9 Radiation

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9.1 Radiation from a Current Moment

In this section, we begin to address the following problem: Given a distribution of impressed current density $\mathbf{J}(\mathbf{r})$, what is the resulting electric field intensity $\mathbf{E}(\mathbf{r})$? One route to an answer is via Maxwell's equations. Viewing Maxwell's equations as a system of differential equations, a rigorous mathematical solution is possible given the appropriate boundary conditions. The rigorous solution following that approach is relatively complicated, and is presented beginning in Section 9.2 of this book.

If we instead limit scope to a sufficiently simple current distribution, a simple informal derivation is possible. This section presents such a derivation. The advantage of tackling a simple special case first is that it will allow us to quickly assess the nature of the solution, which will turn out to be useful once we do eventually address the more general problem. Furthermore, the results presented in this section will turn out to be sufficient to tackle many commonly-encountered applications.

The simple current distribution considered in this section is known as a *current moment*. An example of a current moment is shown in Figure 9.1 and in this case is defined as follows:

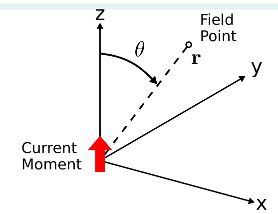

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Figure 9.1. *A $+\hat{\mathbf{z}}$-directed current moment located at the origin.*

$$\Delta\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}}\; I\; \Delta l\; \delta(\mathbf{r}) \tag{9.1}$$

where $I\;\Delta l$ is the scalar component of the current moment, having units of current times length (SI base units of A·m); and $\delta(\mathbf{r})$ is the volumetric sampling function[1] defined as follows:

$$\delta(\mathbf{r}) \triangleq 0 \quad \text{for} \quad \mathbf{r} \neq 0; \quad \text{and} \tag{9.2}$$

$$\int_{\mathcal{V}} \delta(\mathbf{r})\; dv \triangleq 1 \tag{9.3}$$

where $\mathcal{V}$ is any volume which includes the origin ($\mathbf{r}=0$). It is evident from Equation 9.3 that $\delta(\mathbf{r})$ has SI base units of m^{-3}. Subsequently, $\Delta\mathbf{J}(\mathbf{r})$ has SI base units of A/m^2, confirming that it is a volume current density. However, it is the *simplest possible* form of volume current density, since – as indicated by Equation 9.2 – it exists only at the origin and nowhere else.

Although some current distributions approximate the current moment, current distributions encountered in common engineering practice generally do not exist in precisely this form. Nevertheless, the current moment turns out to be generally useful as a “building block” from which practical distributions of current can be constructed, via the principle of superposition. Radiation from current distributions constructed in this manner is calculated simply by summing the radiation from each of the constituent current moments.

Now let us consider the electric field intensity $\Delta\mathbf{E}(\mathbf{r})$ that is created by this current distribution. First, if the current is steady (i.e., “DC”), this problem falls within the domain of magnetostatics; i.e., the outcome is completely described by the magnetic field, and there can be no radiation. Therefore, let us limit our attention to the “AC” case, for which radiation is possible. It will be convenient to employ phasor representation. In phasor representation, the current density is

$$\Delta\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{z}}\ \tilde{I}\ \Delta l\ \delta(\mathbf{r}) \tag{9.4}$$

where $\tilde{I}\ \Delta l$ is simply the scalar current moment expressed as a phasor.

Now we are ready to address the question “What is $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ due to $\Delta\widetilde{\mathbf{J}}(\mathbf{r})$?” Without doing any math, we know quite a bit about $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$. For example:

- Since electric fields are proportional to the currents that give rise to them, we expect $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ to be proportional to $\left|\tilde{I}\ \Delta l\right|$.
- If we are sufficiently far from the origin, we expect $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ to be approximately proportional to $1/r$ where $r \triangleq |\mathbf{r}|$ is the distance from the source current. This is because point sources give rise to spherical waves, and the power density in a spherical wave would be proportional to $1/r^2$. Since time-average power density is proportional to $\left|\Delta\widetilde{\mathbf{E}}(\mathbf{r})\right|^2$, $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ must be proportional to $1/r$.
- If we are sufficiently far from the origin, and the loss due to the medium is negligible, then we expect the phase of $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ to change approximately at rate β where β is the phase propagation constant $2\pi/\lambda$. Since we expect spherical phasefronts, $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ should therefore contain the factor $e^{-j\beta r}$.
- Ampere’s law indicates that a $\hat{\mathbf{z}}$-directed current at the origin should give rise to a $\hat{\phi}$-directed magnetic field in the $z=0$ plane.[2] At the same time, Poynting’s theorem requires the cross product of the electric and magnetic fields to point in the direction of power flow. In the present problem, this direction is away from the source; i.e., $+\hat{\mathbf{r}}$. Therefore, $\Delta\widetilde{\mathbf{E}}(z=0)$ points in the $-\hat{\mathbf{z}}$ direction. The same principle applies outside of the $z=0$ plane, so in general we expect $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ to point in the $\hat{\theta}$ direction.
- We expect $\Delta\widetilde{\mathbf{E}}(\mathbf{r})=0$ along the z axis. Subsequently $\left|\Delta\widetilde{\mathbf{E}}(\hat{\mathbf{r}})\right|$ must increase from zero at $\theta=0$ and return to zero at $\theta=\pi$. The symmetry of the problem suggests $\left|\Delta\widetilde{\mathbf{E}}(\hat{\mathbf{r}})\right|$ is maximum at $\theta=\pi/2$. This magnitude must vary in the simplest possible way, leading us to conclude that $\Delta\widetilde{\mathbf{E}}(\hat{\mathbf{r}})$ is proportional to $\sin\theta$. Furthermore, the radial symmetry of the problem means that $\Delta\widetilde{\mathbf{E}}(\hat{\mathbf{r}})$ should not depend at all on ϕ.

Putting these ideas together, we conclude that the radiated electric field has the following form:

$$\Delta\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} C \left(\tilde{I}\ \Delta l\right) (\sin\theta)\, \frac{e^{-j\beta r}}{r} \tag{9.5}$$

where C is a constant which accounts for all of the constants of proportionality identified in the preceding analysis. Since the units of $\Delta\widetilde{\mathbf{E}}(\mathbf{r})$ are V/m, the units of C must be Ω/m. We have not yet accounted for the wave impedance of the medium η, which has units of Ω, so it would be a good bet based on the units that C is proportional to η. However, here the informal analysis reaches a dead end, so we shall simply state the result from the rigorous solution: $C = j\eta\beta/4\pi$. The units are correct, and we finally obtain:

$$\Delta\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta}\frac{j\eta\beta}{4\pi}\left(\tilde{I}\ \Delta l\right)(\sin\theta)\frac{e^{-j\beta r}}{r} \tag{9.6}$$

Additional evidence that this solution is correct comes from the fact that it satisfies the wave equation $\nabla^2\Delta\widetilde{\mathbf{E}}(\mathbf{r}) + \beta^2\Delta\widetilde{\mathbf{E}}(\mathbf{r}) = 0$.[3]

Note that the expression we have obtained for the radiated electric field is approximate (hence the “$\approx$”). This is due in part to our presumption of a simple spherical wave, which may only be valid at distances far from the source. But how far? An educated guess would be distances much greater than a wavelength (i.e., $r \gg \lambda$). This will do for now; in another section, we shall show rigorously that this guess is essentially correct.

We conclude this section by noting that the current distribution analyzed in this section is sometimes referred to as a *Hertzian dipole*. A Hertzian dipole is typically defined as a straight infinitesimally-thin filament of current with length which is very small relative to a wavelength, but not precisely zero. This interpretation does not change the solution obtained in this section, thus we may view the current moment and the Hertzian dipole as effectively the same in practical engineering applications.

Additional Reading:

- “Dirac delta function (https://en.wikipedia.org/wiki/Dirac_delta_function)” on Wikipedia.
- “Dipole antenna (https://en.wikipedia.org/wiki/Dipole_antenna#Hertzian_dipole)” (section entitled “Hertzian Dipole”) on Wikipedia.

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1. Also a form of the *Dirac delta function*; see “Additional Reading” at the end of this section. ↩
2. This is sometimes described as the “right hand rule” of Ampere’s law. ↩
3. Confirming this is straightforward (simply substitute and evaluate) and is left as an exercise for the student. ↩

9.2 Magnetic Vector Potential

A common problem in electromagnetics is to determine the fields radiated by a specified current distribution. This problem can be solved using Maxwell’s equations along with the appropriate electromagnetic boundary conditions. For time-harmonic (sinusoidally-varying) currents, we use phasor representation.[1] Given the specified current distribution $\widetilde{\mathbf{J}}$ and the desired electromagnetic fields $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$, the appropriate equations are:

$$\nabla\cdot\widetilde{\mathbf{E}} = \tilde{\rho}_v/\epsilon \tag{9.7}$$

$$\nabla\times\widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}} \tag{9.8}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0 \tag{9.9}$$

$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\epsilon\widetilde{\mathbf{E}} \tag{9.10}$$

where $\tilde{\rho}_v$ is the volume charge density. In most engineering problems, one is concerned with propagation through media which are well-modeled as homogeneous media with neutral charge, such as free space.[2] Therefore, in this section, we shall limit our scope to problems in which $\tilde{\rho}_v = 0$. Thus, Equation 9.7 simplifies to:

$$\nabla \cdot \widetilde{\mathbf{E}} = 0 \tag{9.11}$$

To solve the linear system of partial differential Equations 9.8–9.11, it is useful to invoke the concept of *magnetic vector potential*. The magnetic vector potential is a vector field that has the useful property that it is able to represent both the electric and magnetic fields as a single field. This allows the formidable system of equations identified above to be reduced to a single equation which is simpler to solve. Furthermore, this single equation turns out to be the wave equation, with the slight difference that the equation will be mathematically inhomogeneous, with the inhomogeneous part representing the source current.

The magnetic vector potential $\widetilde{\mathbf{A}}$ is defined by the following relationship:

$$\boxed{\widetilde{\mathbf{B}} \triangleq \nabla \times \widetilde{\mathbf{A}}} \tag{9.12}$$

where $\widetilde{\mathbf{B}} = \mu\widetilde{\mathbf{H}}$ is the magnetic flux density. The magnetic field appears in three of Maxwell's equations. For Equation 9.12 to be a reasonable definition, $\nabla \times \widetilde{\mathbf{A}}$ must yield reasonable results when substituted for $\mu\widetilde{\mathbf{H}}$ in each of these equations. Let us first check for consistency with Gauss' law for magnetic fields, Equation 9.9. Making the substitution, we obtain:

$$\nabla \cdot \left(\nabla \times \widetilde{\mathbf{A}}\right) = 0 \tag{9.13}$$

This turns out to be a mathematical identity that applies to *any* vector field (see Equation B.24 in Appendix B.3). Therefore, Equation 9.12 is consistent with Gauss' law for magnetic fields.

Next we check for consistency with Equation 9.8. Making the substitution:

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\left(\nabla \times \widetilde{\mathbf{A}}\right) \tag{9.14}$$

Gathering terms on the left, we obtain

$$\nabla \times \left(\widetilde{\mathbf{E}} + j\omega\widetilde{\mathbf{A}}\right) = 0 \tag{9.15}$$

Now, for reasons that will become apparent in just a moment, we define a new scalar field $\widetilde{V}$ and require it to satisfy the following relationship:

$$-\nabla\widetilde{V} \triangleq \widetilde{\mathbf{E}} + j\omega\widetilde{\mathbf{A}} \tag{9.16}$$

Using this definition, Equation 9.15 becomes:

$$\nabla \times \left(-\nabla\widetilde{V}\right) = 0 \tag{9.17}$$

which is simply

$$\nabla \times \nabla\widetilde{V} = 0 \tag{9.18}$$

Once again we have obtained a mathematical identity that applies to *any* vector field (see Equation B.25 in Appendix B.3). Therefore, $\widetilde{V}$ can be *any* mathematically-valid scalar field. Subsequently, Equation 9.12 is consistent with Equation 9.8 (Maxwell's curl equation for the electric field) for any choice of $\widetilde{V}$ that we are inclined to make.

Astute readers might already realize what we're up to here. Equation 9.16 is very similar to the relationship $\mathbf{E} = -\nabla V$ from electrostatics,[3] in which V is the scalar electric potential field. Evidently Equation 9.16 is an enhanced version of that relationship that accounts for the coupling with $\mathbf{H}$ (here, represented by $\mathbf{A}$) in the time-varying (decidedly non-static) case. That assessment is correct, but let's not get too far ahead of ourselves: As demonstrated in the previous paragraph, we are not yet compelled to make any particular choice for $\widetilde{V}$, and this freedom will be exploited later in this section.

Next we check for consistency with Equation 9.10. Making the substitution:

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \widetilde{\mathbf{A}}\right) = \widetilde{\mathbf{J}} + j\omega\epsilon\widetilde{\mathbf{E}} \tag{9.19}$$

Multiplying both sides of the equation by μ:

$$\nabla \times \nabla \times \widetilde{\mathbf{A}} = \mu\widetilde{\mathbf{J}} + j\omega\mu\epsilon\widetilde{\mathbf{E}} \tag{9.20}$$

Next we use Equation 9.16 to eliminate $\widetilde{\mathbf{E}}$, yielding:

$$\nabla \times \nabla \times \widetilde{\mathbf{A}} = \mu\widetilde{\mathbf{J}} + j\omega\mu\epsilon\left(-\nabla\widetilde{V} - j\omega\widetilde{\mathbf{A}}\right) \tag{9.21}$$

After a bit of algebra, we obtain

$$\nabla \times \nabla \times \widetilde{\mathbf{A}} = \omega^2\mu\epsilon\widetilde{\mathbf{A}} - j\omega\mu\epsilon\nabla\widetilde{V} + \mu\widetilde{\mathbf{J}} \tag{9.22}$$

Now we replace the left side of this equation using vector identity B.29 in Appendix B.3:

$$\nabla \times \nabla \times \widetilde{\mathbf{A}} \equiv \nabla \left(\nabla \cdot \widetilde{\mathbf{A}} \right) - \nabla^2 \widetilde{\mathbf{A}} \tag{9.23}$$

Equation 9.22 becomes:

$$\nabla \left(\nabla \cdot \widetilde{\mathbf{A}} \right) - \nabla^2 \widetilde{\mathbf{A}} = \omega^2 \mu\epsilon \widetilde{\mathbf{A}} - j\omega\mu\epsilon \nabla \widetilde{V} + \mu \widetilde{\mathbf{J}} \tag{9.24}$$

Now multiplying both sides by -1 and rearranging terms:

$$\nabla^2 \widetilde{\mathbf{A}} + \omega^2 \mu\epsilon \widetilde{\mathbf{A}} = \nabla \left(\nabla \cdot \widetilde{\mathbf{A}} \right) + j\omega\mu\epsilon \nabla \widetilde{V} - \mu \widetilde{\mathbf{J}} \tag{9.25}$$

Combining terms on the right side:

$$\nabla^2 \widetilde{\mathbf{A}} + \omega^2 \mu\epsilon \widetilde{\mathbf{A}} = \nabla \left(\nabla \cdot \widetilde{\mathbf{A}} + j\omega\mu\epsilon \widetilde{V} \right) - \mu \widetilde{\mathbf{J}} \tag{9.26}$$

Now consider the expression $\nabla \cdot \widetilde{\mathbf{A}} + j\omega\mu\epsilon \widetilde{V}$ appearing in the parentheses on the right side of the equation. We established earlier that $\widetilde{V}$ can be essentially any scalar field – from a mathematical perspective, we are free to choose. Invoking this freedom, we now require $\widetilde{V}$ to satisfy the following expression:

$$\nabla \cdot \widetilde{\mathbf{A}} + j\omega\mu\epsilon \widetilde{V} = 0 \tag{9.27}$$

Clearly this is advantageous in the sense that Equation 9.26 is now dramatically simplified. This equation becomes:

$$\boxed{\nabla^2 \widetilde{\mathbf{A}} + \omega^2 \mu\epsilon \widetilde{\mathbf{A}} = -\mu \widetilde{\mathbf{J}}} \tag{9.28}$$

Note that this expression is a *wave equation*. In fact it is the *same* wave equation that determines $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ in source-free regions, *except* the right-hand side is not zero. Using mathematical terminology, we have obtained an equation for $\widetilde{\mathbf{A}}$ in the form of an *inhomogeneous* partial differential equation, where the inhomogeneous part includes – no surprise here – the source current $\widetilde{\mathbf{J}}$.

Now we have what we need to find the electromagnetic fields radiated by a current distribution. The procedure is simply as follows:

1. Solve the partial differential Equation 9.28 for $\widetilde{\mathbf{A}}$ along with the appropriate electromagnetic boundary conditions.
2. $\widetilde{\mathbf{H}} = (1/\mu) \nabla \times \widetilde{\mathbf{A}}$
3. $\widetilde{\mathbf{E}}$ may now be determined from $\widetilde{\mathbf{H}}$ using Equation 9.10.

Summarizing:

> The magnetic vector potential $\widetilde{\mathbf{A}}$ is a vector field, defined by Equation 9.12, that is able to represent both the electric and magnetic fields simultaneously.

Also:

> To determine the electromagnetic fields radiated by a current distribution $\widetilde{\mathbf{J}}$, one may solve Equation 9.28 for $\widetilde{\mathbf{A}}$ and then use Equation 9.12 to determine $\widetilde{\mathbf{H}}$ and subsequently $\widetilde{\mathbf{E}}$.

Specific techniques for performing this procedure – in particular, for solving the differential equation – vary depending on the problem, and are discussed in other sections of this book.

We conclude this section with a few comments about Equation 9.27. This equation is known as the *Lorenz gauge condition*. This constraint is not quite as arbitrary as the preceding derivation implies; rather, there is some deep physics at work here. Specifically, the Lorenz gauge leads to the classical interpretation of $\widetilde{V}$ as the familiar scalar electric potential, as noted previously in this section. (For additional information on that idea, recommended starting points are included in "Additional Reading" at the end of this section.)

At this point, it should be clear that the electric and magnetic fields are not merely coupled quantities, but in fact two aspects of the same field; namely, the magnetic vector potential. In fact modern physics (quantum mechanics) yields the magnetic vector potential as a description of the "electromagnetic force," a single entity which constitutes one of the four fundamental forces recognized in modern physics; the others being gravity, the strong nuclear force, and the weak nuclear force. For more information on that concept, an excellent starting point is the video "Quantum Invariance & The Origin of The Standard Model" referenced at the end of this section.

Additional Reading:

- "Lorenz gauge condition (https://en.wikipedia.org/wiki/Lorenz_gauge_condition)" on Wikipedia.
- "Magnetic potential (https://en.wikipedia.org/wiki/Magnetic_potential)" on Wikipedia.
- PBS Space Time video "Quantum Invariance (https://youtu.be/V5kgruUjVBs) & The Origin of The Standard Model," available on YouTube.

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1. Recall that there is no loss of generality in doing so, since any other time-domain variation in the current distribution can be represented using sums of time-harmonic solutions via the Fourier transform. ↩
2. A counter-example would be propagation through a plasma, which by definition consists of non-zero net charge. ↩
3. Note: No tilde in this expression. ↩

9.3 Solution of the Wave Equation for Magnetic Vector Potential

The magnetic vector potential $\widetilde{\mathbf{A}}$ due to a current density $\widetilde{\mathbf{J}}$ is given by the following wave equation:

$$\nabla^2\widetilde{\mathbf{A}} - \gamma^2\widetilde{\mathbf{A}} = -\mu\widetilde{\mathbf{J}} \tag{9.29}$$

where γ is the *propagation constant*, defined in the usual manner[1]

$$\gamma^2 \triangleq -\omega^2\mu\epsilon \tag{9.30}$$

Equation 9.29 is a partial differential equation which is inhomogeneous (in the mathematical sense) and can be solved given appropriate boundary conditions. In this section, we present the solution for arbitrary distributions of current in free space.

The strategy is to first identify a solution for a distribution of current that exists at a single point. This distribution is the *current moment*. An example of a current moment is shown in Figure 9.2.

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Figure 9.2. *An example of a current moment located at the origin. In this case,* $\hat{\mathbf{l}} = \hat{\mathbf{z}}$.

A general expression for a current moment located at the origin is:

$$\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{l}}\ \tilde{I}\ \Delta l\ \delta(\mathbf{r}) \tag{9.31}$$

where $\tilde{I}$ has units of current (SI base units of A), Δl has units of length (SI base units of m), $\hat{\mathbf{l}}$ is the direction of current flow, and $\delta(\mathbf{r})$ is the volumetric sampling function[2] defined as follows:

$$\delta(\mathbf{r}) \triangleq 0 \quad \text{for} \quad \mathbf{r} \neq 0; \quad \text{and} \tag{9.32}$$
$$\int_{\mathcal{V}} \delta(\mathbf{r})\ dv \triangleq 1 \tag{9.33}$$

where $\mathcal{V}$ is any volume which includes the origin ($\mathbf{r} = 0$). It is evident from Equation 9.33 that $\delta(\mathbf{r})$ has SI base units of m^{-3}; i.e., inverse volume. Subsequently $\mathbf{J}(\mathbf{r})$ has SI base units of A/m^2, indicating that it is a volume current density.[3] – as indicated by Equation 9.32 – it exists only at the origin and nowhere else.

Substituting Equation 9.31 into Equation 9.29, we obtain:

$$\nabla^2\widetilde{\mathbf{A}} - \gamma^2\widetilde{\mathbf{A}} = -\mu\hat{\mathbf{l}}\ \tilde{I}\ \Delta l\ \delta(\mathbf{r}) \tag{9.34}$$

The general solution to this equation presuming homogeneous and time-invariant media (i.e., μ and ϵ constant with respect to space and time) is:

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{l}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{\pm\gamma r}}{4\pi r} \tag{9.35}$$

To confirm that this is a solution to Equation 9.34, substitute this expression into Equation 9.29 and observe that the equality holds.

Note that Equation 9.35 indicates two solutions, corresponding to the signs of the exponent in the factor $e^{\pm\gamma r}$. We can actually be a little more specific by applying some common-sense physics. Recall that γ in general may be expressed in terms of real and imaginary components as follows:

$$\gamma = \alpha + j\beta \tag{9.36}$$

where α is a real-valued positive constant known as the *attenuation constant* and β is a real-valued positive constant known as the *phase propagation constant*. Thus, we may rewrite Equation 9.35 as follows:

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{l}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{\pm\alpha r}e^{\pm j\beta r}}{4\pi r} \tag{9.37}$$

Now consider the factor $e^{\pm\alpha r}/r$, which determines the dependence of magnitude on distance r. If we choose the negative sign in the exponent, this factor decays exponentially with increasing distance from the origin, ultimately reaching zero at $r \to \infty$. This is precisely the expected behavior, since we expect the magnitude of a radiated field to diminish with increasing distance from the source. If on the other hand we choose the positive sign in the exponent, this factor increases to infinity as $r \to \infty$. That outcome can be ruled out on physical grounds, since it implies the introduction of energy independently from the source. The requirement that field magnitude diminishes to zero as distance from the source increases to infinity is known as the *radiation condition*, and is essentially a boundary condition that applies at $r \to \infty$. Invoking the radiation condition, Equation 9.35 becomes:

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{l}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{-\gamma r}}{4\pi r} \tag{9.38}$$

In the loss-free ($\alpha = 0$) case, we cannot rely on the radiation condition to constrain the sign of γ. However, in practical engineering work there is always some media loss; i.e., α might be negligible but is not quite zero. Thus, we normally assume that the solution for lossless (including free space) conditions is given by Equation 9.38 with $\alpha = 0$:

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{l}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{-j\beta r}}{4\pi r} \tag{9.39}$$

Now consider the factor $e^{-j\beta r}$ in Equation 9.39. This factor exclusively determines the dependence of the phase of $\widetilde{\mathbf{A}}(\mathbf{r})$ with increasing distance r from the source. In this case, we observe that surfaces of constant phase correspond to spherical shells which are concentric with the source. Thus, $\widetilde{\mathbf{A}}(\mathbf{r})$ is a spherical wave.

Let us now consider a slightly more complicated version of the problem in which the current moment is no longer located at the origin, but rather is located at $\mathbf{r}'$. This is illustrated in Figure 9.3.

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Figure 9.3. *Current moment displaced from the origin.*

This current distribution can be expressed as follows:

$$\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{l}}\ \tilde{I}\ \Delta l\ \delta(\mathbf{r}-\mathbf{r}') \tag{9.40}$$

The solution in this case amounts to a straightforward modification of the existing solution. To see this, note that $\widetilde{\mathbf{A}}$ depends only on r, and not at all on θ or ϕ. In other words, $\widetilde{\mathbf{A}}$ depends only on the distance between the "field point" $\mathbf{r}$ at which we observe $\widetilde{\mathbf{A}}$, and the "source point" $\mathbf{r}'$ at which the current moment lies. Thus we replace r with this distance, which is $|\mathbf{r}-\mathbf{r}'|$. The solution becomes:

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{l}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \tag{9.41}$$

Continuing to generalize the solution, let us now consider a scenario consisting of a filament of current following a path $\mathcal{C}$ through space. This is illustrated in Figure 9.4.

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Figure 9.4. *A filament of current lying along the path $\mathcal{C}$. This current distribution may be interpreted as a collection of current moments lying along $\mathcal{C}$.*

Such a filament may be viewed as a collection of a large number N of discrete current moments distributed along the path. The contribution of the n^{th} current moment, located at $\mathbf{r}_n$, to the total magnetic vector potential is:

$$\Delta\widetilde{\mathbf{A}}(\mathbf{r};\mathbf{r}_n) = \hat{\mathbf{l}}(\mathbf{r}_n)\ \mu\ \tilde{I}(\mathbf{r}_n)\ \Delta l\ \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}_n|}}{4\pi\,|\mathbf{r}-\mathbf{r}_n|} \tag{9.42}$$

Note that we are allowing both the current $\tilde{I}$ and current direction $\hat{\mathbf{l}}$ to vary with position along $\mathcal{C}$. Assuming the medium is linear, superposition applies. So we add these contributions to obtain the total magnetic vector potential:

$$\widetilde{\mathbf{A}}(\mathbf{r}) \approx \sum_{n=1}^{N} \Delta\widetilde{\mathbf{A}}(\mathbf{r};\mathbf{r}_n) \tag{9.43}$$

$$\approx \frac{\mu}{4\pi}\ \sum_{n=1}^{N} \hat{\mathbf{l}}(\mathbf{r}_n)\ \tilde{I}(\mathbf{r}_n)\ \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}_n|}}{|\mathbf{r}-\mathbf{r}_n|}\ \Delta l \tag{9.44}$$

Now letting $\Delta l \to 0$ so that we may replace Δl with the differential length dl:

$$\boxed{\widetilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{\mathcal{C}} \hat{\mathbf{l}}(\mathbf{r}')\ \tilde{I}(\mathbf{r}')\ \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ dl} \tag{9.45}$$

where we have also replaced $\mathbf{r}_n$ with the original notation $\mathbf{r}'$ since we once again have a continuum (as opposed to a discrete set) of source locations.

> The magnetic vector potential corresponding to radiation from a line distribution of current is given by Equation 9.45.

Given $\widetilde{\mathbf{A}}(\mathbf{r})$, the magnetic and electric fields may be determined using the procedure developed in Section 9.2.

Additional Reading:

- "Magnetic potential (https://en.wikipedia.org/wiki/Magnetic_potential)" on Wikipedia.
- "Dirac delta function (https://en.wikipedia.org/wiki/Dirac_delta_function)" on Wikipedia.

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1. Alternatively, $\gamma^2 \triangleq -\omega^2\mu\epsilon_c$ accounting for the possibility of lossy media. ↩
2. Also a form of the *Dirac delta function*; see "Additional Reading" at the end of this section. ↩
3. Remember, the density of a volume current is with respect to the *area* through which it flows, therefore the units are A/m^2. ↩

9.4 Radiation from a Hertzian Dipole

Section 9.1 presented an informal derivation of the electromagnetic field radiated by a Hertzian dipole represented by a zero-length current moment. In this section, we provide a rigorous derivation using the concept of magnetic vector potential discussed in Sections 9.2 and 9.3. A review of those sections is recommended before tackling this section.

A Hertzian dipole is commonly defined as an electrically-short and infinitesimally-thin straight filament of current, in which the density of the current is uniform over its length. The Hertzian dipole is commonly used as a "building block" for constructing physically-realizable distributions of current as exhibited by devices such as wire antennas. The method is to model these relatively complex distributions of current as the sum of Hertzian dipoles, which reduces the problem to that of summing the contributions of the individual Hertzian dipoles, with each Hertzian dipole having the appropriate (i.e., different) position, magnitude, and phase.

To facilitate use of the Hertzian dipole as a building block suitable for constructing physically-realizable distributions of current, we choose to represent the Hertzian dipole using an essentially equivalent current distribution which is mathematically more versatile. This description of the Hertzian dipole replaces the notion of constant current over finite length with the notion of a *current moment* located at a single point. This is shown in Figure 9.5,

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Figure 9.5. *A Hertzian dipole located at the origin, represented as a current moment. In this case,* $\hat{\mathbf{l}} = \hat{\mathbf{z}}$.

and is given by:

$$\Delta\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{l}}\ \tilde{I}\ \Delta l\ \delta(\mathbf{r}) \tag{9.46}$$

where the product $\tilde{I}\Delta l$ (SI base units of A·m) is the current moment, $\hat{\mathbf{l}}$ is the direction of current flow, and $\delta(\mathbf{r})$ is the volumetric sampling function defined as follows:

$$\delta(\mathbf{r}) \triangleq 0 \quad \text{for} \quad \mathbf{r} \neq 0; \quad \text{and} \tag{9.47}$$

$$\int_{\mathcal{V}} \delta(\mathbf{r})\ dv \triangleq 1 \tag{9.48}$$

where $\mathcal{V}$ is any volume which includes the origin ($\mathbf{r} = 0$). In this description, the Hertzian dipole is located at the origin.

The solution for the magnetic vector potential due to a $\hat{\mathbf{z}}$-directed Hertzian dipole located at the origin was presented in Section 9.3. In the present scenario, it is:

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{z}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{-\gamma r}}{4\pi r} \tag{9.49}$$

where the propagation constant $\gamma = \alpha + j\beta$ as usual. Assuming lossless media ($\alpha = 0$), we have

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{z}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{-j\beta r}}{4\pi r} \tag{9.50}$$

We obtain the magnetic field intensity using the definition of magnetic vector potential:

$$\widetilde{\mathbf{H}} \triangleq (1/\mu)\nabla \times \widetilde{\mathbf{A}} \tag{9.51}$$

$$= \frac{\tilde{I}\ \Delta l}{4\pi}\ \nabla \times \hat{\mathbf{z}}\ \frac{e^{-j\beta r}}{r} \tag{9.52}$$

To proceed, it is useful to convert $\hat{\mathbf{z}}$ into the spherical coordinate system. To do this, we find the component of $\hat{\mathbf{z}}$ that is parallel to $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$; and then sum the results:

$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}\right) + \hat{\theta}\left(\hat{\theta} \cdot \hat{\mathbf{z}}\right) + \hat{\phi}\left(\hat{\phi} \cdot \hat{\mathbf{z}}\right) \tag{9.53}$$

$$= \hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta + 0 \tag{9.54}$$

Equation 9.52 requires computation of the following quantity:

$$\nabla \times \hat{\mathbf{z}} \frac{e^{-j\beta r}}{r} = \nabla \times \left[\begin{array}{l} \hat{\mathbf{r}} \left(\cos\theta\right) \dfrac{e^{-j\beta r}}{r} \\ -\hat{\theta} \left(\sin\theta\right) \dfrac{e^{-j\beta r}}{r} \end{array} \right] \tag{9.55}$$

At this point, it is convenient to make the following definitions:

$$C_r \triangleq \left(\cos\theta\right) \frac{e^{-j\beta r}}{r} \tag{9.56}$$

$$C_\theta \triangleq -\left(\sin\theta\right) \frac{e^{-j\beta r}}{r} \tag{9.57}$$

These definitions allow Equation 9.55 to be written compactly as follows:

$$\nabla \times \hat{\mathbf{z}} \frac{e^{-j\beta r}}{r} = \nabla \times \left[\hat{\mathbf{r}} C_r + \hat{\theta} C_\theta \right] \tag{9.58}$$

The right side of Equation 9.58 is evaluated using Equation B.18 (Appendix B.2). Although the complete expression consists of 6 terms, only 2 terms are non-zero.[1] This leaves:

$$\nabla \times \hat{\mathbf{z}} \frac{e^{-j\beta r}}{r} = \hat{\phi} \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r C_\theta\right) - \frac{\partial}{\partial \theta} C_r \right] \tag{9.59}$$

$$= \hat{\phi} \left(\sin\theta\right) \frac{e^{-j\beta r}}{r} \left(j\beta + \frac{1}{r} \right) \tag{9.60}$$

Substituting this result into Equation 9.52, we obtain:

$$\widetilde{\mathbf{H}} = \hat{\phi} \frac{\tilde{I}\ \Delta l}{4\pi} \ \left(\sin\theta\right) \frac{e^{-j\beta r}}{r} \left(j\beta + \frac{1}{r} \right) \tag{9.61}$$

Let us further limit our scope to the field far from the antenna. Specifically, let us assume $r \gg \lambda$. Now we use the relationship $\beta = 2\pi/\lambda$ and determine the following:

$$j\beta + \frac{1}{r} = j\frac{2\pi}{\lambda} + \frac{1}{r} \tag{9.62}$$

$$\approx j\frac{2\pi}{\lambda} = j\beta \tag{9.63}$$

Equation 9.61 becomes:

$$\boxed{\widetilde{\mathbf{H}} \approx \hat{\phi} j \frac{\tilde{I} \cdot \beta \Delta l}{4\pi} \ \left(\sin\theta\right) \frac{e^{-j\beta r}}{r}} \tag{9.64}$$

where the approximation holds for low-loss media and $r \gg \lambda$. This expression is known as a *far field approximation*, since it is valid only for distances "far" (relative to a wavelength) from the source.

Now let us take a moment to interpret this result:

- Notice the factor $\beta\Delta l$ has units of radians; that is, it is *electrical length*. This tells us that the magnitude of the radiated field depends on the electrical length of the current moment.
- The factor $e^{-j\beta r}/r$ indicates that this is a spherical wave; that is, surfaces of constant phase correspond to concentric spheres centered on the source, and magnitude is inversely proportional to distance.
- The direction of the magnetic field vector is always $\hat{\phi}$, which is precisely what we expect; for example, using the Biot-Savart law.
- Finally, note the factor $\sin\theta$. This indicates that the field magnitude is zero along the direction in which the source current flows, and is a maximum in the plane perpendicular to this direction.

Now let us determine the electric field radiated by the Hertzian dipole. The direct method is to employ Ampere's law. That is,

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\epsilon}\nabla \times \widetilde{\mathbf{H}} \tag{9.65}$$

where $\widetilde{\mathbf{H}}$ is given by Equation 9.64. At field points far from the dipole, the radius of curvature of the spherical phasefronts is very large and so appear to be *locally planar*. That is, from the perspective of an observer far from the dipole, the arriving wave appears to be a plane wave. In this case, we may employ the *plane wave relationships*. The appropriate relationship in this case is:

$$\widetilde{\mathbf{E}} = -\eta\hat{\mathbf{r}} \times \widetilde{\mathbf{H}} \tag{9.66}$$

where η is the wave impedance. So we find:

$$\boxed{\widetilde{\mathbf{E}} \approx \hat{\theta} j\eta \frac{\tilde{I} \cdot \beta\Delta l}{4\pi} \ (\sin\theta) \ \frac{e^{-j\beta r}}{r}} \tag{9.67}$$

Summarizing:

The electric and magnetic fields far (i.e., $\gg \lambda$) from a $\hat{\mathbf{z}}$-directed Hertzian dipole having constant current $\tilde{I}$ over length Δl, located at the origin, are given by Equations 9.67 and 9.64, respectively.

Additional Reading:

- "Dipole antenna (https://en.wikipedia.org/wiki/Dipole_antenna#Hertzian_dipole)" (section entitled "Hertzian Dipole") on Wikipedia.

1. Specifically, two terms are zero because there is no $\hat{\phi}$ component in the argument of the curl function; and another two terms are zero because the argument of the curl function is independent of ϕ, so partial derivatives with respect to ϕ are zero. ↩

9.5 Radiation from an Electrically-Short Dipole

The simplest distribution of radiating current that is encountered in common practice is the electrically-short dipole (ESD). This current distribution is shown in Figure 9.6.

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Figure 9.6. *Current distribution of the electrically-short dipole (ESD).*

The two characteristics that define the ESD are (1) the current is aligned along a straight line, and (2) the length L of the line is much less than one-half of a wavelength; i.e., $L \ll \lambda/2$. The latter characteristic is what we mean by "electrically-short."[1]

The current distribution of an ESD is approximately triangular in magnitude, and approximately constant in phase. How do we know this? First, note that distributions of current cannot change in a complex or rapid way over such distances which are much less than a wavelength. If this is not immediately apparent, recall the behavior of transmission lines: The current standing wave on a transmission line exhibits a period of $\lambda/2$, regardless the source or termination. For the ESD, $L \ll \lambda/2$ and so we expect an even simpler variation. Also, we know that the current at the ends of the dipole must be zero, simply because the dipole ends there. These considerations imply that the current distribution of the ESD is well-approximated as triangular in magnitude.[2] Expressed mathematically:

$$\tilde{I}(z) \approx I_0 \left(1 - \frac{2}{L}\left|z\right|\right) \tag{9.68}$$

where I_0 (SI base units of A) is a complex-valued constant indicating the maximum current magnitude and phase.

There are two approaches that we might consider in order to find the electric field radiated by an ESD. The first approach is to calculate the magnetic vector potential $\widetilde{\mathbf{A}}$ by integration over the current distribution, calculate $\widetilde{\mathbf{H}} = (1/\mu)\nabla \times \widetilde{\mathbf{A}}$, and finally calculate $\widetilde{\mathbf{E}}$ from $\widetilde{\mathbf{H}}$ using Ampere's law. We shall employ a simpler approach, shown in Figure 9.7.

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Figure 9.7. *Current distribution of the electrically-short dipole (ESD) approximated as a large number of Hertzian dipoles.*

Imagine the ESD as a collection of many shorter segments of current that radiate independently. The total field is then the sum of these short segments. Because these segments are very short relative to the length of the dipole as well as being short relative to a wavelength, we may approximate the current over each segment as approximately constant. In other words, we may interpret each of these segments as being, to a good approximation, a Hertzian dipole.

The advantage of this approach is that we already have a solution for each of the segments. In Section 9.4, it is shown that a $\hat{\mathbf{z}}$-directed Hertzian dipole at the origin radiates the electric field

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\eta \frac{\tilde{I} \cdot \beta\Delta l}{4\pi} \; (\sin\theta) \, \frac{e^{-j\beta r}}{r} \tag{9.69}$$

where $\tilde{I}$ and Δl may be interpreted as the current and length of the dipole, respectively. In this expression, η is the wave impedance of medium in which the dipole radiates (e.g., $\approx 377\ \Omega$ for free space), and we have presumed lossless media such that the attenuation constant $\alpha \approx 0$ and the phase propagation constant $\beta = 2\pi/\lambda$. This expression also assumes field points far from the dipole; specifically, distances r that are much greater than λ. Repurposing this expression for the present problem, the segment at the origin radiates the electric field:

$$\widetilde{\mathbf{E}}(\mathbf{r}; z'=0) \approx \hat{\theta} j\eta \frac{I_0 \cdot \beta\Delta l}{4\pi} \; (\sin\theta) \, \frac{e^{-j\beta r}}{r} \tag{9.70}$$

where the notation $z' = 0$ indicates the Hertzian dipole is located at the origin. Letting the length Δl of this segment shrink to differential length dz', we may describe the contribution of this segment to the field radiated by the ESD as follows:

$$d\widetilde{\mathbf{E}}(\mathbf{r}; z'=0) \approx \hat{\theta} j\eta \frac{I_0 \cdot \beta dz'}{4\pi} \; (\sin\theta) \, \frac{e^{-j\beta r}}{r} \tag{9.71}$$

Using this approach, the electric field radiated by *any* segment can be written:

$$d\widetilde{\mathbf{E}}(\mathbf{r}; z') \approx \hat{\theta}' j\eta\beta \frac{\tilde{I}(z')}{4\pi} \, (\sin\theta') \, \frac{e^{-j\beta\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}}{\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|} \; dz' \tag{9.72}$$

Note that θ is replaced by θ' since the ray $\mathbf{r} - \hat{\mathbf{z}}z'$ forms a different angle (i.e., θ') with respect to $\hat{\mathbf{z}}$. Similarly, $\hat{\theta}$ is replaced by $\hat{\theta}'$, since it also varies with z'. The electric field radiated by the ESD is obtained by integration over these contributions:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \int_{-L/2}^{+L/2} d\widetilde{\mathbf{E}}(\hat{\mathbf{r}}; z') \tag{9.73}$$

yielding:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx j\frac{\eta\beta}{4\pi} \int_{-L/2}^{+L/2} \hat{\theta}' \tilde{I}(z') \; (\sin\theta') \, \frac{e^{-j\beta\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}}{\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|} dz' \tag{9.74}$$

Given some of the assumptions we have already made, this expression can be further simplified. For example, note that $\theta' \approx \theta$ since $L \ll r$. For the same reason, $\hat{\theta}' \approx \hat{\theta}$. Since these variables are approximately constant over the length of the dipole, we may move them outside the integral, yielding:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j \frac{\eta\beta}{4\pi} \ (\sin\theta) \int_{-L/2}^{+L/2} \tilde{I}(z') \frac{e^{-j\beta|\mathbf{r}-\hat{\mathbf{z}}z'|}}{|\mathbf{r}-\hat{\mathbf{z}}z'|} dz' \tag{9.75}$$

It is also possible to simplify the expression $|\mathbf{r}-\hat{\mathbf{z}}z'|$. Consider Figure 9.8.

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Figure 9.8. *Parallel ray approximation for an ESD.*

Since we have already assumed that $r \gg L$ (i.e., the distance to field points is much greater than the length of the dipole), the vector $\mathbf{r}$ is approximately parallel to the vector $\mathbf{r}-\hat{\mathbf{z}}z'$. Subsequently, it must be true that

$$|\mathbf{r}-\hat{\mathbf{z}}z'| \approx r - \hat{\mathbf{r}} \cdot \hat{\mathbf{z}}z' \tag{9.76}$$

Note that the magnitude of $r - \hat{\mathbf{r}} \cdot \hat{\mathbf{z}}z'$ must be approximately equal to r, since $r \gg L$. So, insofar as $|\mathbf{r}-\hat{\mathbf{z}}z'|$ determines the *magnitude* of $\widetilde{\mathbf{E}}(\mathbf{r})$, we may use the approximation:

$$|\mathbf{r}-\hat{\mathbf{z}}z'| \approx r \quad \text{(magnitude)} \tag{9.77}$$

Insofar as $|\mathbf{r}-\hat{\mathbf{z}}z'|$ determines *phase*, we have to be a bit more careful. The part of the integrand of Equation 9.75 that exhibits varying phase is $e^{-j\beta|\mathbf{r}-\hat{\mathbf{z}}z'|}$. Using Equation 9.76, we find

$$e^{-j\beta|\mathbf{r}-\hat{\mathbf{z}}z'|} \approx e^{-j\beta r} e^{+j\beta\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}z'} \tag{9.78}$$

The worst case in terms of phase variation within the integral is for field points along the z axis. For these points, $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \pm 1$ and subsequently $|\mathbf{r}-\hat{\mathbf{z}}z'|$ varies from $z - L/2$ to $z + L/2$ where z is the location of the field point. However, since $L \ll \lambda$ (i.e., because the dipole is electrically short), this difference in lengths is much less than $\lambda/2$. Therefore, the phase $\beta\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}z'$ varies by much less than π radians, and subsequently $e^{-j\beta\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}z'} \approx 1$. We conclude that under these conditions,

$$e^{-j\beta|\mathbf{r}-\hat{\mathbf{z}}z'|} \approx e^{-j\beta r} \quad \text{(phase)} \tag{9.79}$$

Applying these simplifications for magnitude and phase to Equation 9.75, we obtain:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j \frac{\eta\beta}{4\pi} (\sin\theta) \ \frac{e^{-j\beta r}}{r} \ \int_{-L/2}^{+L/2} \tilde{I}(z') dz' \tag{9.80}$$

The integral in this equation is very easy to evaluate; in fact, from inspection (Figure 9.6), we determine it is equal to $I_0 L/2$. Finally, we obtain:

$$\boxed{\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\eta \frac{I_0 \cdot \beta L}{8\pi}\ (\sin\theta)\ \frac{e^{-j\beta r}}{r}} \tag{9.81}$$

Summarizing:

The electric field intensity radiated by an ESD located at the origin and aligned along the z axis is given by Equation 9.81. This expression is valid for $r \gg \lambda$.

It is worth noting that the variation in magnitude, phase, and polarization of the ESD with field point location is identical to that of a single Hertzian dipole having current moment $\hat{\mathbf{z}} I_0 L/2$ (Section 9.4). However, the magnitude of the field radiated by the ESD is exactly one-half that of the Hertzian dipole. Why one-half? Simply because the integral over the triangular current distribution assumed for the ESD is one-half the integral over the uniform current distribution that defines the Hertzian dipole. This similarly sometimes causes confusion between Hertzian dipoles and ESDs. Remember that ESDs are physically realizable, whereas Hertzian dipoles are not.

It is common to eliminate the factor of β in the magnitude using the relationship $\beta = 2\pi/\lambda$, yielding:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j \frac{\eta I_0}{4} \frac{L}{\lambda}\ (\sin\theta)\ \frac{e^{-j\beta r}}{r} \tag{9.82}$$

At field points $r \gg \lambda$, the wave appears to be locally planar. Therefore, we are justified using the plane wave relationship $\widetilde{\mathbf{H}} = \frac{1}{\eta}\hat{\mathbf{r}} \times \widetilde{\mathbf{E}}$ to calculate $\widetilde{\mathbf{H}}$. The result is:

$$\widetilde{\mathbf{H}}(\mathbf{r}) \approx \hat{\phi} j \frac{I_0}{4} \frac{L}{\lambda}\ (\sin\theta)\ \frac{e^{-j\beta r}}{r} \tag{9.83}$$

Finally, let us consider the spatial characteristics of the radiated field. Figures 9.9 and 9.10 show the result in a plane of constant ϕ. Figures 9.11 and 9.12 show the result in the $z = 0$ plane. Note that the orientations of the electric and magnetic field vectors indicate a Poynting vector $\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}$ that is always directed radially outward from the location of the dipole. This confirms that power flow is always directed radially outward from the dipole. Due to the symmetry of the problem, Figures 9.9–9.12 provide a complete characterization of the relative magnitudes and orientations of the radiated fields.

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Figure 9.9. *Magnitude of the radiated field in any plane of constant ϕ.*

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Figure 9.10. *Orientation of the electric and magnetic fields in any plane of constant ϕ.*

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Figure 9.11. *Magnitude of the radiated field in any plane of constant z.*

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Figure 9.12. *Orientation of the electric and magnetic fields in the $z = 0$ plane.*

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1. A potential source of confusion is that the *Hertzian dipole* is also a "dipole" which is "electrically-short." The distinction is that the current comprising a Hertzian dipole is *constant* over its length. This condition is rarely and only approximately seen in practice, whereas the triangular magnitude distribution is a relatively good approximation to a broad class of commonly-encountered electrically-short wire antennas. Thus, the term "electrically-short dipole," as used in this book, refers to the triangular distribution unless noted otherwise. ↩
2. A more rigorous analysis leading to the same conclusion is possible, but is beyond the scope of this book. ↩

9.6 Far-Field Radiation from a Thin Straight Filament of Current

A simple distribution of radiating current that is encountered in common practice is the thin straight current filament, shown in Figure 9.13.

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Figure 9.13. *A thin straight distribution of radiating current.*

The defining characteristic of this distribution is that the current filament is aligned along a straight line, and that the maximum dimension of the cross-section of the filament is much less than a wavelength. The latter characteristic is what we mean by "thin" – i.e., the filament is "electrically thin." Physical distributions of current that meet this description include the electrically-short dipole (Section 9.5) and the half-wave dipole (Section 9.7) among others. A gentler introduction to this distribution is via Section 9.5 ("Radiation from an Electrically-Short Dipole"), so students may want to review that section first. This section presents the more general case.

Let the magnitude and phase of current along the filament be given by the phasor quantity $\tilde{I}(z)$ (SI base units of A). In principle, the only constraint on $\tilde{I}(z)$ that applies generally is that it must be zero at the ends of the distribution; i.e., $\tilde{I}(z) = 0$ for $|z| \geq L/2$. Details beyond this constraint depend on L and perhaps other factors. Nevertheless, the characteristics of radiation from members of this class of current distributions have much in common, regardless of L. These become especially apparent if we limit our scope to field points far from the current, as we shall do here.

There are two approaches that we might consider in order to find the electric field radiated by this distribution. The first approach is to calculate the magnetic vector potential $\widetilde{\mathbf{A}}$ by integration over the current distribution (Section 9.3), calculate $\widetilde{\mathbf{H}} = (1/\mu)\nabla \times \widetilde{\mathbf{A}}$, and finally calculate $\widetilde{\mathbf{E}}$ from $\widetilde{\mathbf{H}}$ using the differential form of Ampere's law. In this section, we shall employ a simpler approach, shown in Figure 9.14.

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Figure 9.14. *Current distribution approximated as a set of Hertzian dipoles.*

Imagine the filament as a collection of many shorter segments of current that radiate independently. The total field is then the sum of these short segments. Because these segments are very short relative to the length of the filament as well as being short relative to a wavelength, we may approximate the current over each segment as constant. In other words, we may interpret each segment as being, to a good approximation, a Hertzian dipole.

The advantage of this approach is that we already have a solution for every segment. In Section 9.4, it is shown that a $\hat{\mathbf{z}}$-directed Hertzian dipole at the origin radiates the electric field

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\eta \frac{\tilde{I}\,(\beta\Delta l)}{4\pi}\ (\sin\theta)\ \frac{e^{-j\beta r}}{r} \tag{9.84}$$

where $\tilde{I}$ and Δl may be interpreted as the current and length of the filament, respectively. In this expression, η is the wave impedance of medium in which the filament radiates (e.g., $\approx 377\ \Omega$ for free space), and we have presumed lossless media such that the attenuation constant $\alpha \approx 0$ and the phase propagation constant $\beta = 2\pi/\lambda$. This expression also assumes field points far from the filament; specifically, distances r that are much greater than λ. Repurposing this expression for the present problem, we note that the segment at the origin radiates the electric field:

$$\widetilde{\mathbf{E}}(\mathbf{r}; z' = 0) \approx \hat{\theta} j\eta \frac{\tilde{I}(0)\,(\beta\Delta l)}{4\pi}\ (\sin\theta)\ \frac{e^{-j\beta r}}{r} \tag{9.85}$$

where the notation $z' = 0$ indicates the Hertzian dipole is located at the origin. Letting the length Δl of this segment shrink to differential length dz', we may describe the contribution of this segment to the field radiated by the ESD as follows:

$$d\widetilde{\mathbf{E}}(\mathbf{r}; z' = 0) \approx \hat{\theta} j\eta \frac{\tilde{I}(0)\,(\beta dz')}{4\pi}\ (\sin\theta)\ \frac{e^{-j\beta r}}{r} \tag{9.86}$$

Using this approach, the electric field radiated by *any* segment can be written:

$$d\widetilde{\mathbf{E}}(\mathbf{r};z') \approx \hat{\theta}' j\eta\beta \frac{\tilde{I}(z')}{4\pi} \left(\sin\theta'\right) \frac{e^{-j\beta\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}}{\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}\ dz' \tag{9.87}$$

Note that θ is replaced by θ' since the ray $\mathbf{r}-\hat{\mathbf{z}}z'$ forms a different angle (i.e., θ') with respect to $\hat{\mathbf{z}}$. Subsequently, $\hat{\theta}$ is replaced by $\hat{\theta}'$, which varies similarly with z'. The electric field radiated by the filament is obtained by integration over these contributions, yielding:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \int_{-L/2}^{+L/2} d\widetilde{\mathbf{E}}(\hat{\mathbf{r}};z') \tag{9.88}$$

Expanding this expression:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx j\frac{\eta\beta}{4\pi}\int_{-L/2}^{+L/2} \hat{\theta}'\tilde{I}(z')\ \left(\sin\theta'\right) \frac{e^{-j\beta\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}}{\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}dz' \tag{9.89}$$

Given some of the assumptions we have already made, this expression can be further simplified. For example, note that $\theta' \approx \theta$ since $L \ll r$. For the same reason, $\hat{\theta}' \approx \hat{\theta}$. Since these variables are approximately constant over the length of the filament, we may move them outside the integral, yielding:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\frac{\eta\beta}{4\pi}\ \left(\sin\theta\right)\int_{-L/2}^{+L/2} \tilde{I}(z') \frac{e^{-j\beta\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}}{\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|}dz' \tag{9.90}$$

It is also possible to simplify the expression $\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|$. Consider Figure 9.15.

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Figure 9.15. *Parallel ray approximation.*

Since we have already assumed that $r \gg L$ (i.e., the distance to field points is much greater than the length of the filament), the vector $\mathbf{r}$ is approximately parallel to the vector $\mathbf{r}-\hat{\mathbf{z}}z'$. Subsequently, it must be true that

$$\left|\mathbf{r}-\hat{\mathbf{z}}z'\right| \approx r - \hat{\mathbf{r}}\cdot\hat{\mathbf{z}}z' \tag{9.91}$$

$$\approx r - z'\cos\theta \tag{9.92}$$

Note that the magnitude of $r-\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}z'$ must be approximately equal to r, since $r \gg L$. So, insofar as $\left|\mathbf{r}-\hat{\mathbf{z}}z'\right|$ determines the *magnitude* of $\widetilde{\mathbf{E}}(\mathbf{r})$, we may use the approximation:

$$\left|\mathbf{r}-\hat{\mathbf{z}}z'\right| \approx r \quad \text{(magnitude)} \tag{9.93}$$

Insofar as $|\mathbf{r} - \hat{\mathbf{z}}z'|$ determines *phase*, we have to be more careful. The part of the integrand of Equation 9.90 that exhibits varying phase is $e^{-j\beta|\mathbf{r}-\hat{\mathbf{z}}z'|}$. Using Equation 9.91, we find

$$e^{-j\beta|\mathbf{r}-\hat{\mathbf{z}}z'|} \approx e^{-j\beta r} e^{+j\beta z' \cos\theta} \tag{9.94}$$

These simplifications are known collectively as a *far field approximation*, since they are valid only for distances "far" from the source.

Applying these simplifications for magnitude and phase to Equation 9.90, we obtain:

$$\begin{aligned} \widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j \frac{\eta\beta}{4\pi} \frac{e^{-j\beta r}}{r} & \; (\sin\theta) \\ \cdot \int_{-L/2}^{+L/2} & \tilde{I}(z') e^{+j\beta z' \cos\theta} dz' \end{aligned} \tag{9.95}$$

Finally, it is common to eliminate the factor of β in the magnitude using the relationship $\beta = 2\pi/\lambda$, yielding:

$$\begin{aligned} \widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j \frac{\eta}{2} \frac{e^{-j\beta r}}{r} & \; (\sin\theta) \\ \cdot \left[\frac{1}{\lambda} \int_{-L/2}^{+L/2} \right. & \left. \tilde{I}(z') e^{+j\beta z' \cos\theta} dz' \right] \end{aligned} \tag{9.96}$$

The electric field radiated by a thin, straight, $\hat{\mathbf{z}}$-directed current filament of length L located at the origin and aligned along the z axis is given by Equation 9.96. This expression is valid for $r \gg L$ and $r \gg \lambda$.

At field points satisfying the conditions $r \gg L$ and $r \gg \lambda$, the wave appears to be locally planar. Therefore, we are justified using the plane wave relationship $\widetilde{\mathbf{H}} = (1/\eta)\hat{\mathbf{r}} \times \widetilde{\mathbf{E}}$ to calculate $\widetilde{\mathbf{H}}$.

As a check, one may readily verify that Equation 9.96 yields the expected result for the electrically-short dipole (Section 9.5).

9.7 Far-Field Radiation from a Half-Wave Dipole

A simple and important current distribution is that of the thin half-wave dipole (HWD), shown in Figure 9.16.

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Figure 9.16. *Current distribution of the half-wave dipole (HWD).*

This is the distribution expected on a thin straight wire having length $L = \lambda/2$, where λ is wavelength. This distribution is described mathematically as follows:

$$\tilde{I}(z) \approx I_0 \cos\left(\pi \frac{z}{L}\right) \quad \text{for} \quad |z| \leq \frac{L}{2} \tag{9.97}$$

where I_0 (SI base units of A) is a complex-valued constant indicating the maximum magnitude of the current and its phase. Note that the current is zero at the ends of the dipole; i.e., $\tilde{I}(z)=0$ for $|z|=L/2$. Note also that this “cosine pulse” distribution is very similar to the triangular distribution of the ESD, and is reminiscent of the sinusoidal variation of current in a standing wave.

Since $L=\lambda/2$ for the HWD, Equation 9.97 may equivalently be written:

$$\tilde{I}(z) \approx I_0 \cos\left(2\pi\frac{z}{\lambda}\right) \tag{9.98}$$

The electromagnetic field radiated by this distribution of current may be calculated using the method described in Section 9.6, in particular:

$$\begin{aligned}\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\frac{\eta}{2}\frac{e^{-j\beta r}}{r} & \ (\sin\theta) \\ & \cdot \left[\frac{1}{\lambda}\int_{-L/2}^{+L/2}\tilde{I}(z')e^{+j\beta z'\cos\theta}dz'\right]\end{aligned} \tag{9.99}$$

which is valid for field points $\mathbf{r}$ far from the dipole; i.e., for $r \gg L$ and $r \gg \lambda$. For the HWD, the quantity in square brackets is

$$\frac{I_0}{\lambda}\int_{-\lambda/4}^{+\lambda/4}\cos\left(2\pi\frac{z'}{\lambda}\right)e^{+j\beta z'\cos\theta}dz' \tag{9.100}$$

The evaluation of this integral is straightforward, but tedious. The integral reduces to

$$\frac{I_0}{\pi}\frac{\cos\left[(\pi/2)\cos\theta\right]}{\sin^2\theta} \tag{9.101}$$

Substitution into Equation 9.99 yields

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\frac{\eta I_0}{2\pi}\frac{\cos\left[(\pi/2)\cos\theta\right]}{\sin\theta}\frac{e^{-j\beta r}}{r} \tag{9.102}$$

The magnetic field may be determined from this result using Ampere’s law. However, a simpler method is to use the fact that the electric field, magnetic field, and direction of propagation $\hat{\mathbf{r}}$ are mutually perpendicular and related by:

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\hat{\mathbf{r}} \times \widetilde{\mathbf{E}} \tag{9.103}$$

This relationship indicates that the magnetic field will be $+\hat{\phi}$-directed.

The magnitude and polarization of the radiated field is similar to that of the electrically-short dipole (ESD; Section 9.5). A comparison of the magnitudes in any radial plane containing the z-axis is shown in Figure 9.17.

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Figure 9.17. *Comparison of the magnitude of the radiated field of the HWD to that of an electrically-short dipole also oriented along the z-axis. This result is for any radial plane that includes the z-axis.*

For either current distribution, the maximum magnitude of the fields occurs in the $z = 0$ plane. For a given terminal current I_0, the maximum magnitude is greater for the HWD than for the ESD. Both current distributions yield zero magnitude along the axis of the dipole. The polarization characteristics of the fields of both current distributions are identical.

Additional Reading:

- "Dipole antenna (https://en.wikipedia.org/wiki/Dipole_antenna#Half-wave_dipole)" (section entitled "Half-wave dipole") on Wikipedia.

9.8 Radiation from Surface and Volume Distributions of Current

In Section 9.3, a solution was developed for radiation from current located at a single point $\mathbf{r}'$. This current distribution was expressed mathematically as a *current moment*, as follows:

$$\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{l}}\ \tilde{I}\ \Delta l\ \delta(\mathbf{r} - \mathbf{r}') \tag{9.104}$$

where $\hat{\mathbf{l}}$ is the direction of current flow, $\tilde{I}$ has units of current (SI base units of A), Δl has units of length (SI base units of m), and $\delta(\mathbf{r})$ is the volumetric sampling function (Dirac "delta" function; SI base units of m^{-3}). In this formulation, $\widetilde{\mathbf{J}}$ has SI base units of A/m^2. The magnetic vector potential radiated by this current, observed at the field point $\mathbf{r}$, was found to be

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{l}}\ \mu\ \tilde{I}\ \Delta l\ \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r} - \mathbf{r}'|} \tag{9.105}$$

This solution was subsequently generalized to obtain the radiation from any distribution of *line* current; i.e., any distribution of current that is constrained to flow along a single path through space, as along an infinitesimally-thin wire.

In this section, we derive an expression for the radiation from current that is constrained to flow along a surface and from current which flows through a volume. The solution in both cases can be obtained by "recycling" the solution for line current as follows. Letting Δl shrink to the differential length dl, Equation 9.104 becomes:

$$d\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{l}}\ \tilde{I}\ dl\ \delta(\mathbf{r} - \mathbf{r}') \tag{9.106}$$

Subsequently, Equation 9.105 becomes:

$$d\widetilde{\mathbf{A}}(\mathbf{r};\mathbf{r}') = \hat{\mathbf{l}}\ \mu\ \tilde{I}\ dl\ \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \tag{9.107}$$

where the notation $d\widetilde{\mathbf{A}}(\mathbf{r};\mathbf{r}')$ is used to denote the magnetic vector potential at the field point $\mathbf{r}$ due to the differential-length current moment at the source point $\mathbf{r}'$. Next, consider that any distribution of current can be described as a distribution of current moments. By the principle of superposition, the radiation from this distribution of current moments can be calculated as the sum of the radiation from the individual current moments. This is expressed mathematically as follows:

$$\widetilde{\mathbf{A}}(\mathbf{r}) = \int d\widetilde{\mathbf{A}}(\mathbf{r};\mathbf{r}') \tag{9.108}$$

where the integral is over $\mathbf{r}'$; i.e., summing over the source current.

If we substitute Equation 9.107 into Equation 9.108, we obtain the solution derived in Section 9.3, which is specific to line distributions of current. To obtain the solution for surface and volume distributions of current, we reinterpret the definition of the differential current moment in Equation 9.106. Note that this current is completely specified by its direction ($\hat{\mathbf{l}}$) and the quantity $\tilde{I}\ dl$, which has SI base units of A·m. We may describe the same current distribution alternatively as follows:

$$d\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{l}}\ \widetilde{J}_s\ ds\ \delta(\mathbf{r}-\mathbf{r}') \tag{9.109}$$

where $\widetilde{J}_s$ has units of surface current density (SI base units of A/m) and ds has units of area (SI base units of m^2). To emphasize that this is precisely the same current moment, note that $\widetilde{J}_s\ ds$, like $\tilde{I}\ dl$, has units of A·m. Similarly,

$$d\widetilde{\mathbf{J}}(\mathbf{r}) = \hat{\mathbf{l}}\ \widetilde{J}\ dv\ \delta(\mathbf{r}-\mathbf{r}') \tag{9.110}$$

where $\widetilde{J}$ has units of volume current density (SI base units of A/m^2) and dv has units of volume (SI base units of m^3). Again, $\widetilde{J}\ dv$ has units of A·m. Summarizing, we have found that

$$\tilde{I}\ dl = \widetilde{J}_s\ ds = \widetilde{J}\ dv \tag{9.111}$$

all describe the same differential current moment.

Thus, we may obtain solutions for surface and volume distributions of current simply by replacing $\tilde{I}\ dl$ in Equation 9.107, and subsequently in Equation 9.108, with the appropriate quantity from Equation 9.111. For a surface current distribution, we obtain:

$$\boxed{\widetilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu}{4\pi}\ \int_{\mathcal{S}}\ \widetilde{\mathbf{J}}_s(\mathbf{r}')\ \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ ds} \tag{9.112}$$

where $\widetilde{\mathbf{J}}_s(\mathbf{r}') \triangleq \hat{\mathbf{l}}(\mathbf{r}')\widetilde{J}_s(\mathbf{r}')$ and where $\mathcal{S}$ is the surface over which the current flows. Similarly for a volume current distribution, we obtain:

$$\boxed{\widetilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{\mathcal{V}} \widetilde{\mathbf{J}}(\mathbf{r}') \, \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, dv} \tag{9.113}$$

where $\widetilde{\mathbf{J}}(\mathbf{r}') \triangleq \hat{\mathbf{l}}(\mathbf{r}')\widetilde{J}(\mathbf{r}')$ and where $\mathcal{V}$ is the volume in which the current flows.

The magnetic vector potential corresponding to radiation from a surface and volume distribution of current is given by Equations 9.112 and 9.113, respectively.

Given $\widetilde{\mathbf{A}}(\mathbf{r})$, the magnetic and electric fields may be determined using the procedure developed in Section 9.2.

End Matter

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