end of this section). These planes are separated by a dielectric material, and the resulting structure exhibits capacitance. This capacitance may be viewed as an equivalent discrete capacitor in parallel with the power supply. The value of this equivalent capacitor may be either negligible, significant and beneficial, or significant and harmful. So, it is useful to know the value of this equivalent capacitor.

For a common type of circuit board, the dielectric thickness is about 1.6 mm and the relative permittivity of the material is about 4.5. If the area in common between the ground and power planes is 25 cm^2 , what is the value of the equivalent capacitor?

Solution. From the problem statement, $\epsilon \cong 4.5\epsilon_0$, $A \cong 25 \text{ cm}^2 = 2.5 \times 10^{-3} \text{ m}^2$, and $d \cong 1.6 \text{ mm}$. Using Equation 5.165, the value of the equivalent capacitor is 62.3 pF.

Additional Reading:

- “Printed circuit board (https://en.wikipedia.org/wiki/Printed_circuit_board)” on Wikipedia.

5.24 Capacitance of a Coaxial Structure

Let us now determine the capacitance of coaxially-arranged conductors, shown in Figure 5.15. Among other applications, this information is useful in the analysis of voltage and current waves on coaxial transmission line, as addressed in Sections 3.4 and 3.10.

For our present purposes, we may model the structure as consisting of two concentric perfectly-conducting cylinders of radii a and b , separated by an ideal dielectric having permittivity ϵ_s . We place the $+z$ axis along the common axis of the concentric cylinders so that the cylinders may be described as constant-coordinate surfaces $\rho = a$ and $\rho = b$.

Figure 5.15. Determining the capacitance of a coaxial structure.

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In this section, we shall find the capacitance by assuming a total charge Q_+ on the inner conductor and integrating over the associated electric field to obtain the voltage between the conductors. Then, capacitance is computed as the ratio of the assumed charge to the resulting potential difference. This strategy is the same as that employed in Section 5.23 for the parallel plate capacitor, so it may be useful to review that section before attempting this derivation.

The first step is to find the electric field inside the structure. This is relatively simple if we assume that the structure has infinite length (i.e., $l \rightarrow \infty$), since then there are no fringing fields and the internal field will be utterly constant with respect to z . In the central region of a finite-length capacitor, however, the field is not much different from the field that exists in the case of infinite length, and if the energy storage in fringing fields is negligible compared to the energy storage in this central region then there is no harm in assuming the internal field is constant with z . Alternatively, we may think of the length l as pertaining to one short section of a much longer structure and thereby obtain the capacitance *per length* as opposed to the total capacitance. Note that the latter is exactly what we need for the transmission line lumped-element equivalent circuit model (Section 3.4).

To determine the capacitance, we invoke the definition (Section 5.22):

$$C \triangleq \frac{Q_+}{V} \quad (5.166)$$

where Q_+ is the charge on the positively-charged conductor and V is the potential measured from the negative conductor to the positive conductor. The charge on the inner conductor is uniformly-distributed with density

$$\rho_l = \frac{Q_+}{l} \quad (5.167)$$

which has units of C/m. Now we will determine the electric field intensity $\mathbf{E}$, integrate $\mathbf{E}$ over a path between conductors to get V , and then apply Equation 5.166 to obtain the capacitance.

The electric field intensity for this scenario was determined in Section 5.6, “Electric Field Due to an Infinite Line Charge using Gauss’ Law,” where we found

$$\mathbf{E} = \hat{\rho} \frac{\rho_l}{2\pi\epsilon_s\rho} \quad (5.168)$$

The reader should note that in that section we were considering merely a line of charge; not a coaxial structure. So, on what basis do we claim the field is the same? This is a consequence of Gauss’ Law (Section 5.5)

$$\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s} = Q_{encl} \quad (5.169)$$

which we used in Section 5.6 to find the field. If in this new problem we specify the same cylindrical surface $\mathcal{S}$ with radius $\rho < b$, then the enclosed charge is the same. Furthermore, the presence of the outer conductor does not change the radial symmetry of the problem, and nothing else remains that can change the outcome. This is worth noting for future reference:

The electric field inside a coaxial structure comprised of concentric conductors and having uniform charge density on the inner conductor is identical to the electric field of a line charge in free space having the same charge density.

Next, we get V using (Section 5.8)

$$V = - \int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{l} \quad (5.170)$$

where $\mathcal{C}$ is any path from the negatively-charged outer conductor to the positively-charged inner conductor. Since this can be *any* such path (Section 5.9), we may as well choose the simplest one. This path is the one that traverses a radial of constant ϕ and z . Thus:

$$\begin{aligned} V &= - \int_{\rho=b}^a \left(\hat{\rho} \frac{\rho_l}{2\pi\epsilon_s\rho} \right) \cdot (\hat{\rho} d\rho) \\ &= - \frac{\rho_l}{2\pi\epsilon_s} \int_{\rho=b}^a \frac{d\rho}{\rho} \\ &= + \frac{\rho_l}{2\pi\epsilon_s} \int_{\rho=a}^b \frac{d\rho}{\rho} \\ &= + \frac{\rho_l}{2\pi\epsilon_s} \ln \left(\frac{b}{a} \right) \end{aligned} \quad (5.171)$$

Wrapping up:

$$C \triangleq \frac{Q_+}{V} = \frac{\rho_l l}{(\rho_l / 2\pi\epsilon_s) \ln(b/a)} \quad (5.172)$$

Note that factors of ρ_l in the numerator and denominator cancel out, leaving:

$$C = \frac{2\pi\epsilon_s l}{\ln(b/a)} \quad (5.173)$$

Note that this expression is dimensionally correct, having units of F. Also note that the expression depends only on materials (through ϵ_s) and geometry (through l , a , and b). The expression does *not* depend on charge or voltage, which would imply non-linear behavior.

To make the connection back to lumped-element transmission line model parameters (Sections [3.4](#) and [3.10](#)), we simply divide by l to get the per-unit length parameter:

$$C' = \frac{2\pi\epsilon_s}{\ln(b/a)} \quad (5.174)$$

Example

Example 5.10

Capacitance of RG-59 coaxial cable.

RG-59 coaxial cable consists of an inner conductor having radius 0.292 mm, an outer conductor having radius 1.855 mm, and a polyethylene spacing material having relative permittivity 2.25. Estimate the capacitance per length of RG-59.

Solution. From the problem statement, $a = 0.292$ mm, $b = 1.855$ mm, and $\epsilon_s = 2.25\epsilon_0$. Using Equation [5.174](#) we find $C' = 67.7$ pF/m.

5.25 Electrostatic Energy

Consider a structure consisting of two perfect conductors, both fixed in position and separated by an ideal dielectric. This could be a capacitor, or it could be one of a variety of capacitive structures that are not explicitly intended to be a capacitor – for example, a printed circuit board. When a potential difference is applied between the two conducting regions, a positive charge Q_+ will appear on the

surface of the conductor at the higher potential, and a negative charge $Q_- = -Q_+$ will appear on the surface of the conductor at the lower potential (Section 5.19). Assuming the conductors are not free to move, potential energy is stored in the electric field associated with the surface charges (Section 5.22).

We now ask the question, what is the energy stored in this field? The answer to this question has relevance in several engineering applications. For example, when capacitors are used as batteries, it is useful to know the amount of energy that can be stored. Also, any system that includes capacitors or has unintended capacitance is using some fraction of the energy delivered by the power supply to charge the associated structures. In many electronic systems – and in digital systems in particular – capacitances are periodically charged and subsequently discharged at a regular rate. Since power is energy per unit time, this cyclic charging and discharging of capacitors consumes power. Therefore, energy storage in capacitors contributes to the power consumption of modern electronic systems. We'll delve into that topic in more detail in Example 5.11.

Since capacitance C relates the charge Q_+ to the potential difference V between the conductors, this is the natural place to start. From the definition of capacitance (Section 5.22):

$$V = \frac{Q_+}{C} \quad (5.175)$$

From Section 5.8, electric potential is defined as the work done (i.e., energy injected) by moving a charged particle, per unit of charge; i.e.,

$$V = \frac{W_e}{q} \quad (5.176)$$

where q is the charge borne by the particle and W_e (units of J) is the work done by moving this particle across the potential difference V . Since we are dealing with charge distributions as opposed to charged particles, it is useful to express this in terms of the contribution ΔW_e made to W_e by a small charge Δq . Letting Δq approach zero we have

$$dW_e = V dq \quad (5.177)$$

Now consider what must happen to transition the system from having zero charge ($q = 0$) to the fully-charged but static condition ($q = Q_+$). This requires moving the differential amount of charge dq across the potential difference between conductors, beginning with $q = 0$ and continuing until $q = Q_+$. Therefore, the total amount of work done in this process is:

$$\begin{aligned}
 W_e &= \int_{q=0}^{Q_+} dW_e \\
 &= \int_0^{Q_+} V dq \\
 &= \int_0^{Q_+} \frac{q}{C} dq \\
 &= \frac{1}{2} \frac{Q_+^2}{C}
 \end{aligned} \tag{5.178}$$

Equation [5.178](#) can be expressed entirely in terms of electrical potential by noting again that $C = Q_+/V$, so

$$W_e = \frac{1}{2} CV^2 \tag{5.179}$$

Since there are no other processes to account for the injected energy, the energy stored in the electric field is equal to W_e . Summarizing:

The energy stored in the electric field of a capacitor (or a capacitive structure) is given by Equation [5.179](#).

Example

Example 5.11

Why multicore computing is power-neutral.

Readers are likely aware that computers increasingly use multicore processors as opposed to single-core processors. For our present purposes, a “core” is defined as the smallest combination of circuitry that performs independent computation. A multicore processor consists of multiple identical cores that run in parallel. Since a multicore processor consists of N identical processors, you might expect power consumption to increase by N relative to a single-core processor. However, this is not the case. To see why, first realize that the power consumption of a modern computing core is dominated by the energy required to continuously charge and discharge the multitude of capacitances within the core. From Equation 5.179, the required energy is $\frac{1}{2}C_0V_0^2$ per clock cycle, where C_0 is the sum capacitance (remember, capacitors in parallel add) and V_0 is the supply voltage. Power is energy per unit time, so the power consumption for a single core is

$$P_0 = \frac{1}{2}C_0V_0^2f_0 \quad (5.180)$$

where f_0 is the clock frequency. In a N -core processor, the sum capacitance is increased by N . However, the frequency is decreased by N since the same amount of computation is (nominally) distributed among the N cores. Therefore, the power consumed by an N -core processor is

$$P_N = \frac{1}{2}(NC_0)V_0^2\left(\frac{f_0}{N}\right) = P_0 \quad (5.181)$$

In other words, the increase in power associated with replication of hardware is nominally offset by the decrease in power enabled by reducing the clock rate. In yet other words, the total energy of the N -core processor is N times the energy of the single core processor at any given time; however, the multicore processor needs to recharge capacitances $1/N$ times as often.

Before moving on, it should be noted that the usual reason for pursuing a multicore design is to increase the amount of computation that can be done; i.e., to increase the product f_0N . Nevertheless, it is extremely helpful that power consumption is proportional to f_0 only, and is independent of N .

The thin parallel plate capacitor (Section 5.23) is representative of a large number of practical applications, so it is instructive to consider the implications of Equation 5.179 for this structure in particular. For the thin parallel plate capacitor,

$$C \approx \frac{\epsilon A}{d} \quad (5.182)$$

where A is the plate area, d is the separation between the plates, and ϵ is the permittivity of the material between the plates. This is an approximation because the fringing field is neglected; we shall proceed as if this is an exact expression. Applying Equation 5.179:

$$W_e = \frac{1}{2} \left(\frac{\epsilon A}{d} \right) (Ed)^2 \quad (5.183)$$

where E is the magnitude of the electric field intensity between the plates. Rearranging factors, we obtain:

$$W_e = \frac{1}{2} \epsilon E^2 (Ad) \quad (5.184)$$

Recall that the electric field intensity in the thin parallel plate capacitor is approximately uniform. Therefore, the density of energy stored in the capacitor is also approximately uniform. Noting that the product Ad is the volume of the capacitor, we find that the energy density is

$$w_e = \frac{W_e}{Ad} = \frac{1}{2} \epsilon E^2 \quad (5.185)$$

which has units of energy per unit volume (J/m^3).

The above expression provides an alternative method to compute the total electrostatic energy. Within a mathematical volume $\mathcal{V}$, the total electrostatic energy is simply the integral of the energy density over $\mathcal{V}$; i.e.,

$$W_e = \int_{\mathcal{V}} w_e \, dv \quad (5.186)$$

This works even if E and ϵ vary with position. So, even though we arrived at this result using the example of the thin parallel-plate capacitor, our findings at this point apply generally. Substituting Equation 5.185 we obtain:

$$\boxed{W_e = \frac{1}{2} \int_{\mathcal{V}} \epsilon E^2 \, dv} \quad (5.187)$$

Summarizing:

The energy stored by the electric field present within a volume is given by Equation 5.187.

It's worth noting that this energy increases with the permittivity of the medium, which makes sense since capacitance is proportional to permittivity.

End Matter

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