6 Waveguides

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6.1 Phase and Group Velocity

Phase velocity is the speed at which a point of constant phase travels as the wave propagates.[1] For a sinusoidally-varying wave, this speed is easy to quantify. To see this, consider the wave:

$$A\cos\left(\omega t - \beta z + \psi\right) \tag{6.1}$$

where $\omega = 2\pi f$ is angular frequency, z is position, and β is the phase propagation constant. At any given time, the distance between points of constant phase is one wavelength λ. Therefore, the phase velocity v_p is

$$v_p = \lambda f \tag{6.2}$$

Since $\beta = 2\pi/\lambda$, this may also be written as follows:

$$\boxed{v_p = \frac{\omega}{\beta}} \tag{6.3}$$

Noting that $\beta = \omega\sqrt{\mu\epsilon}$ for simple matter, we may also express v_p in terms of the constitutive parameters μ and ϵ as follows:

$$v_p = \frac{1}{\sqrt{\mu\epsilon}} \tag{6.4}$$

Since v_p in this case depends *only* on the constitutive properties μ and ϵ, it is reasonable to view phase velocity also as a property of matter.

Central to the concept of phase velocity is uniformity over space and time. Equations 6.2–6.4 presume a wave having the form of Equation 6.1, which exhibits precisely the same behavior over all possible time t from $-\infty$ to $+\infty$ and over all possible z from $-\infty$ to $+\infty$. This uniformity over all space and time precludes the use of such a wave to send information. To send information, the source of the wave needs to vary at least one parameter as a function of time; for example A (resulting in amplitude modulation), ω (resulting in frequency modulation), or ψ (resulting in phase modulation). In other words, information can be transmitted only by making the wave non-uniform in some respect. Furthermore, some materials and structures can cause changes in ψ or other combinations of parameters which vary with position or time. Examples include dispersion and propagation within waveguides. Regardless of the cause, varying the parameters ω or ψ as a function of time means that the instantaneous distance between points of constant phase may be very different from λ. Thus, the *instantaneous* frequency of variation as a function of time and position may be very different from f. In this case Equations 6.2–6.4 may not necessarily provide a meaningful value for the speed of propagation.

Some other concept is required to describe the speed of propagation of such waves. That concept is *group velocity*, v_g, defined as follows:

> *Group velocity*, v_g, is the ratio of the apparent change in frequency ω to the associated change in the phase propagation constant β; i.e., $\Delta\omega/\Delta\beta$.

Letting $\Delta\beta$ become vanishingly small, we obtain

$$\boxed{v_g \triangleq \frac{\partial \omega}{\partial \beta}} \tag{6.5}$$

Note the similarity to the definition of phase velocity in Equation 6.3. Group velocity can be interpreted as the speed at which a *disturbance* in the wave propagates. Information may be conveyed as meaningful disturbances relative to a steady-state condition, so group velocity is also the speed of information in a wave.

Note Equation 6.5 yields the expected result for waves in the form of Equation 6.1:

$$\begin{aligned} v_g &= \left(\frac{\partial \beta}{\partial \omega}\right)^{-1} = \left(\frac{\partial}{\partial \omega}\omega\sqrt{\mu\epsilon}\right)^{-1} \\ &= \frac{1}{\sqrt{\mu\epsilon}} = v_p \end{aligned} \tag{6.6}$$

In other words, the group velocity of a wave in the form of Equation 6.1 is equal to its phase velocity.

To observe a difference between v_p and v_g, β must somehow vary as a function of something other than just ω and the constitutive parameters. Again, modulation (introduced by the source of the wave) and dispersion (frequency-dependent constitutive parameters) are examples in which v_g is not necessarily equal to v_p. Here's an example involving dispersion:

Example

Example 6.1

Phase and group velocity for a material exhibiting square-law dispersion.

A broad class of non-magnetic dispersive media exhibit relative permittivity ϵ_r that varies as the square of frequency over a narrow range of frequencies centered at ω_0. For these media we presume

$$\epsilon_r = K\left(\frac{\omega}{\omega_0}\right)^2 \tag{6.7}$$

where K is a real-valued positive constant. What is the phase and group velocity for a sinusoidally-varying wave in this material?

Solution. First, note

$$\begin{aligned}\beta &= \omega\sqrt{\mu_0\epsilon} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_r}\\ &= \frac{\sqrt{K}\cdot\omega^2}{\omega_0}\sqrt{\mu_0\epsilon_0}\end{aligned} \tag{6.8}$$

The phase velocity is:

$$v_p = \frac{\omega}{\beta} = \frac{\omega_0}{\sqrt{K}\cdot\omega\sqrt{\mu_0\epsilon_0}} \tag{6.9}$$

Whereas the group velocity is:

$$v_g = \frac{\partial\omega}{\partial\beta} = \left(\frac{\partial\beta}{\partial\omega}\right)^{-1} \tag{6.10}$$

$$= \left(\frac{\partial}{\partial\omega}\frac{\sqrt{K}\cdot\omega^2}{\omega_0}\sqrt{\mu_0\epsilon_0}\right)^{-1} \tag{6.11}$$

$$= \left(2\frac{\sqrt{K}\cdot\omega}{\omega_0}\sqrt{\mu_0\epsilon_0}\right)^{-1} \tag{6.12}$$

Now simplifying using Equation 6.8:

$$v_g = \left(2\frac{\beta}{\omega}\right)^{-1} \tag{6.13}$$

$$= \frac{1}{2}\frac{\omega}{\beta} \tag{6.14}$$

$$= \frac{1}{2}v_p \tag{6.15}$$

Thus, we see that in this case the group velocity is always half the phase velocity.

Another commonly-encountered example for which v_g is not necessarily equal to v_p is the propagation of guided waves; e.g., waves within a waveguide. In fact, such waves may exhibit phase velocity greater than the speed of light in a vacuum, c. However, the group velocity remains less than c, which means the speed at which information may propagate in a waveguide is less than c. No physical laws are violated, since the universal "speed limit" c applies to information, and not simply points of constant phase. (See "Additional Reading" at the end of this section for more on this concept.)

Additional Reading:

- "Group velocity (https://en.wikipedia.org/wiki/Group_velocity)" on Wikipedia.
- "Phase velocity (https://en.wikipedia.org/wiki/Phase_velocity)" on Wikipedia.
- "Speed of light (https://en.wikipedia.org/wiki/Speed_of_light)" on Wikipedia.

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1. Formally, "velocity" is a vector which indicates both the direction and rate of motion. It is common practice to use the terms "phase velocity" and "group velocity" even though we are actually referring merely to rate of motion. The direction is, of course, in the direction of propagation. ↩

6.2 Parallel Plate Waveguide: Introduction

A parallel plate waveguide is a device for guiding the propagation of waves between two perfectly-conducting plates. Our primary interest in this structure is as a rudimentary model applicable to a broad range of engineering problems. Examples of such problems include analysis of the fields within microstrip line and propagation of radio waves in the ionosphere.

Figure 6.1 shows the geometry of interest. Here the plates are located at $z = 0$ and $z = a$. The plates are assumed to be infinite in extent, and therefore there is no need to consider fields in the regions $z < 0$ or $z > a$. For this analysis, the region between the plates is assumed to consist of an ideal (lossless) material exhibiting real-valued permeability μ and real-valued permittivity ϵ.

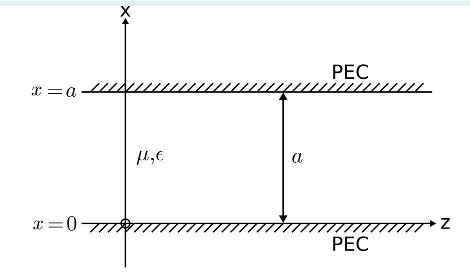

Figure 6.1. *Geometry for analysis of fields in a parallel plate waveguide.*

Let us limit our attention to a region within the waveguide which is free of sources. Expressed in phasor form, the electric field intensity is governed by the wave equation

$$\nabla^2\widetilde{\mathbf{E}}+\beta^2\widetilde{\mathbf{E}}=0 \tag{6.16}$$

where

$$\beta=\omega\sqrt{\mu\epsilon} \tag{6.17}$$

Equation 6.16 is a partial differential equation. This equation, combined with boundary conditions imposed by the perfectly-conducting plates, is sufficient to determine a unique solution. This is most easily done in Cartesian coordinates, as we shall now demonstrate. First we express $\widetilde{\mathbf{E}}$ in Cartesian coordinates:

$$\widetilde{\mathbf{E}}=\hat{\mathbf{x}}\widetilde{E}_x+\hat{\mathbf{y}}\widetilde{E}_y+\hat{\mathbf{z}}\widetilde{E}_z \tag{6.18}$$

This facilitates the decomposition of Equation 6.16 into separate equations governing the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ components of $\widetilde{\mathbf{E}}$:

$$\nabla^2\widetilde{E}_x+\beta^2\widetilde{E}_x=0 \tag{6.19}$$
$$\nabla^2\widetilde{E}_y+\beta^2\widetilde{E}_y=0 \tag{6.20}$$
$$\nabla^2\widetilde{E}_z+\beta^2\widetilde{E}_z=0 \tag{6.21}$$

Next we observe that the operator ∇^2 may be expressed in Cartesian coordinates as follows:

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{6.22}$$

so the equations governing the Cartesian components of $\widetilde{\mathbf{E}}$ may be written as follows:

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_x + \frac{\partial^2}{\partial y^2}\widetilde{E}_x + \frac{\partial^2}{\partial z^2}\widetilde{E}_x = -\beta^2\widetilde{E}_x \tag{6.23}$$

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_y + \frac{\partial^2}{\partial y^2}\widetilde{E}_y + \frac{\partial^2}{\partial z^2}\widetilde{E}_y = -\beta^2\widetilde{E}_y \tag{6.24}$$

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_z + \frac{\partial^2}{\partial y^2}\widetilde{E}_z + \frac{\partial^2}{\partial z^2}\widetilde{E}_z = -\beta^2\widetilde{E}_z \tag{6.25}$$

Let us restrict our attention to scenarios that can be completely described in two dimensions; namely x and z, and for which there is no variation in y. This is not necessarily required, however, it is representative of a broad class of relevant problems, and allows Equations 6.23–6.25 to be considerably simplified. If there is no variation in y, then partial derivatives with respect to y are zero, yielding:

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_x + \frac{\partial^2}{\partial z^2}\widetilde{E}_x = -\beta^2\widetilde{E}_x \tag{6.26}$$

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_y + \frac{\partial^2}{\partial z^2}\widetilde{E}_y = -\beta^2\widetilde{E}_y \tag{6.27}$$

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_z + \frac{\partial^2}{\partial z^2}\widetilde{E}_z = -\beta^2\widetilde{E}_z \tag{6.28}$$

The solution to the parallel plate waveguide problem may now be summarized as follows: The electric field $\widetilde{\mathbf{E}}$ (Equation 6.18) is the solution to simultaneous Equations 6.26–6.28 subject to the boundary conditions that apply at the PEC surfaces at $z = 0$ and $z = d$; namely, that the tangent component of $\widetilde{\mathbf{E}}$ is zero at these surfaces.

At this point, the problem has been reduced to a routine exercise in the solution of partial differential equations. However, a somewhat more useful approach is to first decompose the total electric field into transverse electric (TE) and transverse magnetic (TM) components,[1] and determine the solutions to these components separately. The TE component of the electric field is parallel to the plates, and therefore transverse (perpendicular) to the plane shown in Figure 6.1. Thus, the TE component has $\widetilde{E}_x = \widetilde{E}_z = 0$, and only $\widetilde{E}_y$ may be non-zero. The TM component of the magnetic field intensity ($\widetilde{\mathbf{H}}$) is parallel to the plates, and therefore transverse to the plane shown in Figure 6.1. Thus, the TM component has $\widetilde{H}_x = \widetilde{H}_z = 0$, and only $\widetilde{H}_y$ may be non-zero. As always, the total field is the sum of the TE and TM components.

The TE and TM solutions are presented in Sections 6.3 (Electric component of the TE solution), 6.4 (Magnetic component of the TE solution), and 6.5 (Electric component of the TM solution). The magnetic component of the TM solution can be determined via a straightforward variation of the preceding three cases, and so is not presented here.

Presentation of the solution for the fields in a parallel plate waveguide under the conditions described in the section continues in Section 6.3.

1. For a refresher on TE-TM decomposition, see Section 5.5. ↩

6.3 Parallel Plate Waveguide: TE Case, Electric Field

In Section 6.2, the parallel plate waveguide was introduced. At the end of that section, we described the decomposition of the problem into its TE and TM components. In this section, we find the electric field component of the TE field in the waveguide.

Figure 6.2 shows the problem addressed in this section. (Additional details and assumptions are addressed in Section 6.2.)

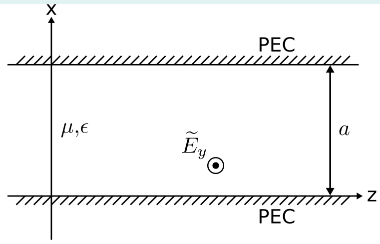

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Figure 6.2. *TE component of the electric field in a parallel plate waveguide.*

Since $\widetilde{E}_x = \widetilde{E}_z = 0$ for the TE component of the electric field, Equations 6.26 and 6.28 are irrelevant, leaving only:

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_y + \frac{\partial^2}{\partial z^2}\widetilde{E}_y = -\beta^2\widetilde{E}_y \tag{6.29}$$

The general solution to this partial differential equation is:

$$\begin{aligned}\widetilde{E}_y = &\quad e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right] \\ &+ e^{+jk_z z}\left[Ce^{-jk_x x} + De^{+jk_x x}\right]\end{aligned} \tag{6.30}$$

where A, B, C, and D are complex-valued constants and k_x and k_z are real-valued constants. We have assigned variable names to these constants with advance knowledge of their physical interpretation; however, at this moment they remain simply unknown constants whose values must be determined by enforcement of boundary conditions.

Note that Equation 6.30 consists of two terms. The first term includes the factor $e^{-jk_z z}$, indicating a wave propagating in the $+\hat{\mathbf{z}}$ direction, and the second term includes the factor $e^{+jk_z z}$, indicating a wave propagating in the $-\hat{\mathbf{z}}$ direction. If we impose the restriction that sources exist only on the left ($z<0$) side of Figure 6.2, and that there be no structure capable of wave scattering (in particular, reflection) on the right ($z>0$) side of Figure 6.2, then there can be no wave components propagating in the $-\hat{\mathbf{z}}$ direction. In this case, $C = D = 0$ and Equation 6.30 simplifies to:

$$\widetilde{E}_y = e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right] \tag{6.31}$$

Before proceeding, let's make sure that Equation 6.31 is actually a solution to Equation 6.29. As we shall see in a moment, performing this check will reveal some additional useful information. First, note:

$$\frac{\partial \widetilde{E}_y}{\partial x} = e^{-jk_z z}\left[-Ae^{-jk_x x} + Be^{+jk_x x}\right](jk_x) \tag{6.32}$$

So:

$$\frac{\partial^2 \widetilde{E}_y}{\partial x^2} = e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right]\left(-k_x^2\right) \tag{6.33}$$

Comparing this to Equation 6.31, we observe the remarkable fact that

$$\frac{\partial^2 \widetilde{E}_y}{\partial x^2} = -k_x^2 \widetilde{E}_y \tag{6.34}$$

Similarly, note:

$$\frac{\partial \widetilde{E}_y}{\partial z} = e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right]\left(-jk_z\right) \tag{6.35}$$

So:

$$\begin{aligned}\frac{\partial^2 \widetilde{E}_y}{\partial z^2} &= e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right]\left(-k_z^2\right) \\ &= -k_z^2 \widetilde{E}_y\end{aligned} \tag{6.36}$$

Now summing these results:

$$\frac{\partial^2 \widetilde{E}_y}{\partial x^2} + \frac{\partial^2 \widetilde{E}_y}{\partial z^2} = -\left(k_x^2 + k_z^2\right)\widetilde{E}_y \tag{6.37}$$

Comparing Equation 6.37 to Equation 6.29, we conclude that Equation 6.31 is indeed a solution to Equation 6.29, but only if:

$$\beta^2 = k_x^2 + k_z^2 \tag{6.38}$$

This confirms that k_x and k_z are in fact the components of the propagation vector

$$\mathbf{k} \triangleq \beta\hat{\mathbf{k}} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z \tag{6.39}$$

where $\hat{\mathrm{k}}$ is the unit vector pointing in the direction of propagation, and $k_y = 0$ in this particular problem.

The solution has now been reduced to finding the constants A, B, and either k_x or k_z. This is accomplished by enforcing the relevant boundary conditions. In general, the component of $\widetilde{\mathrm{E}}$ that is tangent to a perfectly-conducting surface is zero. Applied to the present problem, this means $\widetilde{E}_y = 0$ at $x = 0$ and $\widetilde{E}_y = 0$ at $x = a$. Referring to Equation 6.31, the boundary condition at $x = 0$ means

$$e^{-jk_z z}\left[A \cdot 1 + B \cdot 1\right] = 0 \tag{6.40}$$

The factor $e^{-jk_z z}$ cannot be zero; therefore, $A + B = 0$. Since $B = -A$, we may rewrite Equation 6.31 as follows:

$$\widetilde{E}_y = e^{-jk_z z} B \left[e^{+jk_x x} - e^{-jk_x x}\right] \tag{6.41}$$

This expression is simplified using a trigonometric identity:

$$\frac{1}{2j}\left[e^{+jk_x x} - e^{-jk_x x}\right] = \sin k_x x \tag{6.42}$$

Let us now make the definition $E_{y0} \triangleq j2B$. Then:

$$\widetilde{E}_y = E_{y0} e^{-jk_z z} \sin k_x x \tag{6.43}$$

Now applying the boundary condition at $x = a$:

$$E_{y0} e^{-jk_z z} \sin k_x a = 0 \tag{6.44}$$

The factor $e^{-jk_z z}$ cannot be zero, and $E_{y0} = 0$ yields only trivial solutions; therefore:

$$\sin k_x a = 0 \tag{6.45}$$

This in turn requires that

$$k_x a = m\pi \tag{6.46}$$

where m is an integer. Note that $m = 0$ is not of interest since this yields $k_x = 0$, which according to Equation 6.43 yields the trivial solution $\widetilde{E}_y = 0$. Also each integer value of m that is less than zero is excluded because the associated solution is different from the solution for the corresponding positive value of m in sign only, which can be absorbed in the arbitrary constant E_{y0}.

At this point we have uncovered a family of solutions given by Equation 6.43 and Equation 6.46 with $m = 1, 2, ...$. Each solution associated with a particular value of m is referred to as a *mode*, which (via Equation 6.46) has a particular value of k_x. The value of k_z for mode m is obtained using Equation 6.38 as follows:

$$\begin{aligned} k_z &= \sqrt{\beta^2 - k_x^2} \\ &= \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2} \end{aligned} \tag{6.47}$$

Since k_z is specified to be real-valued, we require:

$$\beta^2 - \left(\frac{m\pi}{a}\right)^2 > 0 \tag{6.48}$$

This constrains β; specifically:

$$\beta > \frac{m\pi}{a} \tag{6.49}$$

Recall that $\beta = \omega\sqrt{\mu\epsilon}$ and $\omega = 2\pi f$ where f is frequency. Solving for f, we find:

$$f > \frac{m}{2a\sqrt{\mu\epsilon}} \tag{6.50}$$

Therefore, each mode exists only above a certain frequency, which is different for each mode. This *cutoff frequency* f_c for mode m is given by

$$\boxed{f_c^{(m)} \triangleq \frac{m}{2a\sqrt{\mu\epsilon}}} \tag{6.51}$$

At frequencies below the cutoff frequency for mode m, modes 1 through $m-1$ exhibit imaginary-valued k_z. The propagation constant must have a real-valued component in order to propagate; therefore, these modes do not propagate and may be ignored.

Let us now summarize the solution. For the scenario depicted in Figure 6.2, the electric field component of the TE solution is given by:

$$\boxed{\hat{\mathbf{y}}\widetilde{E}_y = \hat{\mathbf{y}} \sum_{m=1}^{\infty} \widetilde{E}_y^{(m)}} \tag{6.52}$$

where

$$\boxed{\widetilde{E}_y^{(m)} \triangleq \begin{cases} 0, & f < f_c^{(m)} \\ E_{y0}^{(m)} e^{-jk_z^{(m)}z} \sin k_x^{(m)} x, & f \geq f_c^{(m)} \end{cases}} \tag{6.53}$$

where m enumerates modes ($m = 1, 2, ...$),

$$\boxed{k_z^{(m)} \triangleq \sqrt{\beta^2 - \left[k_x^{(m)}\right]^2}} \tag{6.54}$$

and

$$\boxed{k_x^{(m)} \triangleq m\pi/a} \tag{6.55}$$

Finally, $E_{y0}^{(m)}$ is a complex-valued constant that depends on sources or boundary conditions to the left of the region of interest.

For the scenario depicted in Figure 6.2, the electric field component of the TE solution is given by Equation 6.52 with modal components determined as indicated by Equations 6.51–6.55. This solution presumes all sources lie to the left of the region of interest, and no scattering occurs to the right of the region of interest.

To better understand this result, let us examine the lowest-order mode, $m = 1$. For this mode $f_c^{(1)} = 1/2a\sqrt{\mu\epsilon}$, so this mode can exist if $f > 1/2a\sqrt{\mu\epsilon}$. Also $k_x^{(1)} = \pi/a$, so

$$k_z^{(1)} = \sqrt{\beta^2 - \left(\frac{\pi}{a}\right)^2} \tag{6.56}$$

Subsequently,

$$\widetilde{E}_y^{(1)} = E_{y0}^{(1)} e^{-jk_z^{(1)}z} \sin\frac{\pi x}{a} \tag{6.57}$$

Note that this mode has the form of a plane wave. The plane wave propagates in the $+\hat{\mathbf{z}}$ direction with phase propagation constant $k_z^{(1)}$. Also, we observe that the apparent plane wave is non-uniform, exhibiting magnitude proportional to $\sin \pi x/a$ within the waveguide. This is shown in Figure 6.3 (left image).

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Figure 6.3. *Magnitude of the two lowest-order TE modes in a parallel plate waveguide. Left:* $m = 1$*, Right:* $m = 2$*.*

In particular, we observe that the magnitude of the wave is zero at the perfectly-conducting surfaces – as is necessary to satisfy the boundary conditions – and is maximum in the center of the waveguide.

Now let us examine the $m = 2$ mode. For this mode, $f_c^{(2)} = 1/a\sqrt{\mu\epsilon}$, so this mode can exist if $f > 1/a\sqrt{\mu\epsilon}$. This frequency is higher than $f_c^{(1)}$, so the $m = 1$ mode can exist at any frequency at which the $m = 2$ mode exists. Also $k_x^{(2)} = 2\pi/a$, so

$$k_z^{(2)} = \sqrt{\beta^2 - \left(\frac{2\pi}{a}\right)^2} \tag{6.58}$$

Subsequently,

$$\widetilde{E}_y^{(2)} = E_{y0}^{(2)} e^{-jk_z^{(2)}z} \sin\frac{2\pi x}{a} \tag{6.59}$$

In this case, the apparent plane wave propagates in the $+\hat{\mathbf{z}}$ direction with phase propagation constant $k_z^{(2)}$, which is less than $k_z^{(1)}$. For $m = 2$, we find magnitude is proportional to $\sin 2\pi x/a$ within the waveguide (Figure 6.3, right image). As in the $m = 1$ case, we observe that the magnitude of the wave is zero at the PEC surfaces; however, for $m = 2$, there are *two* maxima with respect to x, and the magnitude in the center of the waveguide is zero.

This pattern continues for higher-order modes. In particular, each successive mode exhibits higher cutoff frequency, smaller propagation constant, and increasing integer number of sinusoidal half-periods in magnitude.

Example

Example 6.2

Single-mode TE propagation in a parallel plate waveguide.

Consider an air-filled parallel plate waveguide consisting of plates separated by 1 cm. Determine the frequency range for which one (and only one) propagating TE mode is assured.

Solution. Single-mode TE propagation is assured by limiting frequency f to greater than the cutoff frequency for $m=1$, but lower than the cutoff frequency for $m=2$. (Any frequency higher than the cutoff frequency for $m=2$ allows at least 2 modes to exist.) Calculating the applicable cutoff frequencies, we find:

$$f_c^{(1)} = \frac{1}{2a\sqrt{\mu_0\epsilon_0}} \cong 15.0 \text{ GHz} \tag{6.60}$$

$$f_c^{(2)} = \frac{2}{2a\sqrt{\mu_0\epsilon_0}} \cong 30.0 \text{ GHz} \tag{6.61}$$

Therefore, 15.0 GHz $\leq f \leq$ 30.0 GHz.

Finally, let us consider the phase velocity v_p within the waveguide. For lowest-order mode $m=1$, this is

$$v_p = \frac{\omega}{k_z^{(1)}} = \frac{\omega}{\sqrt{\omega^2\mu\epsilon - \left(\frac{\pi}{a}\right)^2}} \tag{6.62}$$

Recall that the speed of an electromagnetic wave in unbounded space (i.e., not in a waveguide) is $1/\sqrt{\mu\epsilon}$. For example, the speed of light in free space is $1/\sqrt{\mu_0\epsilon_0} = c$. However, the phase velocity indicated by Equation 6.62 is *greater* than $1/\sqrt{\mu\epsilon}$; e.g., faster than light would travel in the same material (presuming it were transparent). At first glance, this may seem to be impossible. However, recall that *information* travels at the group velocity v_g, and not necessarily the phase velocity. (See Section 6.1 for a refresher.) Although we shall not demonstrate this here, the group velocity in the parallel plate waveguide is always *less* than $1/\sqrt{\mu\epsilon}$, so no physical laws are broken, and signals travel somewhat slower than the speed of light, as they do in any other structure used to convey signals.

Also remarkable is that the speed of propagation is different for each mode. In fact, we find that the phase velocity increases and the group velocity decreases as m increases. This phenomenon is known as *dispersion*, and sometimes specifically as *mode dispersion* or *modal dispersion*.

6.4 Parallel Plate Waveguide: TE Case, Magnetic Field

In Section 6.2, the parallel plate waveguide was introduced. In Section 6.3, we determined the TE component of the electric field. In this section, we determine the TE component of the magnetic field. The reader should be familiar with Section 6.3 before attempting this section.

In Section 6.3, the TE component of the electric field was determined to be:

$$\hat{\mathbf{y}}\widetilde{E}_y = \hat{\mathbf{y}} \sum_{m=1}^{\infty} \widetilde{E}_y^{(m)} \tag{6.63}$$

The TE component of the magnetic field may be obtained from the Maxwell-Faraday equation:

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}} \tag{6.64}$$

Thus:

$$\begin{aligned} \widetilde{\mathbf{H}} &= \frac{j}{\omega\mu}\ \nabla \times \widetilde{\mathbf{E}} \\ &= \frac{j}{\omega\mu}\ \nabla \times \left(\hat{\mathbf{y}}\widetilde{E}_y\right) \end{aligned} \tag{6.65}$$

The relevant form of the curl operator is Equation B.16 (Appendix B.2). Although the complete expression consists of 6 terms, all but 2 of these terms are zero because the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components of $\widetilde{\mathbf{E}}$ are zero. The two remaining terms are $-\hat{\mathbf{x}}\partial\widetilde{E}_y/\partial z$ and $+\hat{\mathbf{z}}\partial\widetilde{E}_y/\partial x$. Thus:

$$\widetilde{\mathbf{H}} = \frac{j}{\omega\mu}\ \left(-\hat{\mathbf{x}}\frac{\partial\widetilde{E}_y}{\partial z} + \hat{\mathbf{z}}\frac{\partial\widetilde{E}_y}{\partial x}\right) \tag{6.66}$$

Recall that $\widetilde{E}_y$ is the sum of modes, as indicated in Equation 6.63. Since differentiation (i.e., $\partial/\partial z$ and $\partial/\partial x$) is a linear operator, we may evaluate Equation 6.66 for modes one at a time, and then sum the results. Using this approach, we find:

$$\begin{aligned} \frac{\partial\widetilde{E}_y^{(m)}}{\partial z} &= \frac{\partial}{\partial z}\ E_{y0}^{(m)} e^{-jk_z^{(m)}z} \sin k_x^{(m)}x \\ &= \left(E_{y0}^{(m)} e^{-jk_z^{(m)}z} \sin k_x^{(m)}x\right)\left(-jk_z^{(m)}\right) \end{aligned} \tag{6.67}$$

and

$$\begin{aligned} \frac{\partial\widetilde{E}_y^{(m)}}{\partial x} &= \frac{\partial}{\partial x}\ E_{y0}^{(m)} e^{-jk_z^{(m)}z} \sin k_x^{(m)}x \\ &= \left(E_{y0}^{(m)} e^{-jk_z^{(m)}z} \cos k_x^{(m)}x\right)\left(+k_x^{(m)}\right) \end{aligned} \tag{6.68}$$

We may now assemble a solution for the magnetic field as follows:

$$\hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{z}}\widetilde{H}_z = \hat{\mathbf{x}}\sum_{m=1}^{\infty}\widetilde{H}_x^{(m)} + \hat{\mathbf{z}}\sum_{m=1}^{\infty}\widetilde{H}_z^{(m)} \tag{6.69}$$

where

$$\widetilde{H}_x^{(m)} = -\frac{k_z^{(m)}}{\omega\mu}E_{y0}^{(m)}e^{-jk_z^{(m)}z}\sin k_x^{(m)}x \tag{6.70}$$

$$\widetilde{H}_z^{(m)} = +j\frac{k_x^{(m)}}{\omega\mu}E_{y0}^{(m)}e^{-jk_z^{(m)}z}\cos k_x^{(m)}x \tag{6.71}$$

and modes may only exist at frequencies greater than the associated cutoff frequencies. Summarizing:

The magnetic field component of the TE solution is given by Equation 6.69 with modal components as indicated by Equations 6.70 and 6.71. Caveats pertaining to the cutoff frequencies and the locations of sources and structures continue to apply.

This result is quite complex, yet some additional insight is possible. At the perfectly-conducting (PEC) surface at $x = 0$, we see

$$\begin{aligned}\widetilde{H}_x^{(m)}(x=0) &= 0\\ \widetilde{H}_z^{(m)}(x=0) &= +j\frac{k_x^{(m)}}{\omega\mu}E_{y0}^{(m)}e^{-jk_z^{(m)}z}\end{aligned} \tag{6.72}$$

Similarly, on the PEC surface at $x = a$, we see

$$\begin{aligned}\widetilde{H}_x^{(m)}(x=a) &= 0\\ \widetilde{H}_z^{(m)}(x=a) &= -j\frac{k_x^{(m)}}{\omega\mu}E_{y0}^{(m)}e^{-jk_z^{(m)}z}\end{aligned} \tag{6.73}$$

Thus, we see the magnetic field vector at the PEC surfaces is non-zero and parallel to the PEC surfaces. Recall that the magnetic field is identically zero *inside* a PEC material. Also recall that boundary conditions require that discontinuity in the component of $\mathbf{H}$ tangent to a surface must be supported by a surface current. We conclude that

Current flows on the PEC surfaces of the waveguide.

If this seems surprising, note that essentially the same thing happens in a coaxial transmission line. That is, signals in a coaxial transmission line can be described equally well in terms of either potentials and currents on the inner and outer conductors, or the electromagnetic fields between the conductors. The parallel plate waveguide is only slightly more complicated because the field in a properly-designed coaxial cable is a single transverse electromagnetic (TEM) mode, whereas the fields in a parallel plate waveguide are combinations of TE and TM modes.

Interestingly, we also find that the magnetic field vector points in different directions depending on position relative to the conducting surfaces. We just determined that the magnetic field is parallel to the conducting surfaces at those surfaces. However, the magnetic field is *perpendicular* to those surfaces at m locations between $x = 0$ and $x = a$. These locations correspond to maxima in the electric field.

6.5 Parallel Plate Waveguide: TM Case, Electric Field

In Section 6.2, the parallel plate waveguide shown in Figure 6.4 was introduced.

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Figure 6.4. *TM component of the electric field in a parallel plate waveguide.*

At the end of that section, we decomposed the problem into its TE and TM components. In this section, we find the TM component of the fields in the waveguide.

“Transverse magnetic” means the magnetic field vector is perpendicular to the plane of interest, and is therefore parallel to the conducting surfaces. Thus, $\widetilde{\mathbf{H}} = \hat{\mathbf{y}}\widetilde{H}_y$, with no component in the $\hat{\mathbf{x}}$ or $\hat{\mathbf{z}}$ directions. Following precisely the same reasoning employed in Section 6.2, we find the governing equation for the magnetic component of TM field is:

$$\frac{\partial^2}{\partial x^2}\widetilde{H}_y + \frac{\partial^2}{\partial z^2}\widetilde{H}_y = -\beta^2\widetilde{H}_y \tag{6.74}$$

The general solution to this partial differential equation is:

$$\begin{aligned}\widetilde{H}_y = \; & e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right] \\ & + e^{+jk_z z}\left[Ce^{-jk_x x} + De^{+jk_x x}\right]\end{aligned} \tag{6.75}$$

where A, B, C, and D are complex-valued constants; and k_x and k_z are real-valued constants. We have assigned variable names to these constants with advance knowledge of their physical interpretation; however, at this moment they remain simply unknown constants whose values must be determined by enforcement of boundary conditions.

Note that Equation 6.75 consists of two terms. The first term includes the factor $e^{-jk_z z}$, indicating a wave propagating in the $+\hat{\mathbf{z}}$ direction, and the second term includes the factor $e^{+jk_z z}$, indicating a wave propagating in the $-\hat{\mathbf{z}}$ direction. If we impose the restriction that sources exist only on the left ($z < 0$) side of Figure 6.4, and that there be no structure capable of wave scattering (in particular, reflection) on the right ($z > 0$) side of Figure 6.4, then there can be no wave components propagating in the $-\hat{\mathbf{z}}$ direction. In this case, $C = D = 0$ and Equation 6.75 simplifies to:

$$\widetilde{H}_y = e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right] \tag{6.76}$$

Before proceeding, let's make sure that Equation 6.76 is actually a solution to Equation 6.74. As in the TE case, this check yields a constraint (in fact, the same constraint) on the as-yet undetermined parameters k_x and k_z. First, note:

$$\frac{\partial \widetilde{H}_y}{\partial x} = e^{-jk_z z}\left[-Ae^{-jk_x x} + Be^{+jk_x x}\right](jk_x) \tag{6.77}$$

So:

$$\begin{aligned}\frac{\partial^2 \widetilde{H}_y}{\partial x^2} &= e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right]\left(-k_x^2\right) \\ &= -k_x^2 \widetilde{H}_y\end{aligned} \tag{6.78}$$

Next, note:

$$\frac{\partial \widetilde{H}_y}{\partial z} = e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right](-jk_z) \tag{6.79}$$

So:

$$\begin{aligned}\frac{\partial^2 \widetilde{H}_y}{\partial z^2} &= e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right]\left(-k_z^2\right) \\ &= -k_z^2 \widetilde{H}_y\end{aligned} \tag{6.80}$$

Now summing these results:

$$\frac{\partial^2 \widetilde{H}_y}{\partial x^2} + \frac{\partial^2 \widetilde{H}_y}{\partial z^2} = -\left(k_x^2 + k_z^2\right)\widetilde{H}_y \tag{6.81}$$

Comparing Equation 6.81 to Equation 6.74, we conclude that Equation 6.76 is a solution to Equation 6.74 under the constraint that:

$$\beta^2 = k_x^2 + k_z^2 \tag{6.82}$$

This is precisely the same constraint identified in the TE case, and confirms that k_x and k_y are in fact the components of the propagation vector

$$\mathbf{k} \triangleq \beta\hat{\mathbf{k}} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z \tag{6.83}$$

where $\hat{\mathbf{k}}$ is the unit vector pointing in the direction of propagation, and $k_y = 0$ in this particular problem.

Our objective in this section is to determine the *electric* field component of the TM field. The electric field may be obtained from the magnetic field using Ampere's law:

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\epsilon\widetilde{\mathbf{E}} \tag{6.84}$$

Thus:

$$\begin{aligned} \widetilde{\mathbf{E}} &= \frac{1}{j\omega\epsilon}\ \nabla \times \widetilde{\mathbf{H}} \\ &= \frac{1}{j\omega\epsilon}\ \nabla \times \left(\hat{\mathbf{y}}\widetilde{H}_y\right) \end{aligned} \tag{6.85}$$

The relevant form of the curl operator is Equation B.16 (Appendix B.2). Although the complete expression consists of 6 terms, all but 2 of these terms are zero because the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components of $\widetilde{\mathbf{H}}$ are zero. The two remaining terms are $-\hat{\mathbf{x}}\partial\widetilde{H}_y/\partial z$ and $+\hat{\mathbf{z}}\partial\widetilde{H}_y/\partial x$. Thus:

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\epsilon}\ \left(-\hat{\mathbf{x}}\frac{\partial\widetilde{H}_y}{\partial z} + \hat{\mathbf{z}}\frac{\partial\widetilde{H}_y}{\partial x}\right) \tag{6.86}$$

We may further develop this expression using Equations 6.77 and 6.79. We find the $\hat{\mathbf{x}}$ component of $\widetilde{\mathbf{E}}$ is:

$$\widetilde{E}_x = \frac{k_z}{\omega\epsilon}\ e^{-jk_z z}\left[Ae^{-jk_x x} + Be^{+jk_x x}\right] \tag{6.87}$$

and the $\hat{\mathbf{z}}$ component of $\widetilde{\mathbf{E}}$ is:

$$\widetilde{E}_z = \frac{k_x}{\omega\epsilon}\ e^{-jk_z z}\left[-Ae^{-jk_x x} + Be^{+jk_x x}\right] \tag{6.88}$$

The solution has now been reduced to the problem of finding the constants A, B, and either k_x or k_z. This is accomplished by enforcing the relevant boundary conditions. In general, the component of the electric field which is tangent to a perfectly-conducting surface is zero. Applied to the present (TM) case, this means $\widetilde{E}_z\left(x=0\right) = 0$ and $\widetilde{E}_z\left(x=a\right) = 0$. Referring to Equation 6.88, the boundary condition at $x = 0$ means

$$\frac{k_x}{\omega\epsilon}\ e^{-jk_z z}\left[-A\left(1\right) + B\left(1\right)\right] = 0 \tag{6.89}$$

The factor $e^{-jk_z z}$ always has unit magnitude, and so cannot be zero. We could require k_x to be zero, but this is unnecessarily restrictive. Instead, we require $A = B$ and we may rewrite Equation 6.88 as follows:

$$\widetilde{E}_z = \frac{Bk_x}{\omega\epsilon}\ e^{-jk_zz}\left[e^{+jk_xx} - e^{-jk_xx}\right] \tag{6.90}$$

This expression is simplified using a trigonometric identity:

$$\sin k_xa = \frac{1}{2j}\left[e^{+jk_xa} - e^{-jk_xa}\right] \tag{6.91}$$

Thus:

$$\widetilde{E}_z = \frac{j2Bk_x}{\omega\epsilon} e^{-jk_zz}\sin k_xx \tag{6.92}$$

Now following up with $\widetilde{E}_x$, beginning from Equation 6.87:

$$\begin{aligned}\widetilde{E}_x &= \frac{k_z}{\omega\epsilon}\ e^{-jk_zz}\left[Ae^{-jk_xx} + Be^{+jk_xx}\right]\\ &= \frac{Bk_z}{\omega\epsilon}\ e^{-jk_zz}\left[e^{-jk_xx} + e^{+jk_xx}\right]\\ &= \frac{2Bk_z}{\omega\epsilon}\ e^{-jk_zz}\cos k_xx\end{aligned} \tag{6.93}$$

For convenience we define the following complex-valued constant:

$$E_{x0} \triangleq \frac{2Bk_z}{\omega\epsilon} \tag{6.94}$$

This yields the following simpler expression:

$$\widetilde{E}_x = E_{x0}\ e^{-jk_zz}\cos k_xx \tag{6.95}$$

Now let us apply the boundary condition at $x=a$ to $\widetilde{E}_z$:

$$\frac{j2Bk_z}{\omega\epsilon} e^{-jk_zz}\sin k_xa = 0 \tag{6.96}$$

Requiring $B=0$ or $k_z=0$ yields only trivial solutions, therefore, it must be true that

$$\sin k_xa = 0 \tag{6.97}$$

This in turn requires that

$$k_xa = m\pi \tag{6.98}$$

where m is an integer. Note that this is precisely the same relationship that we identified in the TE case. There is an important difference, however. In the TE case, $m = 0$ was not of interest because this yields $k_x = 0$, and the associated field turned out to be identically zero. In the present (TM) case, $m = 0$ also yields $k_x = 0$, but the associated field is not necessarily zero. That is, for $m = 0$, $\widetilde{E}_z = 0$ but $\widetilde{E}_x$ is not necessarily zero. Therefore, $m = 0$ as well as $m = 1$, $m = 2$, and so on are of interest in the TM case.

At this point, we have uncovered a family of solutions with $m = 0, 1, 2, ...$. Each solution is referred to as a *mode*, and is associated with a particular value of k_x. In the discussion that follows, we shall find that the consequences are identical to those identified in the TE case, except that $m = 0$ is now also allowed. Continuing: The value of k_z for mode m is obtained using Equation 6.82 as follows:

$$\begin{aligned} k_z &= \sqrt{\beta^2 - k_x^2} \\ &= \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2} \end{aligned} \tag{6.99}$$

Since k_z is specified to be real-valued, we require:

$$\beta^2 - \left(\frac{m\pi}{a}\right)^2 > 0 \tag{6.100}$$

This constrains β; specifically:

$$\beta > \frac{m\pi}{a} \tag{6.101}$$

Recall that $\beta = \omega\sqrt{\mu\epsilon}$ and $\omega = 2\pi f$ where f is frequency. Solving for f, we find:

$$f > \frac{m}{2a\sqrt{\mu\epsilon}} \tag{6.102}$$

Thus, each mode exists only above a certain frequency, which is different for each mode. This *cutoff frequency* f_c for mode m is given by

$$\boxed{f_c^{(m)} \triangleq \frac{m}{2a\sqrt{\mu\epsilon}}} \tag{6.103}$$

At frequencies below the cutoff frequency for mode m, modes 0 through $m-1$ exhibit imaginary-valued k_z and therefore do no propagate. Also, note that the cutoff frequency for $m = 0$ is zero, and so this mode is always able to propagate. That is, the $m = 0$ mode may exist for any $a > 0$ and any $f > 0$. Once again, this is a remarkable difference from the TE case, for which $m = 0$ is not available.

Let us now summarize the solution. With respect to Figure 6.4, we find that the electric field component of the TM field is given by:

$$\boxed{\widetilde{\mathbf{E}} = \sum_{m=0}^{\infty} \left[\hat{\mathbf{x}} \widetilde{E}_x^{(m)} + \hat{\mathbf{z}} \widetilde{E}_z^{(m)} \right]} \tag{6.104}$$

where

$$\boxed{\widetilde{E}_x^{(m)} \triangleq \begin{cases} 0, & f < f_c^{(m)} \\ E_{x0}^{(m)} e^{-jk_z^{(m)} z} \cos k_x^{(m)} x, & f \geq f_c^{(m)} \end{cases}} \tag{6.105}$$

and

$$\boxed{\widetilde{E}_z^{(m)} \triangleq \begin{cases} 0, & f < f_c^{(m)} \\ j \frac{k_x^{(m)}}{k_z^{(m)}} E_{x0}^{(m)} e^{-jk_z^{(m)} z} \sin k_x^{(m)} x, & f \geq f_c^{(m)} \end{cases}} \tag{6.106}$$

where m enumerates modes ($m = 0, 1, 2, ...$) and

$$\boxed{k_z^{(m)} \triangleq \sqrt{\beta^2 - \left[k_x^{(m)} \right]^2}} \tag{6.107}$$

$$\boxed{k_x^{(m)} \triangleq m\pi/a} \tag{6.108}$$

Finally, the coefficients $E_{x0}^{(m)}$ depend on sources and/or boundary conditions to the left of the region of interest.

For the scenario depicted in Figure 6.4, the electric field component of the TM solution is given by Equation 6.104 with modal components determined as indicated by Equations 6.103–6.108. This solution presumes all sources lie to the left of the region of interest, with no additional sources or boundary conditions to the right of the region of interest.

The $m = 0$ mode, commonly referred to as the "TM$_0$" mode, is of particular importance in the analysis of microstrip transmission line, and is addressed in Section 6.6.

6.6 Parallel Plate Waveguide: The TM$_0$ Mode

In Section 6.2, the parallel plate waveguide (also shown in Figure 6.5) was introduced.

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Figure 6.5. *TM_0 mode in a parallel plate waveguide.*

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At the end of that section we decomposed the problem into its constituent TE and TM fields. In Section 6.5, we determined the electric field component of the TM field, which was found to consist of a set of discrete modes. In this section, we address the lowest-order ($m = 0$) mode of the TM field, which has special relevance in a number of applications including microstrip transmission lines. This mode is commonly referred to as the "TM_0" mode.

The TM electric field intensity in the waveguide is given by Equation 6.104 with modal components determined as indicated by Equations 6.103–6.108 (Section 6.5). Recall that the $m = 0$ mode can only exist only in the TM case, and does not exist in the TE case. We also noted that the cutoff frequency for this mode is zero, so it may exist at any frequency, and within any non-zero plate separation a. For this mode $k_x^{(0)} = 0$, $k_z^{(0)} = \beta$, and we find

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}} E_{x0}^{(0)} e^{-j\beta z} \quad (\text{TM}_0 \text{ mode}) \tag{6.109}$$

Remarkably, we find that this mode has the form of a uniform plane wave which propagates in the $+\hat{\mathbf{z}}$ direction; i.e., squarely between the plates. The phase and group velocities of this wave are equal to each other and ω/β, precisely as expected for a uniform plane wave in unbounded media; i.e., as if the plates did not exist. This observation allows us to easily determine the associated *magnetic* field: Using the plane wave relationship,

$$\widetilde{\mathbf{H}} = \frac{1}{\eta} \hat{\mathbf{k}} \times \widetilde{\mathbf{E}} \tag{6.110}$$

$$= \frac{1}{\eta} \hat{\mathbf{z}} \times \left(\hat{\mathbf{x}} E_{x0}^{(0)} e^{-j\beta z} \right) \tag{6.111}$$

$$= \hat{\mathbf{y}} \frac{E_{x0}^{(0)}}{\eta} e^{-j\beta z} \quad (\text{TM}_0 \text{ mode}) \tag{6.112}$$

Example

Example 6.3

Guided waves in a printed circuit board (PCB).

A very common form of PCB consists of a 1.575 mm-thick slab of low-loss dielectric having relative permittivity ≈ 4.5 sandwiched between two copper planes. Characterize the electromagnetic field in a long strip of this material. Assume a single source exists at one end of the strip, and operates at frequencies below 10 GHz.

Solution. Let us assume that the copper planes exhibit conductivity that is sufficiently high that the inward-facing surfaces may be viewed as perfectly-conducting. Also, let us limit scope to the field deep within the "sandwich," and neglect the region near the edges of the PCB. Under these conditions, the PCB is well-modeled as a parallel-plate waveguide. Thus, the electromagnetic field consists of a combination of TE and TM modes. The active (non-zero) modes depend on the source (a mode must be "stimulated" by the source in order to propagate) and modal cutoff frequencies. The cutoff frequency for mode m is

$$f_c^{(m)} = \frac{m}{2a\sqrt{\mu\epsilon}} \tag{6.113}$$

In this case, $a = 1.575$ mm, $\mu \approx \mu_0$, and $\epsilon \approx 4.5\epsilon_0$. Therefore:

$$f_c^{(m)} \approx (44.9 \text{ GHz})\, m \tag{6.114}$$

Since the cutoff frequency for $m = 1$ is much greater than 10 GHz, we may rest assured that the only mode that can propagate inside the PCB is TM_0. Therefore, the field deep inside the PCB may be interpreted as a single plane wave having the TM_0 structure shown in Figure 6.5, propagating away from the source end of the PCB. The phase velocity is simply that of the apparent plane wave:

$$v_p = \frac{1}{\sqrt{\mu\epsilon}} \approx 1.41 \times 10^8 \text{ m/s} \approx 0.47c \tag{6.115}$$

The scenario described in this example is essentially a very rudimentary form of microstrip transmission line.

6.7 General Relationships for Unidirectional Waves

Analysis of electromagnetic waves in enclosed spaces – and in waveguides in particular – is quite difficult. The task is dramatically simplified if it can be assumed that the wave propagates in a single direction; i.e., is unidirectional. This does not necessarily entail loss of generality. For example: Within a straight waveguide, waves can travel either “forward” or “backward.” The principle of superposition allows one to consider these two unidirectional cases separately, and then to simply sum the results.

In this section, the equations that relate the various components of a unidirectional wave are derived. The equations will be derived in Cartesian coordinates, anticipating application to rectangular waveguides. However, the underlying strategy is generally applicable.

We begin with Maxwell’s curl equations:

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}} \tag{6.116}$$
$$\nabla \times \widetilde{\mathbf{H}} = +j\omega\epsilon\widetilde{\mathbf{E}} \tag{6.117}$$

Let us consider Equation 6.117 first. Solving for $\widetilde{\mathbf{E}}$, we have:

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\epsilon}\nabla \times \widetilde{\mathbf{H}} \tag{6.118}$$

This is actually three equations; that is, one each for the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ components of $\widetilde{\mathbf{E}}$, respectively. To extract these equations, let us define the components as follows:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z \tag{6.119}$$
$$\widetilde{\mathbf{H}} = \hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{y}}\widetilde{H}_y + \hat{\mathbf{z}}\widetilde{H}_z \tag{6.120}$$

Now applying the equation for curl in Cartesian coordinates (Equation B.16 in Appendix B.2), we find:

$$\widetilde{E}_x = \frac{1}{j\omega\epsilon}\left(\frac{\partial\widetilde{H}_z}{\partial y} - \frac{\partial\widetilde{H}_y}{\partial z}\right) \tag{6.121}$$
$$\widetilde{E}_y = \frac{1}{j\omega\epsilon}\left(\frac{\partial\widetilde{H}_x}{\partial z} - \frac{\partial\widetilde{H}_z}{\partial x}\right) \tag{6.122}$$
$$\widetilde{E}_z = \frac{1}{j\omega\epsilon}\left(\frac{\partial\widetilde{H}_y}{\partial x} - \frac{\partial\widetilde{H}_x}{\partial y}\right) \tag{6.123}$$

Without loss of generality, we may assume that the single direction in which the wave is traveling is in the $+\hat{\mathbf{z}}$ direction. If this is the case, then $\widetilde{\mathbf{H}}$, and subsequently each component of $\widetilde{\mathbf{H}}$, contains a factor of $e^{-jk_z z}$ where k_z is the phase propagation constant in the direction of travel. The remaining factors are independent of z; i.e., depend only on x and y. With this in mind, we further decompose components of $\widetilde{\mathbf{H}}$ as follows:

$$\widetilde{H}_x = \tilde{h}_x(x,y)e^{-jk_z z} \tag{6.124}$$
$$\widetilde{H}_y = \tilde{h}_y(x,y)e^{-jk_z z} \tag{6.125}$$
$$\widetilde{H}_z = \tilde{h}_z(x,y)e^{-jk_z z} \tag{6.126}$$

where $\tilde{h}_x(x,y)$, $\tilde{h}_y(x,y)$, and $\tilde{h}_z(x,y)$ represent the remaining factors. One advantage of this decomposition is that partial derivatives with respect to z reduce to algebraic operations; i.e.:

$$\frac{\partial \widetilde{H}_x}{\partial z} = -jk_z\tilde{h}_x(x,y)e^{-jk_z z} = -jk_z\widetilde{H}_x \tag{6.127}$$

$$\frac{\partial \widetilde{H}_y}{\partial z} = -jk_z\tilde{h}_y(x,y)e^{-jk_z z} = -jk_z\widetilde{H}_y \tag{6.128}$$

$$\frac{\partial \widetilde{H}_z}{\partial z} = -jk_z\tilde{h}_z(x,y)e^{-jk_z z} = -jk_z\widetilde{H}_z \tag{6.129}$$

We now substitute Equations 6.124–6.126 into Equations 6.121–6.123 and then use Equations 6.127–6.129 to eliminate partial derivatives with respect to z. This yields:

$$\widetilde{E}_x = \frac{1}{j\omega\epsilon}\left(\frac{\partial \widetilde{H}_z}{\partial y} + jk_z\widetilde{H}_y\right) \tag{6.130}$$

$$\widetilde{E}_y = \frac{1}{j\omega\epsilon}\left(-jk_z\widetilde{H}_x - \frac{\partial \widetilde{H}_z}{\partial x}\right) \tag{6.131}$$

$$\widetilde{E}_z = \frac{1}{j\omega\epsilon}\left(\frac{\partial \widetilde{H}_y}{\partial x} - \frac{\partial \widetilde{H}_x}{\partial y}\right) \tag{6.132}$$

Applying the same procedure to the curl equation for $\widetilde{\mathbf{H}}$ (Equation 6.116), one obtains:

$$\widetilde{H}_x = \frac{1}{-j\omega\mu}\left(\frac{\partial \widetilde{E}_z}{\partial y} + jk_z\widetilde{E}_y\right) \tag{6.133}$$

$$\widetilde{H}_y = \frac{1}{-j\omega\mu}\left(-jk_z\widetilde{E}_x - \frac{\partial \widetilde{E}_z}{\partial x}\right) \tag{6.134}$$

$$\widetilde{H}_z = \frac{1}{-j\omega\mu}\left(\frac{\partial \widetilde{E}_y}{\partial x} - \frac{\partial \widetilde{E}_x}{\partial y}\right) \tag{6.135}$$

Equations 6.130–6.135 constitute a set of simultaneous equations that represent Maxwell's curl equations in the special case of a unidirectional (specifically, a $+\hat{\mathbf{z}}$-traveling) wave. With just a little bit of algebraic manipulation of these equations, it is possible to obtain expressions for the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components of $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ which depend only on the $\hat{\mathbf{z}}$ components of $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$. Here they are:[1]

$$\widetilde{E}_x = \frac{-j}{k_\rho^2}\left(+k_z\frac{\partial \widetilde{E}_z}{\partial x} + \omega\mu\frac{\partial \widetilde{H}_z}{\partial y}\right) \tag{6.136}$$

$$\widetilde{E}_y = \frac{+j}{k_\rho^2}\left(-k_z\frac{\partial \widetilde{E}_z}{\partial y} + \omega\mu\frac{\partial \widetilde{H}_z}{\partial x}\right) \tag{6.137}$$

$$\widetilde{H}_x = \frac{+j}{k_\rho^2}\left(\omega\epsilon\frac{\partial \widetilde{E}_z}{\partial y} - k_z\frac{\partial \widetilde{H}_z}{\partial x}\right) \tag{6.138}$$

$$\widetilde{H}_y = \frac{-j}{k_\rho^2}\left(\omega\epsilon\frac{\partial \widetilde{E}_z}{\partial x} + k_z\frac{\partial \widetilde{H}_z}{\partial y}\right) \tag{6.139}$$

where

$$k_\rho^2 \triangleq \beta^2 - k_z^2 \tag{6.140}$$

Why define a parameter called "k_ρ"? Note from the definition that $\beta^2 = k_\rho^2 + k_z^2$. Further, note that this equation is an expression of the Pythagorean theorem, which relates the lengths of the sides of a right triangle. Since β is the "overall" phase propagation constant, and k_z is the phase propagation constant for propagation in the $\hat{\mathbf{z}}$ direction in the waveguide, k_ρ must be associated with variation in fields in directions perpendicular to $\hat{\mathbf{z}}$. In the cylindrical coordinate system, this is the $\hat{\rho}$ direction, hence the subscript "ρ." We shall see later that k_ρ plays a special role in determining the structure of fields within the waveguide, and this provides additional motivation to identify this quantity explicitly in the field equations.

Summarizing: If you know the wave is unidirectional, then knowledge of the components of $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ in the direction of propagation is sufficient to determine each of the remaining components of $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$. There is one catch, however: For this to yield sensible results, k_ρ may not be zero. So, ironically, these expressions don't work in the case of a uniform plane wave, since such a wave would have $\widetilde{E}_z = \widetilde{H}_z = 0$ and $k_z = \beta$ (so $k_\rho = 0$), yielding values of zero divided by zero for each remaining field component. The same issue arises for any other transverse electromagnetic (TEM) wave. For non-TEM waves – and, in particular, unidirectional waves in waveguides – either $\widetilde{E}_z$ or $\widetilde{H}_z$ must be non-zero and k_ρ will be non-zero. In this case, Equations 6.136–6.139 are both usable and useful since they allow determination of all field components given just the $\mathbf{z}$ components.

In fact, we may further exploit this simplicity by taking one additional step: Decomposition of the unidirectional wave into transverse electric (TE) and transverse magnetic (TM) components. In this case, TE means simply that $\widetilde{E}_z = 0$, and TM means simply that $\widetilde{H}_z = 0$. For a wave which is either TE or TM, Equations 6.136–6.139 reduce to one term each.

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1. Students are encouraged to derive these for themselves – only algebra is required. Tip to get started: To get $\widetilde{E}_x$, begin with Equation 6.130, eliminate $\widetilde{H}_y$ using Equation 6.134, and then solve for $\widetilde{E}_x$. ↩

6.8 Rectangular Waveguide: TM Modes

A rectangular waveguide is a conducting cylinder of rectangular cross section used to guide the propagation of waves. Rectangular waveguide is commonly used for the transport of radio frequency signals at frequencies in the SHF band (3–30 GHz) and higher. The fields in a rectangular waveguide consist of a number of propagating modes which depends on the electrical dimensions of the waveguide. These modes are broadly classified as either transverse magnetic (TM) or transverse electric (TE). In this section, we consider the TM modes.

Figure 6.6 shows the geometry of interest. Here the walls are located at $x = 0$, $x = a$, $y = 0$, and $y = b$; thus, the cross-sectional dimensions of the waveguide are a and b. The interior of the waveguide is presumed to consist of a lossless material exhibiting real-valued permeability μ and real-valued permittivity ϵ, and the walls are assumed to be perfectly-conducting.

Figure 6.6. *Geometry for analysis of fields in a rectangular waveguide.*

Let us limit our attention to a region within the waveguide which is free of sources. Expressed in phasor form, the electric field intensity within the waveguide is governed by the wave equation

$$\nabla^2 \widetilde{\mathbf{E}} + \beta^2 \widetilde{\mathbf{E}} = 0 \tag{6.141}$$

where

$$\beta = \omega\sqrt{\mu\epsilon} \tag{6.142}$$

Equation 6.141 is a partial differential equation. This equation, combined with boundary conditions imposed by the perfectly-conducting plates, is sufficient to determine a unique solution. This solution is most easily determined in Cartesian coordinates, as we shall now demonstrate. First we express $\widetilde{\mathbf{E}}$ in Cartesian coordinates:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z \tag{6.143}$$

This facilitates the decomposition of Equation 6.141 into separate equations governing the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ components of $\widetilde{\mathbf{E}}$:

$$\nabla^2\widetilde{E}_x + \beta^2\widetilde{E}_x = 0 \tag{6.144}$$
$$\nabla^2\widetilde{E}_y + \beta^2\widetilde{E}_y = 0 \tag{6.145}$$
$$\nabla^2\widetilde{E}_z + \beta^2\widetilde{E}_z = 0 \tag{6.146}$$

Next we observe that the operator ∇^2 may be expressed in Cartesian coordinates as follows:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{6.147}$$

so the equations governing the Cartesian components of $\widetilde{\mathbf{E}}$ may be written as follows:

$$\frac{\partial^2}{\partial x^2}\widetilde{E}_x + \frac{\partial^2}{\partial y^2}\widetilde{E}_x + \frac{\partial^2}{\partial z^2}\widetilde{E}_x + \beta^2\widetilde{E}_x = 0 \tag{6.148}$$
$$\frac{\partial^2}{\partial x^2}\widetilde{E}_y + \frac{\partial^2}{\partial y^2}\widetilde{E}_y + \frac{\partial^2}{\partial z^2}\widetilde{E}_y + \beta^2\widetilde{E}_y = 0 \tag{6.149}$$
$$\frac{\partial^2}{\partial x^2}\widetilde{E}_z + \frac{\partial^2}{\partial y^2}\widetilde{E}_z + \frac{\partial^2}{\partial z^2}\widetilde{E}_z + \beta^2\widetilde{E}_z = 0 \tag{6.150}$$

In general, we expect the total field in the waveguide to consist of unidirectional waves propagating in the $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ directions. We may analyze either of these waves; then the other wave is easily derived via symmetry, and the total field is simply a linear combination (superposition) of these waves. With this in mind, we limit our focus to the wave propagating in the $+\hat{\mathbf{z}}$ direction.

In Section 6.7, it is shown that all components of the electric and magnetic fields can be easily calculated once $\widetilde{E}_z$ and $\widetilde{H}_z$ are known. The problem is further simplified by decomposing the unidirectional wave into TM and TE components. In this decomposition, the TM component is defined by the property that $\widetilde{H}_z = 0$; i.e., is transverse (perpendicular) to the direction of propagation. Thus, the TM component is completely determined by $\widetilde{E}_z$. Equations 6.136–6.139 simplify to become:

$$\widetilde{E}_x = -j\frac{k_z}{k_\rho^2}\frac{\partial \widetilde{E}_z}{\partial x} \tag{6.151}$$

$$\widetilde{E}_y = -j\frac{k_z}{k_\rho^2}\frac{\partial \widetilde{E}_z}{\partial y} \tag{6.152}$$

$$\widetilde{H}_x = +j\frac{\omega\mu}{k_\rho^2}\frac{\partial \widetilde{E}_z}{\partial y} \tag{6.153}$$

$$\widetilde{H}_y = -j\frac{\omega\mu}{k_\rho^2}\frac{\partial \widetilde{E}_z}{\partial x} \tag{6.154}$$

where

$$k_\rho^2 \triangleq \beta^2 - k_z^2 \tag{6.155}$$

and k_z is the phase propagation constant; i.e., the wave is assumed to propagate according to $e^{-jk_z z}$.

Now let us address the problem of finding $\widetilde{E}_z$, which will then completely determine the TM field. As in Section 6.7, we recognize that $\widetilde{E}_z$ can be represented as the propagation factor $e^{-jk_z z}$ times a factor that describes variation with respect to the remaining spatial dimensions x and y:

$$\widetilde{E}_z = \tilde{e}_z(x,y)e^{-jk_z z} \tag{6.156}$$

Substitution of this expression into Equation 6.150 and dividing out the common factor of $e^{-jk_z z}$ yields:

$$\frac{\partial^2}{\partial x^2}\tilde{e}_z + \frac{\partial^2}{\partial y^2}\tilde{e}_z - k_z^2\tilde{e}_z + \beta^2\tilde{e}_z = 0 \tag{6.157}$$

The last two terms may be combined using Equation 6.155, yielding:

$$\frac{\partial^2}{\partial x^2}\tilde{e}_z + \frac{\partial^2}{\partial y^2}\tilde{e}_z + k_\rho^2\tilde{e}_z = 0 \tag{6.158}$$

This is a partial differential equation for $\tilde{e}_z$ in the variables x and y. This equation may be solved using the technique of *separation of variables*. In this technique, we recognize that $\tilde{e}_z(x,y)$ can be written as the product of a function $X(x)$ which depends only on x, and a function $Y(y)$ that depends only on y. That is,

$$\tilde{e}_z(x,y) = X(x)Y(y) \tag{6.159}$$

Substituting this expression into Equation 6.158, we obtain:

$$Y\frac{\partial^2}{\partial x^2}X + X\frac{\partial^2}{\partial y^2}Y + k_\rho^2 XY = 0 \tag{6.160}$$

Next dividing through by XY, we obtain:

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X+\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y+k_\rho^2=0 \tag{6.161}$$

Note that the first term depends only on x, the second term depends only on y, and the remaining term is a constant. Therefore, the sum of the first and second terms is a constant; namely $-k_\rho^2$. Since these terms depend on either x or y, and not both, the first term must equal some constant, the second term must equal some constant, and these constants must sum to $-k_\rho^2$. Therefore, we are justified in separating the equation into two equations as follows:

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X+k_x^2=0 \tag{6.162}$$

$$\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y+k_y^2=0 \tag{6.163}$$

where the new constants k_x^2 and k_y^2 must satisfy

$$k_x^2+k_y^2=k_\rho^2 \tag{6.164}$$

Now multiplying Equations 6.162 and 6.163 by X and Y, respectively, we find:

$$\frac{\partial^2}{\partial x^2}X+k_x^2X=0 \tag{6.165}$$

$$\frac{\partial^2}{\partial y^2}Y+k_y^2Y=0 \tag{6.166}$$

These are familiar one-dimensional differential equations. The solutions are:[1]

$$X=A\cos\left(k_x x\right)+B\sin\left(k_x x\right) \tag{6.167}$$

$$Y=C\cos\left(k_y y\right)+D\sin\left(k_y y\right) \tag{6.168}$$

where A, B, C, and D – like k_x and k_y – are constants to be determined. At this point, we observe that the wave we seek can be expressed as follows:

$$\begin{aligned}\widetilde{E}_z&=\tilde{e}_z(x,y)e^{-jk_z z}\\&=X(x)\ Y(y)\ e^{-jk_z z}\end{aligned} \tag{6.169}$$

The solution is essentially complete except for the values of the constants A, B, C, D, k_x, and k_y. The values of these constants are determined by applying the relevant electromagnetic boundary condition. In this case, it is required that the component of $\widetilde{\mathbf{E}}$ that is tangent to a perfectly-conducting wall must be zero. Note that the $\hat{\mathbf{z}}$ component of $\widetilde{\mathbf{E}}$ is tangent to all four walls; therefore:

$$\widetilde{E}_z\,(x=0)=0 \tag{6.170}$$
$$\widetilde{E}_z\,(x=a)=0 \tag{6.171}$$
$$\widetilde{E}_z\,(y=0)=0 \tag{6.172}$$
$$\widetilde{E}_z\,(y=b)=0 \tag{6.173}$$

Referring to Equation 6.169, these boundary conditions in turn require:

$$X\,(x=0)=0 \tag{6.174}$$
$$X\,(x=a)=0 \tag{6.175}$$
$$Y\,(y=0)=0 \tag{6.176}$$
$$Y\,(y=b)=0 \tag{6.177}$$

Evaluating these conditions using Equations 6.167 and 6.168 yields:

$$A\cdot 1+B\cdot 0=0 \tag{6.178}$$
$$A\cos\left(k_xa\right)+B\sin\left(k_xa\right)=0 \tag{6.179}$$
$$C\cdot 1+D\cdot 0=0 \tag{6.180}$$
$$C\cos\left(k_yb\right)+D\sin\left(k_yb\right)=0 \tag{6.181}$$

Equations 6.178 and 6.180 can be satisfied only if $A=0$ and $C=0$, respectively. Subsequently, Equations 6.179 and 6.181 reduce to:

$$\sin\left(k_xa\right)=0 \tag{6.182}$$
$$\sin\left(k_yb\right)=0 \tag{6.183}$$

This in turn requires:

$$k_x=\frac{m\pi}{a}\ ,\quad m=0,1,2... \tag{6.184}$$
$$k_y=\frac{n\pi}{b}\ ,\quad n=0,1,2... \tag{6.185}$$

Each positive integer value of m and n leads to a valid expression for $\widetilde{E}_z$ known as a *mode*. Solutions for which $m=0$ or $n=0$ yield $k_x=0$ or $k_y=0$, respectively. These correspond to zero-valued fields, and therefore are not of interest. The most general expression for $\widetilde{E}_z$ must account for all non-trivial modes. Summarizing:

$$\widetilde{E}_z=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\widetilde{E}_z^{(m,n)} \tag{6.186}$$

where

$$\widetilde{E}_z^{(m,n)}\triangleq E_0^{(m,n)}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z^{(m,n)}z} \tag{6.187}$$

where $E_0^{(m,n)}$ is an arbitrary constant (consolidating the constants B and D), and, since $k_x^2+k_y^2=k_\rho^2\triangleq\beta^2-k_z^2$:

$$k_z^{(m,n)} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{6.188}$$

Summarizing:

The TM ($\widetilde{H}_z = 0$) component of the unidirectional ($+\hat{\mathbf{z}}$-traveling) wave in a rectangular waveguide is completely determined by Equation 6.186, and consists of modes as defined by Equations 6.187, 6.188, 6.184, and 6.185. The remaining non-zero field components can be determined using Equations 6.151–6.154.

It is customary and convenient to refer to the TM modes in a rectangular waveguide using the notation "TM$_{mn}$." For example, the mode TM$_{12}$ is given by Equation 6.187 with $m = 1$ and $n = 2$.

Finally, note that values of $k_z^{(m,n)}$ obtained from Equation 6.188 are not necessarily real-valued. It is apparent that for any given value of m, $k_z^{(m,n)}$ will be imaginary-valued for all values of n greater than some value. Similarly, it is apparent that for any given value of n, $k_z^{(m,n)}$ will be imaginary-valued for all values of m greater than some value. This phenomenon is common to both TM and TE components, and so is addressed in a separate section (Section 6.10).

Additional Reading:

- "Waveguide (radio frequency) (https://en.wikipedia.org/wiki/Waveguide_(radio_frequency))" on Wikipedia.
- "Separation of variables (https://en.wikipedia.org/wiki/Separation_of_variables)" on Wikipedia.

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1. Students are encouraged to confirm that these are correct by confirming that they are solutions to Equations 6.165 and 6.165, respectively. ↩

6.9 Rectangular Waveguide: TE Modes

A rectangular waveguide is a conducting cylinder of rectangular cross section used to guide the propagation of waves. Rectangular waveguide is commonly used for the transport of radio frequency signals at frequencies in the SHF band (3–30 GHz) and higher. The fields in a rectangular waveguide consist of a number of propagating modes which depends on the electrical dimensions of the waveguide. These modes are broadly classified as either transverse magnetic (TM) or transverse electric (TE). In this section, we consider the TE modes.

Figure 6.7 shows the geometry of interest. Here the walls are located at $x = 0$, $x = a$, $y = 0$, and $y = b$; thus, the cross-sectional dimensions of the waveguide are a and b. The interior of the waveguide is presumed to consist of a lossless material exhibiting real-valued permeability μ and real-valued permittivity ϵ, and the walls are assumed to be perfectly-conducting.

Figure 6.7. *Geometry for analysis of fields in a rectangular waveguide.*

Let us limit our attention to a region within the waveguide which is free of sources. Expressed in phasor form, the magnetic field intensity within the waveguide is governed by the wave equation:

$$\nabla^2\widetilde{\mathbf{H}} + \beta^2\widetilde{\mathbf{H}} = 0 \tag{6.189}$$

where

$$\beta = \omega\sqrt{\mu\epsilon} \tag{6.190}$$

Equation 6.189 is a partial differential equation. This equation, combined with boundary conditions imposed by the perfectly-conducting plates, is sufficient to determine a unique solution. This solution is most easily determined in Cartesian coordinates, as we shall now demonstrate. First we express $\widetilde{\mathbf{H}}$ in Cartesian coordinates:

$$\widetilde{\mathbf{H}} = \hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{y}}\widetilde{H}_y + \hat{\mathbf{z}}\widetilde{H}_z \tag{6.191}$$

This facilitates the decomposition of Equation 6.189 into separate equations governing the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ components of $\widetilde{\mathbf{H}}$:

$$\nabla^2\widetilde{H}_x + \beta^2\widetilde{H}_x = 0 \tag{6.192}$$
$$\nabla^2\widetilde{H}_y + \beta^2\widetilde{H}_y = 0 \tag{6.193}$$
$$\nabla^2\widetilde{H}_z + \beta^2\widetilde{H}_z = 0 \tag{6.194}$$

Next we observe that the operator ∇^2 may be expressed in Cartesian coordinates as follows:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{6.195}$$

so the equations governing the Cartesian components of $\widetilde{\mathbf{H}}$ may be written as follows:

$$\frac{\partial^2}{\partial x^2}\widetilde{H}_x + \frac{\partial^2}{\partial y^2}\widetilde{H}_x + \frac{\partial^2}{\partial z^2}\widetilde{H}_x + \beta^2\widetilde{H}_x = 0 \tag{6.196}$$
$$\frac{\partial^2}{\partial x^2}\widetilde{H}_y + \frac{\partial^2}{\partial y^2}\widetilde{H}_y + \frac{\partial^2}{\partial z^2}\widetilde{H}_y + \beta^2\widetilde{H}_y = 0 \tag{6.197}$$
$$\frac{\partial^2}{\partial x^2}\widetilde{H}_z + \frac{\partial^2}{\partial y^2}\widetilde{H}_z + \frac{\partial^2}{\partial z^2}\widetilde{H}_z + \beta^2\widetilde{H}_z = 0 \tag{6.198}$$

In general, we expect the total field in the waveguide to consist of unidirectional waves propagating in the $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ directions. We may analyze either of these waves; then the other wave is easily derived via symmetry, and the total field is simply a linear combination (superposition) of these waves. With this in mind, we limit our focus to the wave propagating in the $+\hat{\mathbf{z}}$ direction.

In Section 6.7, it is shown that all components of the electric and magnetic fields can be easily calculated once $\widetilde{E}_z$ and $\widetilde{H}_z$ are known. The problem is further simplified by decomposing the unidirectional wave into TM and TE components. In this decomposition, the TE component is defined by the property that $\widetilde{E}_z = 0$; i.e., is transverse (perpendicular) to the direction of propagation. Thus, the TE component is completely determined by $\widetilde{H}_z$. Equations 6.136–6.139 simplify to become:

$$\widetilde{E}_x = -j\frac{\omega\mu}{k_\rho^2}\frac{\partial \widetilde{H}_z}{\partial y} \tag{6.199}$$

$$\widetilde{E}_y = +j\frac{\omega\mu}{k_\rho^2}\frac{\partial \widetilde{H}_z}{\partial x} \tag{6.200}$$

$$\widetilde{H}_x = -j\frac{k_z}{k_\rho^2}\frac{\partial \widetilde{H}_z}{\partial x} \tag{6.201}$$

$$\widetilde{H}_y = -j\frac{k_z}{k_\rho^2}\frac{\partial \widetilde{H}_z}{\partial y} \tag{6.202}$$

where

$$k_\rho^2 \triangleq \beta^2 - k_z^2 \tag{6.203}$$

and k_z is the phase propagation constant; i.e., the wave is assumed to propagate according to $e^{-jk_z z}$.

Now let us address the problem of finding $\widetilde{H}_z$, which will then completely determine the TE field. As in Section 6.7, we recognize that $\widetilde{H}_z$ can be represented as the propagation factor $e^{-jk_z z}$ times a factor that describes variation with respect to the remaining spatial dimensions x and y:

$$\widetilde{H}_z = \tilde{h}_z(x,y)e^{-jk_z z} \tag{6.204}$$

Substitution of this expression into Equation 6.198 and dividing out the common factor of $e^{-jk_z z}$ yields:

$$\frac{\partial^2}{\partial x^2}\tilde{h}_z + \frac{\partial^2}{\partial y^2}\tilde{h}_z - k_z^2\tilde{h}_z + \beta^2\tilde{h}_z = 0 \tag{6.205}$$

The last two terms may be combined using Equation 6.203, yielding:

$$\frac{\partial^2}{\partial x^2}\tilde{h}_z + \frac{\partial^2}{\partial y^2}\tilde{h}_z + k_\rho^2\tilde{h}_z = 0 \tag{6.206}$$

This is a partial differential equation for $\tilde{h}_z$ in the variables x and y. This equation may be solved using the technique of *separation of variables*. In this technique, we recognize that $\tilde{h}_z(x,y)$ can be written as the product of a function $X(x)$ which depends only on x, and a function $Y(y)$ that depends only on y. That is,

$$\tilde{h}_z(x,y) = X(x)Y(y) \tag{6.207}$$

Substituting this expression into Equation 6.206, we obtain:

$$Y\frac{\partial^2}{\partial x^2}X + X\frac{\partial^2}{\partial y^2}Y + k_\rho^2 XY = 0 \tag{6.208}$$

Next dividing through by XY, we obtain:

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X+\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y+k_\rho^2=0 \tag{6.209}$$

Note that the first term depends only on x, the second term depends only on y, and the remaining term is a constant. Therefore, the sum of the first and second terms is a constant; namely $-k_\rho^2$. Since these terms depend on either x or y, and not both, the first term must equal some constant, the second term must equal some constant, and these constants must sum to $-k_\rho^2$. Therefore, we are justified in separating the equation into two equations as follows:

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X+k_x^2=0 \tag{6.210}$$
$$\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y+k_y^2=0 \tag{6.211}$$

where the new constants k_x^2 and k_y^2 must satisfy

$$k_x^2+k_y^2=k_\rho^2 \tag{6.212}$$

Now multiplying Equations 6.210 and 6.211 by X and Y, respectively, we find:

$$\frac{\partial^2}{\partial x^2}X+k_x^2X=0 \tag{6.213}$$
$$\frac{\partial^2}{\partial y^2}Y+k_y^2Y=0 \tag{6.214}$$

These are familiar one-dimensional differential equations. The solutions are:[1]

$$X=A\cos\left(k_x x\right)+B\sin\left(k_x x\right) \tag{6.215}$$
$$Y=C\cos\left(k_y y\right)+D\sin\left(k_y y\right) \tag{6.216}$$

where A, B, C, and D – like k_x and k_y – are constants to be determined. At this point, we observe that the wave we seek can be expressed as follows:

$$\begin{aligned}\widetilde{H}_z&=\tilde{h}_z(x,y)e^{-jk_z z}\\&=X(x)\ Y(y)\ e^{-jk_z z}\end{aligned} \tag{6.217}$$

The solution is essentially complete except for the values of the constants A, B, C, D, k_x, and k_y. The values of these constants are determined by applying the relevant electromagnetic boundary condition. In this case, it is required that any component of $\widetilde{\mathrm{E}}$ that is tangent to a perfectly-conducting wall must be zero. Therefore:

$$\widetilde{E}_y\left(x=0\right)=0 \tag{6.218}$$
$$\widetilde{E}_y\left(x=a\right)=0 \tag{6.219}$$
$$\widetilde{E}_x\left(y=0\right)=0 \tag{6.220}$$
$$\widetilde{E}_x\left(y=b\right)=0 \tag{6.221}$$

Referring to Equation 6.217 and employing Equations 6.199–6.202, we obtain:

$$\frac{\partial}{\partial x}X\left(x=0\right)=0 \tag{6.222}$$
$$\frac{\partial}{\partial x}X\left(x=a\right)=0 \tag{6.223}$$
$$\frac{\partial}{\partial y}Y\left(y=0\right)=0 \tag{6.224}$$
$$\frac{\partial}{\partial y}Y\left(y=b\right)=0 \tag{6.225}$$

Evaluating the partial derivatives and dividing out common factors of k_x and k_y, we find:

$$-A\sin(k_x\cdot 0)+B\cos(k_x\cdot 0)=0 \tag{6.226}$$
$$-A\sin(k_x\cdot a)+B\cos(k_x\cdot a)=0 \tag{6.227}$$
$$-C\sin(k_y\cdot 0)+D\cos(k_y\cdot 0)=0 \tag{6.228}$$
$$-C\sin(k_y\cdot b)+D\cos(k_y\cdot b)=0 \tag{6.229}$$

Evaluating:

$$-A\cdot 0+B\cdot 1=0 \tag{6.230}$$
$$-A\sin\left(k_x a\right)+B\cos\left(k_x a\right)=0 \tag{6.231}$$
$$-C\cdot 0+D\cdot 1=0 \tag{6.232}$$
$$-C\sin\left(k_y b\right)+D\cos\left(k_y b\right)=0 \tag{6.233}$$

Equations 6.230 and 6.232 can be satisfied only if $B=0$ and $D=0$, respectively. Subsequently, Equations 6.231 and 6.233 reduce to:

$$\sin\left(k_x a\right)=0 \tag{6.234}$$
$$\sin\left(k_y b\right)=0 \tag{6.235}$$

This in turn requires:

$$k_x=\frac{m\pi}{a}\ , \quad m=0,1,2... \tag{6.236}$$
$$k_y=\frac{n\pi}{b}\ , \quad n=0,1,2... \tag{6.237}$$

Each positive integer value of m and n leads to a valid expression for $\widetilde{H}_z$ known as a *mode*. Summarizing:

$$\widetilde{H}_z=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\widetilde{H}_z^{(m,n)} \tag{6.238}$$

where

$$\widetilde{H}_z^{(m,n)}\triangleq H_0^{(m,n)}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z^{(m,n)}z} \tag{6.239}$$

where $H_0^{(m,n)}$ is an arbitrary constant (consolidating the constants A and C), and, since $k_x^2 + k_y^2 = k_\rho^2 \triangleq \beta^2 - k_z^2$:

$$k_z^{(m,n)} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{6.240}$$

Summarizing:

The TE ($\widetilde{E}_z = 0$) component of the unidirectional ($+\hat{\mathbf{z}}$-traveling) wave in a rectangular waveguide is completely determined by Equation 6.238, and consists of modes as defined by Equations 6.239, 6.240, 6.236, and 6.237. The remaining non-zero field components can be determined using Equations 6.199–6.202.

It is customary and convenient to refer to the TE modes in a rectangular waveguide using the notation "TE$_{mn}$." For example, the mode TE$_{12}$ is given by Equation 6.239 with $m = 1$ and $n = 2$.

Although Equation 6.238 implies the existence of a TE$_{00}$ mode, it should be noted that this wave has no non-zero electric field components. This can be determined mathematically by following the procedure outlined above. However, this is also readily confirmed as follows: $\widetilde{E}_x$ is constant for the TE$_{00}$ mode because $k_\rho = 0$ for this mode, however, $\widetilde{E}_x$ must be zero to meet the boundary conditions on the walls at $y = 0$ and $y = b$. Similarly, $\widetilde{E}_y$ is constant for the TE$_{00}$ mode, however, $\widetilde{E}_y$ must be zero to meet the boundary conditions on the walls at $x = 0$ and $x = b$.

Finally, note that values of $k_z^{(m,n)}$ obtained from Equation 6.240 are not necessarily real-valued. It is apparent that for any given value of m, $k_z^{(m,n)}$ will be imaginary-valued for all values of n greater than some value. Similarly, it is apparent that for any given value of n, $k_z^{(m,n)}$ will be imaginary-valued for all values of m greater than some value. This phenomenon is common to both TE and TM components, and so is addressed in a separate section (Section 6.10).

Additional Reading:

- "Waveguide (radio frequency) (https://en.wikipedia.org/wiki/Waveguide_(radio_frequency))" on Wikipedia.
- "Separation of variables (https://en.wikipedia.org/wiki/Separation_of_variables)" on Wikipedia.

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1. Students are encouraged to confirm that these are correct by confirming that they are solutions to Equations 6.213 and 6.213, respectively. ↩

6.10 Rectangular Waveguide: Propagation Characteristics

In this section, we consider the propagation characteristics of TE and TM modes in rectangular waveguides. Because these modes exhibit the same phase dependence on z, findings of this section apply equally to both sets of modes. Recall that the TM modes in a rectangular waveguide are given by:

$$\widetilde{E}_z^{(m,n)} = E_0^{(m,n)} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-jk_z^{(m,n)}z} \tag{6.241}$$

where $E_0^{(m,n)}$ is an arbitrary constant (determined in part by sources), and:

$$k_z^{(m,n)} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{6.242}$$

The TE modes in a rectangular waveguide are:

$$\widetilde{H}_z^{(m,n)} = H_0^{(m,n)} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-jk_z^{(m,n)}z} \tag{6.243}$$

where $H_0^{(m,n)}$ is an arbitrary constant (determined in part by sources).

Cutoff frequency. First, observe that values of $k_z^{(m,n)}$ obtained from Equation 6.242 are not necessarily real-valued. For any given value of m, $(k_z^{(m,n)})^2$ will be negative for all values of n greater than some value. Similarly, for any given value of n, $(k_z^{(m,n)})^2$ will be negative for all values of m greater than some value. Should either of these conditions occur, we find:

$$\begin{aligned}\left(k_z^{(m,n)}\right)^2 &= \omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \quad < 0 \\ &= -\left|\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right| \\ &= -\alpha^2\end{aligned} \tag{6.244}$$

where α is a positive real-valued constant. So:

$$k_z^{(m,n)} = \pm j\alpha \tag{6.245}$$

Subsequently:

$$\begin{aligned}e^{-jk_z^{(m,n)}z} &= e^{-j(\pm j\alpha)z} \\ &= e^{\pm\alpha z}\end{aligned} \tag{6.246}$$

The "$+$" sign option corresponds to a wave that grows exponentially in magnitude with increasing z, which is non-physical behavior. Therefore:

$$e^{-jk_z^{(m,n)}z} = e^{-\alpha z} \tag{6.247}$$

Summarizing: When values of m or n are such that $(k_z^{(m,n)})^2 < 0$, the magnitude of the associated wave is no longer constant with z. Instead, the magnitude of the wave decreases exponentially with increasing z. Such a wave does not effectively convey power through the waveguide, and is said to be *cut off*.

Since waveguides are normally intended for the efficient transfer of power, it is important to know the criteria for a mode to be cut off. Since cutoff occurs when $(k_z^{(m,n)})^2 < 0$, cutoff occurs when:

$$\omega^2\mu\epsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{6.248}$$

Since $\omega = 2\pi f$:

$$f > \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{6.249}$$

$$= \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{6.250}$$

Note that $1/\sqrt{\mu\epsilon}$ is the phase velocity v_p for the medium used in the waveguide. With this in mind, let us define:

$$v_{pu} \triangleq \frac{1}{\sqrt{\mu\epsilon}} \tag{6.251}$$

This is the phase velocity in an unbounded medium having the same permeability and permittivity as the interior of the waveguide. Thus:

$$f > \frac{v_{pu}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{6.252}$$

In other words, the mode (m, n) avoids being cut off if the frequency is high enough to meet this criterion. Thus, it is useful to make the following definition:

$$f_{mn} \triangleq \frac{v_{pu}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{6.253}$$

The *cutoff frequency* f_{mn} (Equation 6.253) is the lowest frequency for which the mode (m, n) is able to propagate (i.e., not cut off).

Example

Example 6.4

Cutoff frequencies for WR-90.

WR-90 is a popular implementation of rectangular waveguide. WR-90 is air-filled with dimensions $a = 22.86$ mm and $b = 10.16$ mm. Determine cutoff frequencies and, in particular, the lowest frequency at which WR-90 can be used.

Solution. Since WR-90 is air-filled, $\mu \approx \mu_0$, $\epsilon \approx \epsilon_0$, and $v_{pu} \approx 1\sqrt{\mu_0\epsilon_0} \cong 3.00 \times 10^8$ m/s. Cutoff frequencies are given by Equation 6.253. Recall that there are no non-zero TE or TM modes with $m = 0$ and $n = 0$. Since $a > b$, the lowest non-zero cutoff frequency is achieved when $m = 1$ and $n = 0$. In this case, Equation 6.253 yields $f_{10} = \underline{6.557 \text{ GHz}}$; this is the lowest frequency that is able to propagate efficiently in the waveguide. The next lowest cutoff frequency is $f_{20} = 13.114$ GHz. The third lowest cutoff frequency is $f_{01} = 14.754$ GHz. The lowest-order TM mode that is non-zero and not cut off is TM$_{11}$ ($f_{11} = 16.145$ GHz).

Phase velocity. The phase velocity for a wave propagating within a rectangular waveguide is greater than that of electromagnetic radiation in unbounded space. For example, the phase velocity of any propagating mode in a vacuum-filled waveguide is greater than c, the speed of light in free space. This is a surprising result. Let us first derive this result and then attempt to make sense of it.

Phase velocity v_p in the rectangular waveguide is given by

$$v_p \triangleq \frac{\omega}{k_z^{(m,n)}} \tag{6.254}$$

$$= \frac{\omega}{\sqrt{\omega^2\mu\epsilon - (m\pi/a)^2 - (n\pi/b)^2}} \tag{6.255}$$

Immediately we observe that phase velocity seems to be different for different modes. Dividing the numerator and denominator by $\beta = \omega\sqrt{\mu\epsilon}$, we obtain:

$$v_p = \frac{1}{\sqrt{\mu\epsilon}} \frac{1}{\sqrt{1 - (\omega^2\mu\epsilon)^{-1}\left[(m\pi/a)^2 + (n\pi/b)^2\right]}} \tag{6.256}$$

Note that $1/\sqrt{\mu\epsilon}$ is v_{pu}, as defined earlier. Employing Equation 6.253 and also noting that $\omega = 2\pi f$, Equation 6.256 may be rewritten in the following form:

$$v_p = \frac{v_{pu}}{\sqrt{1 - (f_{mn}/f)^2}} \tag{6.257}$$

For any propagating mode, $f > f_{mn}$; subsequently, $v_p > v_{pu}$. In particular, $v_p > c$ for a vacuum-filled waveguide.

How can this not be a violation of fundamental physics? As noted in Section 6.1, phase velocity is *not* necessarily the speed at which information travels, but is merely the speed at which a point of constant phase travels. To send information, we must create a disturbance in the otherwise sinusoidal excitation presumed in the analysis so far. The complex field structure creates points of constant phase that travel faster than the disturbance is able to convey information, so there is no violation of physical principles.

Group velocity. As noted in Section 6.1, the speed at which information travels is given by the group velocity v_g. In unbounded space, $v_g = v_p$, so the speed of information is equal to the phase velocity in that case. In a rectangular waveguide, the situation is different. We find:

$$v_g = \left(\frac{\partial k_z^{(m,n)}}{\partial \omega} \right)^{-1} \tag{6.258}$$

$$= v_{pu} \sqrt{1 - (f_{mn}/f)^2} \tag{6.259}$$

which is always *less* than v_{pu} for a propagating mode.

Note that group velocity in the waveguide depends on frequency in two ways. First, because f_{mn} takes on different values for different modes, group velocity is different for different modes. Specifically, higher-order modes propagate more slowly than lower-order modes having the same frequency. This is known as *modal dispersion*. Secondly, note that the group velocity of any given mode depends on frequency. This is known as *chromatic dispersion*.

The speed of a signal within a rectangular waveguide is given by the group velocity of the associated mode (Equation 6.259). This speed is less than the speed of propagation in unbounded media having the same permittivity and permeability. Speed depends on the ratio f_{mn}/f, and generally decreases with increasing frequency for any given mode.

Example

Example 6.5

Speed of propagating in WR-90.

Revisiting WR-90 waveguide from Example 6.4: What is the speed of propagation for a narrowband signal at 10 GHz?

Solution. Let us assume that “narrowband” here means that the bandwidth is negligible relative to the center frequency, so that we need only consider the center frequency. As previously determined, the lowest-order propagating mode is TE$_{10}$, for which $f_{10} = 6.557\ \mathrm{GHz}$. The next-lowest cutoff frequency is $f_{20} = 13.114\ \mathrm{GHz}$. Therefore, only the TE$_{10}$ mode is available for this signal. The group velocity for this mode at the frequency of interest is given by Equation 6.259. Using this equation, the speed of propagation is found to be $\cong 2.26 \times 10^8$ m/s, which is about 75.5% of c.

End Matter

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