5 Wave Reflection and Transmission

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5.1 Plane Waves at Normal Incidence on a Planar Boundary

When a plane wave encounters a discontinuity in media, reflection from the discontinuity and transmission into the second medium is possible. In this section, we consider the scenario illustrated in Figure 5.1: a uniform plane wave which is normally incident on the planar boundary between two semi-infinite material regions. By "normally-incident" we mean the direction of propagation $\hat{\mathbf{k}}$ is perpendicular to the boundary. We shall assume that the media are "simple" and lossless (i.e., the imaginary component of permittivity ϵ'' is equal to zero) and therefore the media are completely defined by a real-valued permittivity and a real-valued permeability.

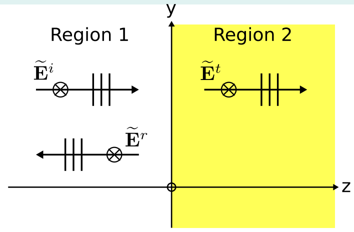

Figure 5.1. *A uniform plane wave normally incident on the planar boundary between two semi-infinite material regions.*

Figure 5.1 shows the wave incident on the boundary, which is located at the $z = 0$ plane. The electric field intensity $\widetilde{\mathbf{E}}^i$ of this wave is given by

$$\widetilde{\mathbf{E}}^i(z) = \hat{\mathbf{x}} E_0^i e^{-j\beta_1 z} \ , \ \ z \leq 0 \tag{5.1}$$

where $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$ is the phase propagation constant in Region 1 and E_0^i is a complex-valued constant. $\widetilde{\mathbf{E}}^i$ serves as the "stimulus" in this problem. That is, all other contributions to the total field may be expressed in terms of $\widetilde{\mathbf{E}}^i$. In fact, all other contributions to the total field may be expressed in terms of E_0^i.

From the plane wave relationships, we determine that the associated magnetic field intensity is

$$\widetilde{\mathbf{H}}^i(z) = \hat{\mathbf{y}} \frac{E_0^i}{\eta_1} e^{-j\beta_1 z} \ , \ \ z \leq 0 \tag{5.2}$$

where $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ is the wave impedance in Region 1.

The possibility of a reflected plane wave propagating in exactly the opposite direction is inferred from two pieces of evidence: The general solution to the wave equation, which includes terms corresponding to waves traveling in both $+\hat{\mathbf{z}}$ or $-\hat{\mathbf{z}}$; and the geometrical symmetry of the problem, which precludes waves traveling in any other directions. The symmetry of the problem also precludes a change of polarization, so the reflected wave should have no $\hat{\mathbf{y}}$ component. Therefore, we may be confident that the reflected electric field has the form

$$\widetilde{\mathbf{E}}^r(z) = \hat{\mathbf{x}} B e^{+j\beta_1 z} \ , \ \ z \leq 0 \tag{5.3}$$

where B is a complex-valued constant that remains to be determined. Since the direction of propagation for the reflected wave is $-\hat{\mathbf{z}}$, we have from the plane wave relationships that

$$\widetilde{\mathbf{H}}^r(z) = -\hat{\mathbf{y}}\frac{B}{\eta_1}e^{+j\beta_1 z}\ , \ \ z \leq 0 \tag{5.4}$$

Similarly, we infer the existence of a “transmitted” plane wave propagating on the $z > 0$ side of the boundary. The symmetry of the problem precludes any direction of propagation other than $+\mathbf{z}$, and with no possibility of a wave traveling in the $-\hat{\mathbf{z}}$ direction for $z > 0$. Therefore, we may be confident that the transmitted electric field has the form:

$$\widetilde{\mathbf{E}}^t(z) = \hat{\mathbf{x}}Ce^{-j\beta_2 z}\ , \ \ z \geq 0 \tag{5.5}$$

and an associated magnetic field having the form

$$\widetilde{\mathbf{H}}^t(z) = \hat{\mathbf{y}}\frac{C}{\eta_2}e^{-j\beta_2 z}\ , \ \ z \geq 0 \tag{5.6}$$

where $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$ and $\eta_2 = \sqrt{\mu_2/\epsilon_2}$ are the phase propagation constant and wave impedance, respectively, in Region 2. The constant C, like B, is a complex-valued constant that remains to be determined.

At this point, the only unknowns in this problem are the complex-valued constants B and C. Once these values are known, the problem is completely solved. These values can be determined by the application of boundary conditions at $z = 0$. First, recall that the tangential component of the total electric field intensity must be continuous across a material boundary. To apply this boundary condition, let us define $\widetilde{\mathbf{E}}_1$ and $\widetilde{\mathbf{E}}_2$ to be the total electric field intensities in Regions 1 and 2, respectively. The total field in Region 1 is the sum of incident and reflected fields, so

$$\widetilde{\mathbf{E}}_1(z) = \widetilde{\mathbf{E}}^i(z) + \widetilde{\mathbf{E}}^r(z) \tag{5.7}$$

The field in Region 2 is simply

$$\widetilde{\mathbf{E}}_2(z) = \widetilde{\mathbf{E}}^t(z) \tag{5.8}$$

Also, we note that all field components are already tangent to the boundary. Thus, continuity of the tangential component of the electric field across the boundary requires $\widetilde{\mathbf{E}}_1(0) = \widetilde{\mathbf{E}}_2(0)$, and therefore

$$\widetilde{\mathbf{E}}^i(0) + \widetilde{\mathbf{E}}^r(0) = \widetilde{\mathbf{E}}^t(0) \tag{5.9}$$

Now employing Equations 5.1, 5.3, and 5.5, we obtain:

$$E_0^i + B = C \tag{5.10}$$

Clearly a second equation is required to determine both B and C. This equation can be obtained by enforcing the boundary condition on the magnetic field. Recall that any discontinuity in the tangential component of the total magnetic field intensity must be supported by a current flowing on the surface. There is no impressed current in this problem, and there is no reason to suspect a current will arise in response to the fields present in the problem. Therefore, the tangential components of the magnetic field must be continuous across the boundary. This becomes the same boundary condition that we applied to the total electric field intensity, so the remaining steps are the same. We define $\widetilde{\mathbf{H}}_1$ and $\widetilde{\mathbf{H}}_2$ to be the total magnetic field intensities in Regions 1 and 2, respectively. The total field in Region 1 is

$$\widetilde{\mathbf{H}}_1(z) = \widetilde{\mathbf{H}}^i(z) + \widetilde{\mathbf{H}}^r(z) \tag{5.11}$$

The field in Region 2 is simply

$$\widetilde{\mathbf{H}}_2(z) = \widetilde{\mathbf{H}}^t(z) \tag{5.12}$$

The boundary condition requires $\widetilde{\mathbf{H}}_1(0) = \widetilde{\mathbf{H}}_2(0)$, and therefore

$$\widetilde{\mathbf{H}}^i(0) + \widetilde{\mathbf{H}}^r(0) = \widetilde{\mathbf{H}}^t(0) \tag{5.13}$$

Now employing Equations 5.2, 5.4, and 5.6, we obtain:

$$\frac{E_0^i}{\eta_1} - \frac{B}{\eta_1} = \frac{C}{\eta_2} \tag{5.14}$$

Equations 5.10 and 5.14 constitute a linear system of two simultaneous equations with two unknowns. A straightforward method of solution is to first eliminate C by substituting the left side of Equation 5.10 into Equation 5.14, and then to solve for B. One obtains:

$$B = \Gamma_{12} E_0^i \tag{5.15}$$

where

$$\boxed{\Gamma_{12} \triangleq \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \tag{5.16}$$

Γ_{12} is known as a *reflection coefficient*. The subscript “12” indicates that this coefficient applies for incidence from Region 1 toward Region 2. We may now solve for C by substituting Equation 5.15 into Equation 5.10. We find:

$$C = (1 + \Gamma_{12})\ E_0^i \tag{5.17}$$

Now summarizing the solution:

$$\boxed{\widetilde{\mathbf{E}}^r(z) = \hat{\mathbf{x}} \Gamma_{12} E_0^i e^{+j\beta_1 z} \quad , \ \ z \leq 0} \tag{5.18}$$

$$\boxed{\widetilde{\mathbf{E}}^t(z) = \hat{\mathbf{x}}\, (1 + \Gamma_{12})\, E_0^i e^{-j\beta_2 z} \quad , \ \ z \geq 0} \tag{5.19}$$

Equations 5.18 and 5.19 are the reflected and transmitted fields, respectively, in response to the incident field given in Equation 5.1 in the normal incidence scenario shown in Figure 5.1.

Expressions for $\widetilde{\mathbf{H}}^r$ and $\widetilde{\mathbf{H}}^t$ may be obtained by applying the plane wave relationships to the preceding expressions.

It is useful to check this solution by examining some special cases. First: If the material in Region 2 is identical to the material in Region 1, then there should be no reflection and $\widetilde{\mathbf{E}}^t = \widetilde{\mathbf{E}}^i$. In this case, $\eta_2 = \eta_1$, so $\Gamma_{12} = 0$, and we obtain the expected result.

A second case of practical interest is when Region 2 is a perfect conductor. First, note that this may seem at first glance to be a violation of the "lossless" assumption made at the beginning of this section. While it is true that we did not explicitly account for the possibility of a perfect conductor in Region 2, let's see what the present analysis has to say about this case. If the material in Region 2 is a perfect conductor, then there should be no transmission since the electric field is zero in a perfect conductor. In this case, $\eta_2 = 0$ since the ratio of electric field intensity to magnetic field intensity is zero in Region 2, and subsequently $\Gamma_{12} = -1$ and $1 + \Gamma_{12} = 0$. As expected, $\widetilde{\mathbf{E}}^t$ is found to be zero, and the reflected electric field experiences a sign change as required to enforce the boundary condition $\widetilde{\mathbf{E}}^i(0) + \widetilde{\mathbf{E}}^r(0) = 0$. Thus, we obtained the correct answer because we were able to independently determine that $\eta_2 = 0$ in a perfect conductor.

When Region 2 is a perfect conductor, the reflection coefficient $\Gamma_{12} = -1$ and the solution described in Equations 5.18 and 5.19 applies.

It may be helpful to note the very strong analogy between reflection of the electric field component of a plane wave from a planar boundary and the reflection of a voltage wave in a transmission line from a terminating impedance. In a transmission line, the voltage reflection coefficient Γ is given by

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{5.20}$$

where Z_L is the load impedance and Z_0 is the characteristic impedance of the transmission line. Comparing this to Equation 5.16, we see η_1 is analogous to Z_0 and η_2 is analogous to Z_L. Furthermore, we see the special case $\eta_2 = \eta_1$, considered previously, is analogous to a matched load, and the special case $\eta_2 = 0$, also considered previously, is analogous to a short-circuit load.

In the case of transmission lines, we were concerned about what fraction of the power was delivered into a load and what fraction of the power was reflected from a load. A similar question applies in the case of plane wave reflection. In this case, we are concerned about what fraction of the power *density* is transmitted into Region 2 and what fraction of the power *density* is reflected from the boundary. The time-average power density S^i_{ave} associated with the incident wave is:

$$S^i_{ave} = \frac{|E^i_0|^2}{2\eta_1} \tag{5.21}$$

presuming that E^i_0 is expressed in peak (as opposed to rms) units. Similarly, the time-average power density S^r_{ave} associated with the reflected wave is:

$$\begin{aligned} S^r_{ave} &= \frac{\left|\Gamma_{12} E^i_0\right|^2}{2\eta_1} \\ &= |\Gamma_{12}|^2 \frac{\left|E^i_0\right|^2}{2\eta_1} \\ &= |\Gamma_{12}|^2 S^i_{ave} \end{aligned} \tag{5.22}$$

From the principle of conservation of power, the power density S^t_{ave} transmitted into Region 2 must be equal to the incident power density minus the reflected power density. Thus:

$$\begin{aligned} S^t_{ave} &= S^i_{ave} - S^r_{ave} \\ &= \left(1 - |\Gamma_{12}|^2\right) S^i_{ave} \end{aligned} \tag{5.23}$$

In other words, the ratio of power density transmitted into Region 2 to power density incident from Region 1 is

$$\boxed{\frac{S^t_{ave}}{S^i_{ave}} = 1 - |\Gamma_{12}|^2} \tag{5.24}$$

Again, the analogy to transmission line theory is evident.

The fractions of power density reflected and transmitted relative to power density incident are given in terms of the reflection coefficient Γ_{12} by Equations 5.22 and 5.24, respectively.

Finally, note that a variety of alternative expressions exist for the reflection coefficient Γ_{12}. Since the wave impedance $\eta = \sqrt{\mu/\epsilon}$ in lossless media, and since most lossless materials are non-magnetic (i.e., exhibit $\mu \approx \mu_0$), it is possible to express Γ_{12} purely in terms of permittivity for these media. Assuming $\mu = \mu_0$, we find:

$$\begin{aligned} \Gamma_{12} &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\ &= \frac{\sqrt{\mu_0/\epsilon_2} - \sqrt{\mu_0/\epsilon_1}}{\sqrt{\mu_0/\epsilon_2} + \sqrt{\mu_0/\epsilon_1}} \\ &= \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \end{aligned} \tag{5.25}$$

Moreover, recall that permittivity can be expressed in terms of relative permittivity; that is, $\epsilon = \epsilon_r \epsilon_0$. Making the substitution above and eliminating all extraneous factors of ϵ_0, we find:

$$\Gamma_{12} = \frac{\sqrt{\epsilon_{r1}} - \sqrt{\epsilon_{r2}}}{\sqrt{\epsilon_{r1}} + \sqrt{\epsilon_{r2}}} \tag{5.26}$$

where ϵ_{r1} and ϵ_{r2} are the relative permittivities in Regions 1 and 2, respectively.

Example

Example 5.1

Radio reflection from the surface of the Moon.

At radio frequencies, the Moon can be modeled as a low-loss dielectric with relative permittivity of about 3. Characterize the efficiency of reflection of a radio wave from the Moon.

Solution. At radio frequencies, a wavelength is many orders of magnitude less than the diameter of the Moon, so we are justified in treating the Moon's surface as a planar boundary between free space and a semi-infinite region of lossy dielectric material. The reflection coefficient is given approximately by Equation 5.26 with $\epsilon_{r1} \approx 1$ and $\epsilon_{r2} \sim 3$. Thus, $\Gamma_{12} \sim -0.27$. Subsequently, the fraction of power reflected from the Moon relative to power incident is $|\Gamma_{12}|^2 \sim 0.07$; i.e., about 7%.

In optics, it is common to make the definition $n \triangleq \sqrt{\epsilon_r}$ where n is known as the *index of refraction* or *refractive index*.[1] In terms of the indices of refraction n_1 and n_2 for Regions 1 and 2, respectively:

$$\Gamma_{12} = \frac{n_1 - n_2}{n_1 + n_2} \tag{5.27}$$

Additional Reading:

- "Refractive index (https://en.wikipedia.org/wiki/Refractive_index)" on Wikipedia.

1. A bit of a misnomer, since the definition applies even when there is no refraction, as in the scenario considered in this section. ↩

5.2 Plane Waves at Normal Incidence on a Material Slab

In Section 5.1, we considered what happens when a uniform plane wave is normally incident on the planar boundary between two semi-infinite media. In this section, we consider the problem shown in Figure 5.2: a uniform plane wave normally incident on a "slab" sandwiched between two semi-infinite media. This scenario arises in many practical engineering problems, including the design and analysis of filters and impedance-matching devices at RF and optical frequencies, the analysis of RF propagation through walls, and the design and analysis of radomes. The findings of Section 5.1 are an important stepping stone to the solution of this problem, so a review of Section 5.1 is recommended before reading this section.

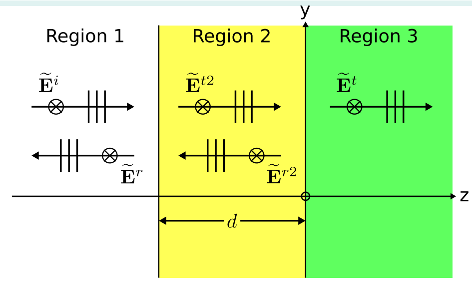

Figure 5.2. *A uniform plane wave normally incident on a slab.*

For consistency of terminology, let us refer to the problem considered in Section 5.1 as the "single-boundary" problem and the present (slab) problem as the "double-boundary" problem. Whereas there are only two regions ("Region 1" and "Region 2") in the single-boundary problem, in the double-boundary problem there is a third region that we shall refer to as "Region 3." We assume that the media comprising each slab are "simple" and lossless (i.e., the imaginary component of permittivity ϵ'' is equal to zero) and therefore the media are completely defined by a real-valued permittivity and a real-valued permeability. The boundary between Regions 1 and 2 is at $z = -d$, and the boundary between Regions 2 and 3 is at $z = 0$. Thus, the thickness of the slab is d. In both problems, we presume an incident wave in Region 1 incident on the boundary with Region 2 having electric field intensity

$$\widetilde{\mathbf{E}}^i(z) = \hat{\mathbf{x}} E_0^i e^{-j\beta_1 z} \quad \text{(Region 1)} \tag{5.28}$$

where $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$ is the phase propagation constant in Region 1. $\widetilde{\mathbf{E}}^i$ serves as the "stimulus" in this problem. That is, all other contributions to the total field may be expressed in terms of $\widetilde{\mathbf{E}}^i$.

From the plane wave relationships, we determine that the associated magnetic field intensity is

$$\widetilde{\mathbf{H}}^i(z) = \hat{\mathbf{y}} \frac{E_0^i}{\eta_1} e^{-j\beta_1 z} \quad \text{(Region 1)} \tag{5.29}$$

where $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ is the wave impedance in Region 1.

The symmetry arguments of the single-boundary problem apply in precisely the same way to the double-boundary problem. Therefore, we presume that the reflected electric field intensity is:

$$\widetilde{\mathbf{E}}^r(z) = \hat{\mathbf{x}} B e^{+j\beta_1 z} \quad \text{(Region 1)} \tag{5.30}$$

where B is a complex-valued constant that remains to be determined; and subsequently the reflected magnetic field intensity is:

$$\widetilde{\mathbf{H}}^r(z) = -\hat{\mathbf{y}} \frac{B}{\eta_1} e^{+j\beta_1 z} \quad \text{(Region 1)} \tag{5.31}$$

Similarly, we infer the existence of a transmitted plane wave propagating in the $+\hat{\mathbf{z}}$ direction in Region 2. The electric and magnetic field intensities of this wave are given by:

$$\widetilde{\mathbf{E}}^{t2}(z) = \hat{\mathbf{x}} C e^{-j\beta_2 z} \quad \text{(Region 2)} \tag{5.32}$$

and an associated magnetic field having the form:

$$\widetilde{\mathbf{H}}^{t2}(z) = \hat{\mathbf{y}}\frac{C}{\eta_2}e^{-j\beta_2 z} \quad \text{(Region 2)} \tag{5.33}$$

where $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$ and $\eta_2 = \sqrt{\mu_2/\epsilon_2}$ are the phase propagation constant and wave impedance, respectively, in Region 2. The constant C, like B, is a complex-valued constant that remains to be determined.

Now let us consider the boundary between Regions 2 and 3. Note that $\widetilde{\mathbf{E}}^{t2}$ is incident on this boundary in precisely the same manner as $\widetilde{\mathbf{E}}^{i}$ is incident on the boundary between Regions 1 and 2. Therefore, we infer a reflected electric field intensity in Region 2 as follows:

$$\widetilde{\mathbf{E}}^{r2}(z) = \hat{\mathbf{x}}De^{+j\beta_2 z} \quad \text{(Region 2)} \tag{5.34}$$

where D is a complex-valued constant that remains to be determined; and an associated magnetic field

$$\widetilde{\mathbf{H}}^{r2}(z) = -\hat{\mathbf{y}}\frac{D}{\eta_2}e^{+j\beta_2 z} \quad \text{(Region 2)} \tag{5.35}$$

Subsequently, we infer a transmitted electric field intensity in Region 3 as follows:

$$\widetilde{\mathbf{E}}^{t}(z) = \hat{\mathbf{x}}Fe^{-j\beta_3 z} \quad \text{(Region 3)} \tag{5.36}$$

and an associated magnetic field having the form

$$\widetilde{\mathbf{H}}^{t}(z) = \hat{\mathbf{y}}\frac{F}{\eta_3}e^{-j\beta_3 z} \quad \text{(Region 3)} \tag{5.37}$$

where $\beta_3 = \omega\sqrt{\mu_3\epsilon_3}$ and $\eta_3 = \sqrt{\mu_3/\epsilon_3}$ are the phase propagation constant and wave impedance, respectively, in Region 3. The constant F, like D, is a complex-valued constant that remains to be determined. We infer no wave traveling in the ($-\hat{\mathbf{z}}$) in Region 3, just as we inferred no such wave in Region 2 of the single-boundary problem.

For convenience, Table 5.1 shows a complete summary of the field components we have just identified.

Table 5.1. *Field components in the double-boundary (slab) problem of Figure 5.2.*

	Electric Field Intensity	Magnetic Field Intensity	Region of Validity
Region 1	$\widetilde{\mathbf{E}}^i(z) = \hat{\mathbf{x}} E_0^i e^{-j\beta_1 z}$	$\widetilde{\mathbf{H}}^i(z)$ $= +\hat{\mathbf{y}} \left(E_0^i/\eta_1\right) e^{-j\beta_1 z}$	$z \le -d$
	$\widetilde{\mathbf{E}}^r(z) = \hat{\mathbf{x}} B e^{+j\beta_1 z}$	$\widetilde{\mathbf{H}}^r(z)$ $= -\hat{\mathbf{y}} \left(B/\eta_1\right) e^{+j\beta_1 z}$	
Region 2	$\widetilde{\mathbf{E}}^{t2}(z) = \hat{\mathbf{x}} C e^{-j\beta_2 z}$	$\widetilde{\mathbf{H}}^{t2}(z)$ $= +\hat{\mathbf{y}} \left(C/\eta_2\right) e^{-j\beta_2 z}$	$-d \le z \le 0$
	$\widetilde{\mathbf{E}}^{r2}(z) = \hat{\mathbf{x}} D e^{+j\beta_2 z}$	$\widetilde{\mathbf{H}}^{r2}(z)$ $= -\hat{\mathbf{y}} \left(D/\eta_2\right) e^{+j\beta_2 z}$	
Region 3	$\widetilde{\mathbf{E}}^t(z) = \hat{\mathbf{x}} F e^{-j\beta_3 z}$	$\widetilde{\mathbf{H}}^t(z)$ $= +\hat{\mathbf{y}} \left(F/\eta_3\right) e^{-j\beta_3 z}$	$z \ge 0$

Note that the *total* field intensity in each region is the *sum* of the field components in that region. For example, the total electric field intensity in Region 2 is $\widetilde{\mathbf{E}}^{t2} + \widetilde{\mathbf{E}}^{r2}$.

Now observe that there are four unknown constants remaining; namely B, C, D, and F. The double-boundary problem is completely solved once we have expressions for these constants in terms of the "given" quantities in the problem statement. Solutions for the unknown constants can be obtained by enforcing boundary conditions on the electric and magnetic fields at each of the two boundaries. As in the single-boundary case, the relevant boundary condition is that the total field should be continuous across each boundary. Applying this condition to the boundary at $z = 0$, we obtain:

$$\widetilde{\mathbf{E}}^{t2}(0) + \widetilde{\mathbf{E}}^{r2}(0) = \widetilde{\mathbf{E}}^{t}(0) \tag{5.38}$$

$$\widetilde{\mathbf{H}}^{t2}(0) + \widetilde{\mathbf{H}}^{r2}(0) = \widetilde{\mathbf{H}}^{t}(0) \tag{5.39}$$

Applying this condition to the boundary at $z = -d$, we obtain:

$$\widetilde{\mathbf{E}}^{i}(-d) + \widetilde{\mathbf{E}}^{r}(-d) = \widetilde{\mathbf{E}}^{t2}(-d) + \widetilde{\mathbf{E}}^{r2}(-d) \tag{5.40}$$

$$\widetilde{\mathbf{H}}^{i}(-d) + \widetilde{\mathbf{H}}^{r}(-d) = \widetilde{\mathbf{E}}^{t2}(-d) + \widetilde{\mathbf{E}}^{r2}(-d) \tag{5.41}$$

Making substitutions from Table 5.1 and dividing out common factors, Equation 5.38 becomes:

$$C + D = F \tag{5.42}$$

Equation 5.39 becomes:

$$\frac{C}{\eta_2} - \frac{D}{\eta_2} = \frac{F}{\eta_3} \tag{5.43}$$

Equation 5.40 becomes:

$$E_0^i e^{+j\beta_1 d} + Be^{-j\beta_1 d} = Ce^{+j\beta_2 d} + De^{-j\beta_2 d} \tag{5.44}$$

Equation 5.41 becomes:

$$\frac{E_0^i}{\eta_1} e^{+j\beta_1 d} - \frac{B}{\eta_1} e^{-j\beta_1 d} = \frac{C}{\eta_2} e^{+j\beta_2 d} - \frac{D}{\eta_2} e^{-j\beta_2 d} \tag{5.45}$$

Equations 5.42–5.45 are recognizable as a system of 4 simultaneous linear equations with the number of unknowns equal to the number of equations. We could simply leave it at that, however, some very useful insights are gained by solving this system of equations in a particular manner. First, note the resemblance between the situation at the $z = 0$ boundary in this problem and the situation at $z = 0$ boundary in the single-boundary problem. In fact, the two are the *same* problem, with the following transformation of variables (single-boundary $\rightarrow$ double-boundary):

$$E_0^i \rightarrow C \quad \text{i.e., the independent variable} \tag{5.46}$$

$$B \rightarrow D \quad \text{i.e., reflection} \tag{5.47}$$

$$C \rightarrow F \quad \text{i.e., transmission} \tag{5.48}$$

It immediately follows that

$$D = \Gamma_{23} C \tag{5.49}$$

and

$$F = (1 + \Gamma_{23})\, C \tag{5.50}$$

where

$$\boxed{\Gamma_{23} \triangleq \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}} \tag{5.51}$$

At this point, we could use the expressions we have just derived to eliminate D and F in the system of four simultaneous equations identified earlier. This would reduce the problem to that of two equations in two unknowns – a dramatic simplification!

However, even this is more work than is necessary, and a little cleverness at this point pays big dividends later. The key idea is that we usually have no interest in the fields internal to the slab; in most problems, we are interested merely in reflection into Region 1 from the $z = -d$ boundary and transmission into Region 3 through the $z = 0$ interface. With this in mind, let us simply replace the two-interface problem in Figure 5.2 with an "equivalent" single-boundary problem shown in Figure 5.3. This problem is "equivalent" in the following sense only: Fields in Region 1 are identical to those in Region 1 of the original problem. The material properties in the region to the right of the $z = -d$ boundary in the equivalent problem seem unlikely to be equal to those in Regions 2 or 3 of the original problem, so we define a new wave impedance η_{eq} to represent this new condition. If we can find an expression for η_{eq}, then we can develop a solution to the original (two-boundary) problem that looks like a solution to the equivalent (simpler, single-boundary) problem.

Figure 5.3. *Representing the double-boundary problem as an equivalent single-boundary problem.*

To obtain an expression for η_{eq}, we invoke the definition of wave impedance: It is simply the ratio of electric field intensity to magnetic field intensity in the medium. Thus:

$$\eta_{eq} \triangleq \frac{\hat{\mathbf{x}} \cdot \widetilde{\mathbf{E}}_2(z_2)}{\hat{\mathbf{y}} \cdot \widetilde{\mathbf{H}}_2(z_2)} = \frac{\hat{\mathbf{x}} \cdot \left[\widetilde{\mathbf{E}}^{t2}(z_2) + \widetilde{\mathbf{E}}^{r2}(z_2)\right]}{\hat{\mathbf{y}} \cdot \left[\widetilde{\mathbf{H}}^{t2}(z_2) + \widetilde{\mathbf{H}}^{r2}(z_2)\right]} \tag{5.52}$$

where z_2 is any point in Region 2. For simplicity, let us choose $z_2 = -d$. Making substitutions, we obtain:

$$\eta_{eq} = \frac{Ce^{+j\beta_2 d} + De^{-j\beta_2 d}}{(C/\eta_2)\, e^{+j\beta_2 d} - (D/\eta_2)\, e^{-j\beta_2 d}} \tag{5.53}$$

Bringing the factor of η_2 to the front and substituting $D = \Gamma_{23} C$:

$$\eta_{eq} = \eta_2 \frac{Ce^{+j\beta_2 d} + \Gamma_{23} Ce^{-j\beta_2 d}}{Ce^{+j\beta_2 d} - \Gamma_{23} Ce^{-j\beta_2 d}} \tag{5.54}$$

Finally, we divide out the common factor of C and multiply numerator and denominator by $e^{-j\beta_2 d}$, yielding:

$$\boxed{\eta_{eq} = \eta_2 \frac{1 + \Gamma_{23} e^{-j2\beta_2 d}}{1 - \Gamma_{23} e^{-j2\beta_2 d}}} \tag{5.55}$$

Equation 5.55 is the wave impedance in the region to the right of the boundary in the equivalent scenario shown in Figure 5.3. “Equivalent” in this case means that the incident and reflected fields in Region 1 are identical to those in the original (slab) problem.

Two comments on this expression before proceeding. First: Note that if the materials in Regions 2 and 3 are identical, then $\eta_2 = \eta_3$, so $\Gamma_{23} = 0$, and thus $\eta_{eq} = \eta_2$, as expected. Second: Note that η_{eq} is, in general, complex-valued. This may initially seem troubling, since the imaginary component of the wave impedance is normally associated with general (e.g., possibly lossy) material. We specifically precluded this possibility in the problem statement, so clearly the complex value of the wave impedance is not indicating loss. Instead, the non-zero phase of η_{eq} represents the ability of the standing wave inside the slab to impart a phase shift between the electric and magnetic fields. This is *precisely* the same effect that one observes at the input of a transmission line: The input impedance Z_{in} is, in general, complex-valued even if the line is lossless and the characteristic impedance and load impedance are real-valued.[1] In fact, the impedance looking into a transmission line is given by an expression of precisely the same form as Equation 5.55. This striking analogy between plane waves at planar boundaries and voltage and current waves in transmission lines applies broadly.

We can now identify an “equivalent reflection coefficient” $\Gamma_{1,eq}$ for the scenario shown in Figure 5.3:

$$\boxed{\Gamma_{1,eq} \triangleq \frac{\eta_{eq} - \eta_1}{\eta_{eq} + \eta_1}} \tag{5.56}$$

The quantity $\Gamma_{1,eq}$ may now be used precisely in the same way as Γ_{12} was used in the single-boundary problem to find the reflected fields and reflected power density in Region 1.

Example

Example 5.2

Reflection of WiFi from a glass pane.

A WiFi (wireless LAN) signal at a center frequency of 2.45 GHz is normally incident on a glass pane which is 1 cm thick and is well-characterized in this application as a lossless dielectric with $\epsilon_r = 4$. The source of the signal is sufficiently distant from the window that the incident signal is well-approximated as a plane wave. Determine the fraction of power reflected from the pane.

Solution. In this case, we identify Regions 1 and 3 as approximately free space, and Region 2 as the pane. Thus

$$\eta_1 = \eta_3 = \eta_0 \cong 376.7\ \Omega \tag{5.57}$$

and

$$\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_r}} \cong 188.4\ \Omega \tag{5.58}$$

The reflection coefficient from Region 2 to Region 3 is

$$\begin{aligned}\Gamma_{23} &= \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \\ &= \frac{\eta_0 - \eta_2}{\eta_0 + \eta_2} \\ &\cong 0.3333\end{aligned} \tag{5.59}$$

Given $f = 2.45$ GHz, the phase propagation constant in the glass is

$$\beta_2 = \frac{2\pi}{\lambda} = \frac{2\pi f\sqrt{\epsilon_r}}{c} \cong 102.6\ \mathrm{rad/m} \tag{5.60}$$

Given $d = 1$ cm, the equivalent wave impedance is

$$\begin{aligned}\eta_{eq} &= \eta_2 \frac{1 + \Gamma_{23} e^{-j2\beta_2 d}}{1 - \Gamma_{23} e^{-j2\beta_2 d}} \\ &\cong 117.9 - j78.4\ \Omega\end{aligned} \tag{5.61}$$

Next we calculate

$$\begin{aligned}\Gamma_{1,eq} &= \frac{\eta_{eq} - \eta_1}{\eta_{eq} + \eta_1} \\ &\cong -0.4859 - j0.2354\end{aligned} \tag{5.62}$$

The ratio of reflected power density to incident power density is simply the squared magnitude of this reflection coefficient, i.e.:

$$\frac{S^r_{ave}}{S^i_{ave}} = |\Gamma_{1,eq}|^2 = 0.292 \cong \boxed{29.2\%} \tag{5.63}$$

where S^r_{ave} and S^i_{ave} are the reflected and incident power densities, respectively.

Since $B = \Gamma_{1,eq} E^i_0$, we have:

$$\widetilde{\mathbf{E}}^r(z) = \hat{\mathbf{x}} \Gamma_{1,eq} E^i_0 e^{+j\beta_1 z} \ , \ \ z \leq -d \tag{5.64}$$

and $\widetilde{\mathbf{H}}^r(z)$ can be obtained from the plane wave relationships. If desired, it is now quite simple to obtain solutions for the electric and magnetic fields in Regions 2 and 3. However, it is the usually the power density transmitted into Region 3 that is of greatest interest. This power density is easily determined from the principle of conservation of power. If the loss in Region 2 is negligible, then no power can be dissipated there. In this case, all power not reflected from the $z = -d$ interface must be transmitted into Region 3. In other words:

$$\boxed{\frac{S^t_{ave}}{S^i_{ave}} = 1 - |\Gamma_{1,eq}|^2} \tag{5.65}$$

where S^t_{ave} is the transmitted power density.

Example

Example 5.3

Transmission of WiFi through a glass pane.

Continuing Example 5.2: What fraction of incident power passes completely through the glass pane?

Solution.

$$\frac{S^t_{ave}}{S^i_{ave}} = 1 - |\Gamma_{1,eq}|^2 \cong \boxed{70.8\%} \tag{5.66}$$

Additional Reading:

- "Radome (https://en.wikipedia.org/wiki/Radome)" on Wikipedia.

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1. See the section "Input Impedance of a Terminated Lossless Transmission Line" for a reminder. This section may appear in a different volume depending on the version of this book. ↩

5.3 Total Transmission Through a Slab

Section 5.2 details the solution of the "single-slab" problem. This problem is comprised of three material regions: A semi-infinite Region 1, from which a uniform plane wave is normally incident; Region 2, the slab, defined by parallel planar boundaries separated by distance d; and a semi-infinite Region 3, through which the plane wave exits. The solution that was developed assumes simple media with negligible loss, so that the media in each region is characterized entirely by permittivity and permeability.

We now focus on a particular class of applications involving this structure. In this class of applications, we seek total transmission through the slab. By "total transmission" we mean 100% of power incident on the slab is transmitted through the slab into Region 3, and 0% of the power is reflected back into Region 1. There are many applications for such a structure. One application is the *radome*, a protective covering which partially or completely surrounds an antenna, but nominally does not interfere with waves being received by or transmitted from the antenna. Another application is RF and optical wave filtering; that is, passing or rejecting waves falling within a narrow range of frequencies. In this section, we will first describe the conditions for total transmission, and then shall provide some examples of these applications.

We begin with characterization of the media. Region 1 is characterized by its permittivity ϵ_1 and permeability μ_1, such that the wave impedance in Region 1 is $\eta_1 = \sqrt{\mu_1/\epsilon_1}$. Similarly, Region 2 is characterized by its permittivity ϵ_2 and permeability μ_2, such that the wave impedance in Region 2 is $\eta_2 = \sqrt{\mu_2/\epsilon_2}$. Region 3 is characterized by its permittivity ϵ_3 and permeability μ_3, such that the wave impedance in Region 3 is $\eta_3 = \sqrt{\mu_3/\epsilon_3}$. The analysis in this section also depends on β_2, the phase propagation constant in Region 2, which is given by $\omega\sqrt{\mu_2\epsilon_2}$.

Recall that reflection from the slab is quantified by the reflection coefficient

$$\Gamma_{1,eq} = \frac{\eta_{eq} - \eta_1}{\eta_{eq} + \eta_1} \tag{5.67}$$

where η_{eq} is given by

$$\eta_{eq} = \eta_2 \frac{1 + \Gamma_{23} e^{-j2\beta_2 d}}{1 - \Gamma_{23} e^{-j2\beta_2 d}} \tag{5.68}$$

and where Γ_{23} is given by

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \tag{5.69}$$

Total transmission requires that $\Gamma_{1,eq} = 0$. From Equation 5.67 we see that $\Gamma_{1,eq}$ is zero when $\eta_1 = \eta_{eq}$. Now employing Equation 5.68, we see that total transmission requires:

$$\eta_1 = \eta_2 \frac{1 + \Gamma_{23} e^{-j2\beta_2 d}}{1 - \Gamma_{23} e^{-j2\beta_2 d}} \tag{5.70}$$

For convenience and clarity, let us define the quantity

$$P \triangleq e^{-j2\beta_2 d} \tag{5.71}$$

Using this definition, Equation 5.70 becomes

$$\eta_1 = \eta_2 \frac{1 + \Gamma_{23} P}{1 - \Gamma_{23} P} \tag{5.72}$$

Also, let us substitute Equation 5.69 for Γ_{23}, and multiply numerator and denominator by $\eta_3 + \eta_2$ (the denominator of Equation 5.69). We obtain:

$$\eta_1 = \eta_2 \frac{\eta_3 + \eta_2 + (\eta_3 - \eta_2) P}{\eta_3 + \eta_2 - (\eta_3 - \eta_2) P} \tag{5.73}$$

Rearranging the numerator and denominator, we obtain:

$$\boxed{\eta_1 = \eta_2 \frac{(1+P)\eta_3 + (1-P)\eta_2}{(1-P)\eta_3 + (1+P)\eta_2}} \tag{5.74}$$

The parameters η_1, η_2, η_3, β_2, and d defining any single-slab structure that exhibits total transmission must satisfy Equation 5.74.

Our challenge now is to identify combinations of parameters that satisfy this condition. There are two general categories of solutions. These categories are known as *half-wave matching* and *quarter-wave matching*.

Half-wave matching applies when we have the same material on either side of the slab; i.e., $\eta_1 = \eta_3$. Let us refer to this common value of η_1 and η_3 as η_{ext}. Then the condition for total transmission becomes:

$$\eta_{ext} = \eta_2 \frac{(1+P)\eta_{ext} + (1-P)\eta_2}{(1-P)\eta_{ext} + (1+P)\eta_2} \tag{5.75}$$

For the above condition to be satisfied, we need the fraction on the right side of the equation to be equal to η_{ext}/η_2. From Equation 5.71, the magnitude of P is always 1, so the first value of P you might think to try is $P = +1$. In fact, this value satisfies Equation 5.75. Therefore, $e^{-j2\beta_2 d} = +1$. This new condition is satisfied when $2\beta_2 d = 2\pi m$, where $m = 1, 2, 3, ...$ (We do not consider $m \leq 0$ to be valid solutions since these would represent zero or negative values of d.) Thus, we find

$$d = \frac{\pi m}{\beta_2} = \frac{\lambda_2}{2} m \text{ , where } m = 1, 2, 3, ... \tag{5.76}$$

where $\lambda_2 = 2\pi/\beta_2$ is the wavelength inside the slab. Summarizing:

Total transmission through a slab embedded in regions of material having equal wave impedance (i.e., $\eta_1 = \eta_3$) may be achieved by setting the thickness of the slab equal to an integer number of half-wavelengths at the frequency of interest. This is known as *half-wave matching*.

A remarkable feature of half-wave matching is that there is *no* restriction on the permittivity or permeability of the slab, and the only constraint on the media in Regions 1 and 3 is that they have equal wave impedance.

Example

Example 5.4

Radome design by half-wave matching.

The antenna for a 60 GHz radar is to be protected from weather by a radome panel positioned directly in front of the radar. The panel is to be constructed from a low-loss material having $\mu_r \approx 1$ and $\epsilon_r = 4$. To have sufficient mechanical integrity, the panel must be at least 3 mm thick. What thickness should be used?

Solution. This is a good application for half-wave matching because the material on either side of the slab is the same (presumably free space) whereas the material used for the slab is unspecified. The phase velocity in the slab is

$$v_p = \frac{c}{\sqrt{\epsilon_r}} \cong 1.5 \times 10^8 \text{ m/s} \tag{5.77}$$

so the wavelength in the slab is

$$\lambda_2 = \frac{v_p}{f} \cong 2.5 \text{ mm} \tag{5.78}$$

Thus, the minimum thickness of the slab that satisfies the half-wave matching condition is $d = \lambda_2/2 \cong 1.25$ mm. However, this does not meet the 3 mm minimum-thickness requirement. Neither does the next available thickness, $d = \lambda_2 \cong 2.5$ mm. The next-thickest option, $d = 3\lambda_2/2 \cong 3.75$ mm, does meet the requirement. Therefore, we select $d \cong 3.75$ mm.

It should be emphasized that designs employing half-wave matching will be *narrowband* – that is, total only for the design frequency. As frequency increases or decreases from the design frequency, there will be increasing reflection and decreasing transmission.

Quarter-wave matching requires that the wave impedances in each region are different and related in a particular way. The quarter-wave solution is obtained by requiring $P = -1$, so that

$$\eta_1 = \eta_2 \frac{(1+P)\eta_3 + (1-P)\eta_2}{(1-P)\eta_3 + (1+P)\eta_2} = \eta_2 \frac{\eta_2}{\eta_3} \tag{5.79}$$

Solving this equation for wave impedance of the slab material, we find

$$\eta_2 = \sqrt{\eta_1 \eta_3} \tag{5.80}$$

Note that $P = -1$ is obtained when $2\beta_2 d = \pi + 2\pi m$ where $m = 0, 1, 2, ...$. Thus, we find

$$\begin{aligned} d &= \frac{\pi}{2\beta_2} + \frac{\pi}{\beta_2} m \\ &= \frac{\lambda_2}{4} + \frac{\lambda_2}{2} m \text{ , where } m = 0, 1, 2, ... \end{aligned} \tag{5.81}$$

Summarizing:

Total transmission is achieved through a slab by selecting $\eta_2 = \sqrt{\eta_1 \eta_3}$ and making the slab one-quarter wavelength at the frequency of interest, or some integer number of half-wavelengths thicker if needed. This is known as *quarter-wave matching*.

Example

Example 5.5

Radome design by quarter-wave matching.

The antenna for a 60 GHz radar is to be protected from weather by a radome panel positioned directly in front of the radar. In this case, however, the antenna is embedded in a lossless material having $\mu_r \approx 1$ and $\epsilon_r = 2$, and the radome panel is to be placed between this material and the outside, which we presume is free space. The material in which the antenna is embedded, and against which the radome panel is installed, is quite rigid so there is no minimum thickness requirement. However, the radome panel must be made from a material which is lossless and non-magnetic. Design the radome panel.

Solution. The radome panel must be comprised of a material having

$$\eta_2 = \sqrt{\eta_1 \eta_3} = \sqrt{\frac{\eta_0}{\sqrt{2}} \cdot \eta_0} \cong 317\ \Omega \tag{5.82}$$

Since the radome panel is required to be non-magnetic, the relative permittivity must be given by

$$\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_r}} \quad \Rightarrow \quad \epsilon_r = \left(\frac{\eta_0}{\eta_2}\right)^2 \cong 1.41 \tag{5.83}$$

The phase velocity in the slab will be

$$v_p = \frac{c}{\sqrt{\epsilon_r}} \cong \frac{3 \times 10^8\ \mathrm{m/s}}{\sqrt{1.41}} \cong 2.53 \times 10^8\ \mathrm{m/s} \tag{5.84}$$

so the wavelength in the slab is

$$\lambda_2 = \frac{v_p}{f} \cong \frac{2.53 \times 10^8\ \mathrm{m/s}}{60 \times 10^9\ \mathrm{Hz}} \cong 4.20\ \mathrm{mm} \tag{5.85}$$

Thus, the minimum possible thickness of the radome panel is $d = \lambda_2/4 \cong 1.05$ mm, and the relative permittivity of the radome panel must be $\epsilon_r \cong 1.41$.

Additional Reading:

- "Radome (https://en.wikipedia.org/wiki/Radome)" on Wikipedia.

5.4 Propagation of a Uniform Plane Wave in an Arbitrary Direction

An example of a uniform plane wave propagating in a lossless medium is shown in Figure 5.4. This wave is expressed in the indicated coordinate system as follows:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}} E_0 e^{-j\beta z} \tag{5.86}$$

%

Figure 5.4. *The plane wave described by Equation 5.86.*

This is a phasor-domain expression for the electric field intensity, so E_0 is a complex-valued number representing the magnitude and reference phase of the sinusoidally-varying wave. The term "reference phase" is defined as the phase of $\widetilde{\mathbf{E}}$ at the origin of the coordinate system. Since the phase propagation constant β is real and positive, this wave is traveling in the $+\hat{\mathbf{z}}$ direction through simple lossless media.

Note that electric field intensity vector is linearly polarized in a direction parallel to $\hat{\mathbf{x}}$. Depending on the position in space (and, for the physical time-domain waveform, time), $\widetilde{\mathbf{E}}$ points either in the $+\hat{\mathbf{x}}$ direction or the $-\hat{\mathbf{x}}$ direction. Let us be a bit more specific about the direction of the vector $\widetilde{\mathbf{E}}$. To do this, let us define the *reference polarization* to be the direction in which $\widetilde{\mathbf{E}}$ points when $\mathrm{Re}\left\{E_0 e^{-j\beta z}\right\} \geq 0$; i.e., when the phase of $\widetilde{\mathbf{E}}$ is between $-\pi/2$ and $+\pi/2$ radians. Thus, the reference polarization of $\widetilde{\mathbf{E}}$ in Equation 5.86 is always $+\hat{\mathbf{x}}$.

Note that Equation 5.86 indicates a specific combination of reference polarization and direction of propagation. However, we may obtain any other combination of reference polarization and direction of propagation by rotation of the Cartesian coordinate system. For example, if we rotate the $+x$ axis of the coordinate system into the position originally occupied by the $+y$ axis, then the very same wave is expressed as

$$\widetilde{\mathbf{E}} = -\hat{\mathbf{y}} E_0 e^{-j\beta z} \tag{5.87}$$

This is illustrated in Figure 5.5.

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Figure 5.5. *The same plane wave described in a rotated coordinate system, yielding Equation 5.87.*

At first glance, it appears that the reference polarization has changed; however, this is due entirely to our choice of coordinate system. That is, the reference polarization is precisely the same; it is only the coordinate system used to describe the reference polarization that has changed.

Let us now rotate the $+z$ axis of the coordinate system around the x axis into the position originally occupied by the $-z$ axis. Now the very same wave is expressed as

$$\widetilde{\mathbf{E}} = +\hat{\mathbf{y}} E_0 e^{+j\beta z} \tag{5.88}$$

This is illustrated in Figure 5.6.

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Figure 5.6. *The same plane wave described in yet another rotation of the coordinate system, yielding Equation 5.88.*

At first glance, it appears that the direction of propagation has reversed; but, again, it is only the coordinate system that has changed, and not the direction of propagation. Summarizing: Equations 5.86, 5.87, and 5.88 all represent the same wave. They only appear to be different due to our choice for the orientation of the coordinate system in each case.

Now consider the same thought experiment for the infinite number of cases in which the wave does not propagate in one of the three basis directions of the Cartesian coordinate system. One situation in which we face this complication is when the wave is obliquely incident on a surface. In this case, it is impossible to select a single orientation of the coordinate system in which the directions of propagation, reference polarization, and surface normal can all be described in terms of one basis vector each. To be clear: There is no fundamental limitation imposed by this slipperiness of the coordinate system. However, practical problems such as the oblique-incidence scenario described above are much easier to analyze if we are able to express waves in a system of coordinates which always yields the same expressions.

Fortunately, this is easily accomplished using *ray-fixed coordinates*. Ray-fixed coordinates are a unique set of coordinates that are determined from the characteristics of the wave, as opposed to being determined arbitrarily and separately from the characteristics of the wave. The ray-fixed representation of a uniform plane wave is:

$$\boxed{\widetilde{\mathbf{E}}(\mathbf{r}) = \hat{\mathbf{e}} E_0 e^{-j\mathbf{k}\cdot\mathbf{r}}} \tag{5.89}$$

This is illustrated in Figure 5.7.

%

Figure 5.7. *A plane wave described in ray-fixed coordinates, yielding Equation 5.89.*

In this representation, $\mathbf{r}$ is the position at which $\widetilde{\mathbf{E}}$ is evaluated, $\hat{\mathbf{e}}$ is the reference polarization expressed as a unit vector, and $\mathbf{k}$ is the unit vector $\hat{\mathbf{k}}$ in the direction of propagation, times β; i.e.:

$$\mathbf{k} \triangleq \hat{\mathbf{k}}\beta \tag{5.90}$$

Consider Equation 5.86 as an example. In this case, $\hat{\mathbf{e}} = \hat{\mathbf{x}}$, $\mathbf{k} = \hat{\mathbf{z}}\beta$, and (as always)

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z \tag{5.91}$$

Thus, $\mathbf{k}\cdot\mathbf{r} = \beta z$, as expected.

In ray-fixed coordinates, a wave can be represented by one – and only one – expression, which is the same expression regardless of the orientation of the "global" coordinate system. Moreover, only two basis directions (namely, $\hat{\mathbf{k}}$ and $\hat{\mathbf{e}}$) must be defined. Should a third coordinate be required, either $\hat{\mathbf{k}} \times \hat{\mathbf{e}}$ or $\hat{\mathbf{e}} \times \hat{\mathbf{k}}$ may be selected as the additional basis direction. Note that the first choice has the possibly-useful feature that is the reference polarization of the magnetic field intensity $\widetilde{\mathbf{H}}$.

A general procedure for recasting the ray-fixed representation into a "coordinate-bound" representation is as follows. First, we represent $\mathbf{k}$ in the new fixed coordinate system; e.g.:

$$\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}} \tag{5.92}$$

where

$$k_x \triangleq \beta\hat{\mathbf{k}} \cdot \hat{\mathbf{x}} \tag{5.93}$$
$$k_y \triangleq \beta\hat{\mathbf{k}} \cdot \hat{\mathbf{y}} \tag{5.94}$$
$$k_z \triangleq \beta\hat{\mathbf{k}} \cdot \hat{\mathbf{z}} \tag{5.95}$$

Then,

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z \tag{5.96}$$

With this expression in hand, Equation 5.89 may be rewritten as:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{e}}E_0 e^{-jk_x x} e^{-jk_y y} e^{-jk_z z} \tag{5.97}$$

If desired, one can similarly decompose $\hat{\mathbf{e}}$ into its Cartesian components as follows:

$$\hat{\mathbf{e}} = (\hat{\mathbf{e}} \cdot \hat{\mathbf{x}})\,\hat{\mathbf{x}} + (\hat{\mathbf{e}} \cdot \hat{\mathbf{y}})\,\hat{\mathbf{y}} + (\hat{\mathbf{e}} \cdot \hat{\mathbf{z}})\,\hat{\mathbf{z}} \tag{5.98}$$

Thus, we see that the ray-fixed representation of Equation 5.89 accommodates all possible combinations of direction of propagation and reference polarization.

Example

Example 5.6

Plane wave propagating away from the z-axis.

A uniform plane wave exhibiting a reference polarization of $\hat{\mathbf{z}}$ propagates away from the z-axis. Develop representations of this wave in ray-fixed and global Cartesian coordinates.

Solution. As always, the ray-fixed representation is given by Equation 5.89. Since the reference polarization is $\hat{\mathbf{z}}$, $\hat{\mathbf{e}} = \hat{\mathbf{z}}$. Propagating away from the z-axis means

$$\hat{\mathbf{k}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi \tag{5.99}$$

where ϕ indicates the specific direction of propagation. For example, $\phi = 0$ yields $\hat{\mathbf{k}} = +\hat{\mathbf{x}}$, $\phi = \pi/2$ yields $\hat{\mathbf{k}} = +\hat{\mathbf{y}}$, and so on. Therefore,

$$\mathbf{k} \triangleq \beta\hat{\mathbf{k}} = \beta\left(\hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi\right) \tag{5.100}$$

For completeness, note that the following factor appears in the phase-determining exponent in Equation 5.89:

$$\mathbf{k} \cdot \mathbf{r} = \beta\left(x \cos\phi + y \sin\phi\right) \tag{5.101}$$

In this case, we see $k_x = \beta\cos\phi$, $k_y = \beta\sin\phi$, and $k_z = 0$. Thus, the wave may be expressed in Cartesian coordinates as follows:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{z}} E_0 e^{-j\beta x\cos\phi} e^{-j\beta y\sin\phi} \tag{5.102}$$

5.5 Decomposition of a Wave into TE and TM Components

A broad range of problems in electromagnetics involve scattering of a plane wave by a planar boundary between dissimilar media. Section 5.1 (“Plane Waves at Normal Incidence on a Planar Boundary Between Lossless Media”) addressed the special case in which the wave arrives in a direction which is perpendicular to the boundary (i.e., “normal incidence”). Analysis of the normal incidence case is simplified by the fact that the directions of field vectors associated with the reflected and transmitted are the same (except possibly with a sign change) as those of the incident wave. For the more general case in which the incident wave is *obliquely* incident (i.e., not necessarily normally-incident), the directions of the field vectors will generally be different. This added complexity is easily handled if we take the effort to represent the incident wave as the sum of two waves having particular polarizations. These polarizations are referred to as *transverse electric* (TE) and *transverse magnetic* (TM). This section describes these polarizations and the method for decomposition of a plane wave into TE and TM components. We will then be prepared to address the oblique incidence case in a later section.

To begin, we define the ray-fixed coordinate system shown in Figure 5.8.

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Figure 5.8. *Coordinate system for TE-TM decomposition. Since both $\hat{\mathbf{k}}^i$ and $\hat{\mathbf{n}}$ lie in the plane of the page, this is also the plane of incidence.*

In this figure, $\hat{\mathbf{k}}^i$ is a unit vector indicating the direction in which the incident wave propagates. The unit normal $\hat{\mathbf{n}}$ is perpendicular to the boundary, and points into the region from which the wave is incident. We now make the following definition:

The plane of incidence is the plane in which *both* the normal to the surface ($\hat{\mathbf{n}}$) and the direction of propagation ($\hat{\mathbf{k}}^i$) lie.

The TE-TM decomposition consists of finding the components of the electric and magnetic fields which are perpendicular ("transverse") to the plane of incidence. Of the two possible directions that are perpendicular to the plane of incidence, we choose $\hat{\mathbf{e}}_\perp$, defined as shown in Figure 5.8. From the figure, we see that:

$$\hat{\mathbf{e}}_\perp \triangleq \frac{\hat{\mathbf{k}}^i \times \hat{\mathbf{n}}}{\left|\hat{\mathbf{k}}^i \times \hat{\mathbf{n}}\right|} \tag{5.103}$$

Defined in this manner, $\hat{\mathbf{e}}_\perp$ is a unit vector which is perpendicular to both $\hat{\mathbf{k}}^i$, and so may serve as a basis vector of a coordinate system which is attached to the incident ray. The remaining basis vector for this ray-fixed coordinate system is chosen as follows:

$$\hat{\mathbf{e}}^i_\parallel \triangleq \hat{\mathbf{e}}_\perp \times \hat{\mathbf{k}}^i \tag{5.104}$$

Defined in this manner, $\hat{\mathbf{e}}^i_\parallel$ is a unit vector which is perpendicular to both $\hat{\mathbf{e}}_\perp$ and $\hat{\mathbf{k}}^i$, and parallel to the plane of incidence.

Let us now examine an incident uniform plane wave in the new, ray-fixed coordinate system. We begin with the following phasor representation of the electric field intensity:

$$\widetilde{\mathbf{E}}^i = \hat{\mathbf{e}}^i E_0^i e^{-j\mathbf{k}^i\cdot\mathbf{r}} \tag{5.105}$$

where $\hat{\mathbf{e}}^i$ is a unit vector indicating the reference polarization, E_0^i is a complex-valued scalar, and $\mathbf{r}$ is a vector indicating the position at which $\widetilde{\mathbf{E}}^i$ is evaluated. We may express $\hat{\mathbf{e}}^i$ in the ray-fixed coordinate system of Figure 5.8 as follows:

$$\begin{aligned}\hat{\mathbf{e}}^i = &\left(\hat{\mathbf{e}}^i\cdot\hat{\mathbf{e}}_\perp\right)\hat{\mathbf{e}}_\perp \\ &+\left(\hat{\mathbf{e}}^i\cdot\hat{\mathbf{e}}^i_\parallel\right)\hat{\mathbf{e}}^i_\parallel \\ &+\left(\hat{\mathbf{e}}^i\cdot\hat{\mathbf{k}}^i\right)\hat{\mathbf{k}}^i\end{aligned} \tag{5.106}$$

The electric field vector is always perpendicular to the direction of propagation, so $\hat{\mathbf{e}}^i\cdot\hat{\mathbf{k}}^i = 0$. This leaves:

$$\hat{\mathbf{e}}^i = \left(\hat{\mathbf{e}}^i\cdot\hat{\mathbf{e}}_\perp\right)\hat{\mathbf{e}}_\perp + \left(\hat{\mathbf{e}}^i\cdot\hat{\mathbf{e}}^i_\parallel\right)\hat{\mathbf{e}}^i_\parallel \tag{5.107}$$

Substituting this expression into Equation 5.105, we obtain:

$$\widetilde{\mathbf{E}}^i = \hat{\mathbf{e}}_\perp E^i_{TE} e^{-j\mathbf{k}^i\cdot\mathbf{r}} + \hat{\mathbf{e}}^i_\parallel E^i_{TM} e^{-j\mathbf{k}^i\cdot\mathbf{r}} \tag{5.108}$$

where

$$E^i_{TE} \triangleq E^i_0 \; \hat{\mathbf{e}}^i \cdot \hat{\mathbf{e}}_\perp \tag{5.109}$$
$$E^i_{TM} \triangleq E^i_0 \; \hat{\mathbf{e}}^i \cdot \hat{\mathbf{e}}^i_\parallel \tag{5.110}$$

The first term in Equation 5.108 is the transverse electric (TE) component of $\widetilde{\mathbf{E}}^i$, so-named because it is the component which is perpendicular to the plane of incidence. The second term in Equation 5.108 is the transverse magnetic (TM) component of $\widetilde{\mathbf{E}}^i$. The term “TM” refers to the fact that the magnetic field associated with this component of the electric field is perpendicular to the plane of incidence. This is apparent since the magnetic field vector is perpendicular to both the direction of propagation and the electric field vector.

Summarizing:

The TE component is the component for which $\widetilde{\mathbf{E}}^i$ is perpendicular to the plane of incidence.

The TM component is the component for which $\widetilde{\mathbf{H}}^i$ is perpendicular to the plane of incidence; i.e., the component for which $\widetilde{\mathbf{E}}^i$ is parallel to the plane of incidence.

Finally, observe that the total wave is the sum of its TE and TM components. Therefore, we may analyze the TE and TM components separately, and know that the result for the combined wave is simply the sum of the results for the TE and TM components.

As stated at the beginning of this section, the utility of the TE-TM decomposition is that it simplifies the analysis of wave reflection. This is because the analysis of the TE and TM cases is relatively simple, whereas direct analysis of the scattering of arbitrarily-polarized waves is relatively difficult.

Note that the nomenclature “TE” and “TM” is commonly but not universally used. Sometimes “TE” is referred to as “perpendicular” polarization, indicated using the subscript “$\perp$” or “s” (short for *senkrecht*, German for “perpendicular”). Correspondingly, “TM” is sometimes referred to as “parallel” polarization, indicated using the subscript “$\parallel$” or “p.”

Also, note that the TE component and TM component are sometimes referred to as the TE *mode* and TM *mode*, respectively. While the terms “component” and “mode” are synonymous for the single plane wave scenarios considered in this section, the terms are not synonymous in general. For example, the wave inside a waveguide may consist of multiple unique TE modes which collectively comprise the TE component of the field, and similarly the wave inside a waveguide may consist of multiple unique TM modes which collectively comprise the TM component of the field.

Finally, consider what happens when a plane wave is normally-incident upon the boundary; i.e., when $\hat{\mathbf{k}}^i = -\hat{\mathbf{n}}$. In this case, Equation 5.103 indicates that $\hat{\mathrm{e}}_\perp = 0$, so the TE-TM decomposition is undefined. The situation is simply that both $\mathbf{E}^i$ and $\mathbf{H}^i$ are already both perpendicular to the boundary, and there is no single plane that can be uniquely identified as the plane of incidence. We refer to this case as *transverse electromagnetic* (TEM).

A wave which is normally-incident on a planar surface is said to be transverse electromagnetic (TEM) with respect to that boundary. There is no unique TE-TM decomposition in this case.

The fact that a unique TE-TM decomposition does not exist in the TEM case is of no consequence, because the TEM case is easily handled as a separate condition (see Section 5.1).

5.6 Plane Waves at Oblique Incidence on a Planar Boundary: TE Case

In this section, we consider the problem of reflection and transmission from a planar boundary between semi-infinite media for a transverse electric (TE) uniform plane wave. Before attempting this section, a review of Sections 5.1 ("Plane Waves at Normal Incidence on a Planar Boundary Between Lossless Media") and 5.5 ("Decomposition of a Wave into TE and TM Components") is recommended. Also, note that this section has much in common with Section 5.7 ("Plane Waves at Oblique Incidence on a Planar Boundary: TM Case"), although it is recommended to attempt the TE case first.

The TE case is illustrated in Figure 5.9.

Figure 5.9. *A TE uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions.*

The boundary between the two semi-infinite and lossless regions is located at the $z = 0$ plane. The wave is incident from Region 1. The electric field intensity $\widetilde{\mathbf{E}}^i_{TE}$ of this wave is given by

$$\widetilde{\mathbf{E}}^i_{TE}(\mathbf{r}) = \hat{\mathbf{y}} E^i_{TE} e^{-j\mathbf{k}^i \cdot \mathbf{r}} \tag{5.111}$$

In this expression, $\mathbf{r}$ is the position at which $\widetilde{\mathbf{E}}^i_{TE}$ is evaluated, and

$$\mathbf{k}^i = \hat{\mathbf{k}}^i \beta_1 \tag{5.112}$$

where $\hat{\mathbf{k}}^i$ is the unit vector indicating the direction of propagation and $\beta_1 = \omega\sqrt{\mu_1 \epsilon_1}$ is the phase propagation constant in Region 1. $\widetilde{\mathbf{E}}^i_{TE}$ serves as the "stimulus" in this problem, so all other contributions to the total field will be expressed in terms of quantities appearing in Equation 5.111.

The presence of reflected and transmitted uniform plane waves is inferred from our experience with the normal incidence scenario (Section 5.1). There, as here, the symmetry of the problem indicates that the reflected and transmitted components of the electric field will have the same polarization as that of the incident electric field. This is because there is nothing present in the problem that could account for a change in polarization. Thus, the reflected and transmitted fields will also be TE. Therefore, we postulate the following expression for the reflected wave:

$$\widetilde{\mathbf{E}}^r(\mathbf{r}) = \hat{\mathbf{y}} B e^{-j\mathbf{k}^r \cdot \mathbf{r}} \tag{5.113}$$

where B is an unknown, possibly complex-valued constant to be determined and

$$\mathbf{k}^r = \hat{\mathbf{k}}^r \beta_1 \tag{5.114}$$

indicates the direction of propagation, which is also currently unknown.

Similarly, we postulate the following expression for the transmitted wave:

$$\widetilde{\mathbf{E}}^t(\mathbf{r}) = \hat{\mathbf{y}} C e^{-j\mathbf{k}^t \cdot \mathbf{r}} \tag{5.115}$$

where C is an unknown, possibly complex-valued constant to be determined;

$$\mathbf{k}^t = \hat{\mathbf{k}}^t \beta_2 \tag{5.116}$$

where $\hat{\mathbf{k}}^t$ is the unit vector indicating the direction of propagation; and $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$ is the phase propagation constant in Region 2.

At this point, the unknowns in this problem are the constants B and C, as well as the directions $\hat{\mathbf{k}}^r$ and $\hat{\mathbf{k}}^t$. We may establish a relationship between E^i_{TE}, B, and C by application of boundary conditions at $z = 0$. First, recall that the tangential component of the total electric field intensity must be continuous across material boundaries. To apply this boundary condition, let us define $\widetilde{\mathbf{E}}_1$ and $\widetilde{\mathbf{E}}_2$ to be the total electric fields in Regions 1 and 2, respectively. The total field in Region 1 is the sum of incident and reflected fields, so

$$\widetilde{\mathbf{E}}_1(\mathbf{r}) = \widetilde{\mathbf{E}}^i_{TE}(\mathbf{r}) + \widetilde{\mathbf{E}}^r(\mathbf{r}) \tag{5.117}$$

The total field in Region 2 is simply

$$\widetilde{\mathbf{E}}_2(\mathbf{r}) = \widetilde{\mathbf{E}}^t(\mathbf{r}) \tag{5.118}$$

Next, note that all electric field components are already tangent to the boundary. Thus, continuity of the tangential component of the electric field across the boundary requires $\widetilde{\mathbf{E}}_1(0) = \widetilde{\mathbf{E}}_2(0)$, and therefore

$$\widetilde{\mathbf{E}}^i_{TE}(\mathbf{r}_0) + \widetilde{\mathbf{E}}^r(\mathbf{r}_0) = \widetilde{\mathbf{E}}^t(\mathbf{r}_0) \tag{5.119}$$

where $\mathbf{r} = \mathbf{r}_0 \triangleq \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ since $z = 0$ on the boundary. Now employing Equations 5.111, 5.113, and 5.115, we obtain:

$$\hat{\mathbf{y}}E^i_{TE}e^{-j\mathbf{k}^i\cdot\mathbf{r}_0} + \hat{\mathbf{y}}Be^{-j\mathbf{k}^r\cdot\mathbf{r}_0} = \hat{\mathbf{y}}Ce^{-j\mathbf{k}^t\cdot\mathbf{r}_0} \tag{5.120}$$

Dropping the vector ($\hat{\mathbf{y}}$) since it is the same in each term, we obtain:

$$E^i_{TE}e^{-j\mathbf{k}^i\cdot\mathbf{r}_0} + Be^{-j\mathbf{k}^r\cdot\mathbf{r}_0} = Ce^{-j\mathbf{k}^t\cdot\mathbf{r}_0} \tag{5.121}$$

For Equation 5.121 to be true at every point $\mathbf{r}_0$ on the boundary, it must be true that

$$\mathbf{k}^i \cdot \mathbf{r}_0 = \mathbf{k}^r \cdot \mathbf{r}_0 = \mathbf{k}^t \cdot \mathbf{r}_0 \tag{5.122}$$

Essentially, we are requiring the phases of each field in Regions 1 and 2 to be matched at every point along the boundary. Any other choice will result in a violation of boundary conditions at some point along the boundary. This phase matching criterion will determine the directions of propagation of the reflected and transmitted fields, which we shall do later.

First, let us use the phase matching criterion to complete the solution for the coefficients B and C. Enforcing Equation 5.122, we observe that Equation 5.121 reduces to:

$$E^i_{TE} + B = C \tag{5.123}$$

A second equation is needed since we currently have only one equation (Equation 5.123) and two unknowns (B and C). The second equation is obtained by applying the appropriate boundary conditions to the magnetic field. The magnetic field associated with each of the electric field components is identified in Figure 5.10.

Figure 5.10. *Incident, reflected, and transmitted magnetic field components associated with the TE electric field components shown in Figure 5.9.*

Note the orientations of the magnetic field vectors may be confirmed using the *plane wave relationships*: Specifically, the cross product of the electric and magnetic fields should point in the direction of propagation. Expressions for each of the magnetic field components is determined formally below.

From the plane wave relationships, we determine that the incident magnetic field intensity is

$$\widetilde{\mathbf{H}}^i(\mathbf{r}) = \frac{1}{\eta_1}\hat{\mathbf{k}}^i \times \widetilde{\mathbf{E}}^i_{TE} \tag{5.124}$$

where $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ is the wave impedance in Region 1. To make progress requires that we express $\hat{\mathbf{k}}^i$ in the global fixed coordinate system. Here it is:

$$\hat{\mathbf{k}}^i = \hat{\mathbf{x}}\sin\psi^i + \hat{\mathbf{z}}\cos\psi^i \tag{5.125}$$

Thus:

$$\widetilde{\mathbf{H}}^i(\mathbf{r}) = \left(\hat{\mathbf{z}}\sin\psi^i - \hat{\mathbf{x}}\cos\psi^i\right)\frac{E^i_{TE}}{\eta_1}e^{-j\mathbf{k}^i\cdot\mathbf{r}} \tag{5.126}$$

Similarly, we determine that the reflected magnetic field has the form:

$$\widetilde{\mathbf{H}}^r(\mathbf{r}) = \frac{1}{\eta_1}\hat{\mathbf{k}}^r \times \widetilde{\mathbf{E}}^r \tag{5.127}$$

In the global coordinate system:

$$\hat{\mathbf{k}}^r = \hat{\mathbf{x}}\sin\psi^r - \hat{\mathbf{z}}\cos\psi^r \tag{5.128}$$

Thus:

$$\widetilde{\mathbf{H}}^r(\mathbf{r}) = \left(\hat{\mathbf{z}}\sin\psi^r + \hat{\mathbf{x}}\cos\psi^r\right)\frac{B}{\eta_1}e^{-j\mathbf{k}^r\cdot\mathbf{r}} \tag{5.129}$$

The transmitted magnetic field has the form:

$$\widetilde{\mathbf{H}}^t(\mathbf{r}) = \frac{1}{\eta_2}\hat{\mathbf{k}}^t \times \widetilde{\mathbf{E}}^t \tag{5.130}$$

In the global coordinate system:

$$\hat{\mathbf{k}}^t = \hat{\mathbf{x}} \sin\psi^t + \hat{\mathbf{z}} \cos\psi^t \tag{5.131}$$

Thus:

$$\widetilde{\mathbf{H}}^t(\mathbf{r}) = \left(\hat{\mathbf{z}} \sin\psi^t - \hat{\mathbf{x}} \cos\psi^t\right) \frac{C}{\eta_2} e^{-j\mathbf{k}^t\cdot\mathbf{r}} \tag{5.132}$$

The total magnetic field in Region 1 is the sum of incident and reflected fields, so

$$\widetilde{\mathbf{H}}_1(\mathbf{r}) = \widetilde{\mathbf{H}}^i(\mathbf{r}) + \widetilde{\mathbf{H}}^r(\mathbf{r}) \tag{5.133}$$

The magnetic field in Region 2 is simply

$$\widetilde{\mathbf{H}}_2(\mathbf{r}) = \widetilde{\mathbf{H}}^t(\mathbf{r}) \tag{5.134}$$

Since there is no current on the boundary, the tangential component of the total magnetic field intensity must be continuous across the boundary. Expressed in terms of the quantities already established, this boundary condition requires:

$$\hat{\mathbf{x}} \cdot \widetilde{\mathbf{H}}^i(\mathbf{r}_0) + \hat{\mathbf{x}} \cdot \widetilde{\mathbf{H}}^r(\mathbf{r}_0) = \hat{\mathbf{x}} \cdot \widetilde{\mathbf{H}}^t(\mathbf{r}_0) \tag{5.135}$$

where "$\hat{\mathbf{x}}\cdot$" selects the component of the magnetic field that is tangent to the boundary. Evaluating this expression, we obtain:

$$\begin{aligned} &-\left(\cos\psi^i\right) \frac{E^i_{TE}}{\eta_1} e^{-j\mathbf{k}^i\cdot\mathbf{r}_0} \\ &+\left(\cos\psi^r\right) \frac{B}{\eta_1} e^{-j\mathbf{k}^r\cdot\mathbf{r}_0} \\ =&-\left(\cos\psi^t\right) \frac{C}{\eta_2} e^{-j\mathbf{k}^t\cdot\mathbf{r}_0} \end{aligned} \tag{5.136}$$

Now employing the phase matching condition expressed in Equation 5.122, we find:

$$\begin{aligned} &-\left(\cos\psi^i\right) \frac{E^i_{TE}}{\eta_1} \\ &+\left(\cos\psi^r\right) \frac{B}{\eta_1} \\ =&-\left(\cos\psi^t\right) \frac{C}{\eta_2} \end{aligned} \tag{5.137}$$

Equations 5.123 and 5.137 comprise a linear system of equations with unknowns B and C. This system of equations is easily solved for B as follows. First, use Equation 5.123 to eliminate C in Equation 5.137. The result is:

$$\begin{aligned}&-\left(\cos\psi^i\right)\frac{E^i_{TE}}{\eta_1}\\&+\left(\cos\psi^r\right)\frac{B}{\eta_1}\\=&-\left(\cos\psi^t\right)\frac{E^i_{TE}+B}{\eta_2}\end{aligned} \tag{5.138}$$

Solving this equation for B, we obtain:

$$B=\frac{\eta_2\cos\psi^i-\eta_1\cos\psi^t}{\eta_2\cos\psi^r+\eta_1\cos\psi^t}\ E^i_{TE} \tag{5.139}$$

We can express this result as a reflection coefficient as follows:

$$B=\Gamma_{TE}E^i_{TE} \tag{5.140}$$

where

$$\Gamma_{TE}\triangleq\frac{\eta_2\cos\psi^i-\eta_1\cos\psi^t}{\eta_2\cos\psi^r+\eta_1\cos\psi^t} \tag{5.141}$$

It is worth noting that Equation 5.141 becomes the reflection coefficient for normal (TEM) incidence when $\psi^i=\psi^r=\psi^t=0$, as expected.

Returning to Equation 5.123, we now find

$$C=\left(1+\Gamma_{TE}\right)E^i_{TE} \tag{5.142}$$

Let us now summarize the solution. Given the TE electric field intensity expressed in Equation 5.111, we find:

$$\boxed{\widetilde{\mathbf{E}}^r(\mathbf{r})=\hat{\mathbf{y}}\Gamma_{TE}E^i_{TE}e^{-j\mathbf{k}^r\cdot\mathbf{r}}} \tag{5.143}$$

$$\boxed{\widetilde{\mathbf{E}}^t(\mathbf{r})=\hat{\mathbf{y}}\left(1+\Gamma_{TE}\right)E^i_{TE}e^{-j\mathbf{k}^t\cdot\mathbf{r}}} \tag{5.144}$$

This solution is complete except that we have not yet determined $\hat{\mathbf{k}}^r$, which is now completely determined by ψ^r via Equation 5.128, and $\hat{\mathbf{k}}^t$, which is now completely determined by ψ^t via Equation 5.131. In other words, we have not yet determined the directions of propagation ψ^r for the reflected wave and ψ^t for the transmitted wave. However, ψ^r and ψ^i can be found using Equation 5.122. Here we shall simply state the result, and in Section 5.8 we shall perform this part of the derivation in detail and with greater attention to the implications. One finds:

$$\psi^r=\psi^i \tag{5.145}$$

and

$$\psi^t=\arcsin\left(\frac{\beta_1}{\beta_2}\sin\psi^i\right) \tag{5.146}$$

Equation 5.145 is the unsurprising result that angle of reflection equals angle of incidence. Equation 5.146 – addressing angle of transmission – is a bit more intriguing. Astute readers may notice that there is something fishy about this equation: It seems possible for the argument of $\arcsin$ to be greater than one. This oddity is addressed in Section 5.8.

Finally, note that Equation 5.145 allows us to eliminate ψ^r from Equation 5.141, yielding:

$$\boxed{\Gamma_{TE} = \frac{\eta_2 \cos\psi^i - \eta_1 \cos\psi^t}{\eta_2 \cos\psi^i + \eta_1 \cos\psi^t}} \tag{5.147}$$

Thus, we obtain what is perhaps the most important finding of this section:

> The electric field reflection coefficient for oblique TE incidence, Γ_{TE}, is given by Equation 5.147.

The following example demonstrates the utility of this result.

Example

Example 5.7

Power transmission at an air-to-glass interface (TE case).

Figure 5.11 illustrates a TE plane wave incident from air onto the planar boundary with glass. The glass exhibits relative permittivity of 2.1. Determine the power reflected and transmitted relative to power incident on the boundary.

Solution. The power reflected relative to power incident is $|\Gamma_{TE}|^2$ whereas the power transmitted relative to power incident is $1 - |\Gamma_{TE}|^2$. Γ_{TE} may be calculated using Equation 5.147. Calculating the quantities that enter into this expression:

$$\eta_1 \approx \eta_0 \cong 376.7\ \Omega \quad \text{(air)} \tag{5.148}$$

$$\eta_2 \approx \frac{\eta_0}{\sqrt{2.1}} \cong 260.0\ \Omega \quad \text{(glass)} \tag{5.149}$$

$$\psi^i = 30^\circ \tag{5.150}$$

Note

$$\frac{\beta_1}{\beta_2} \approx \frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\sqrt{\mu_0 \cdot 2.1\epsilon_0}} \cong 0.690 \tag{5.151}$$

so

$$\psi^t = \arcsin\left(\frac{\beta_1}{\beta_2}\sin\psi^i\right) \cong 20.2^\circ \tag{5.152}$$

Now substituting these values into Equation 5.147, we obtain

$$\Gamma_{TE} \cong -0.2220 \tag{5.153}$$

Subsequently, the fraction of power reflected relative to power incident is $|\Gamma_{TE}|^2 \cong 0.049$; i.e., about 4.9%. $1 - |\Gamma_{TE}|^2 \cong$ 95.1% of the power is transmitted into the glass.

Figure 5.11. *A TE uniform plane wave incident from air to glass.*

5.7 Plane Waves at Oblique Incidence on a Planar Boundary: TM Case

In this section, we consider the problem of reflection and transmission from a planar boundary between semi-infinite media for a transverse magnetic (TM) uniform plane wave. Before attempting this section, a review of Sections 5.1 ("Plane Waves at Normal Incidence on a Planar Boundary Between Lossless Media") and 5.5 ("Decomposition of a Wave into TE and TM Components") is recommended. Also, note that this section has much in common with Section 5.6 ("Plane Waves at Oblique Incidence on a Planar Boundary: TE Case"), and it is recommended to attempt the TE case first.

In this section, we consider the scenario illustrated in Figure 5.12.

Figure 5.12. *A TM uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions.*

The boundary between the two semi-infinite and lossless regions is located at the $z = 0$ plane. The wave is incident from Region 1. The magnetic field intensity $\widetilde{\mathbf{H}}^i_{TM}$ of this wave is given by

$$\widetilde{\mathbf{H}}^i_{TM}(\mathbf{r}) = \hat{\mathbf{y}} H^i_{TM} e^{-j\mathbf{k}^i\cdot\mathbf{r}} \tag{5.154}$$

In this expression, $\mathbf{r}$ is the position at which $\widetilde{\mathbf{H}}^i_{TM}$ is evaluated. Also,

$$\mathbf{k}^i = \hat{\mathbf{k}}^i \beta_1 \tag{5.155}$$

where $\hat{\mathbf{k}}^i$ is the unit vector indicating the direction of propagation and $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$ is the phase propagation constant in Region 1. $\widetilde{\mathbf{H}}^i_{TM}$ serves as the "stimulus" in this problem, and all other contributions to the total field may be expressed in terms of parameters associated with Equation 5.154.

The presence of reflected and transmitted uniform plane waves is inferred from our experience with the normal incidence scenario (Section 5.1). There, as here, the symmetry of the problem indicates that the reflected and transmitted components of the magnetic field will have the same polarization as that of the incident electric field. This is because there is nothing present in the problem that could account for a change in polarization. Thus, the reflected and transmitted fields will also be TM. So we postulate the following expression for the reflected wave:

$$\widetilde{\mathbf{H}}^r(\mathbf{r}) = -\hat{\mathbf{y}} B e^{-j\mathbf{k}^r\cdot\mathbf{r}} \tag{5.156}$$

where B is an unknown, possibly complex-valued constant to be determined and

$$\mathbf{k}^r = \hat{\mathbf{k}}^r \beta_1 \tag{5.157}$$

indicates the direction of propagation.

The reader may wonder why we have chosen $-\hat{\mathbf{y}}$, as opposed to $+\hat{\mathbf{y}}$, as the reference polarization for $\widetilde{\mathbf{H}}^r$. In fact, either $+\hat{\mathbf{y}}$ or $-\hat{\mathbf{y}}$ could be used. However, the choice is important because the form of the results we obtain in this section – specifically, the reflection coefficient – will be determined with respect to this specific convention, and will be incorrect with respect to the opposite convention. We choose $-\hat{\mathbf{y}}$ because it has a particular advantage which we shall point out at the end of this section.

Continuing, we postulate the following expression for the transmitted wave:

$$\widetilde{\mathbf{H}}^t(\mathbf{r}) = \hat{\mathbf{y}} C e^{-j\mathbf{k}^t\cdot\mathbf{r}} \tag{5.158}$$

where C is an unknown, possibly complex-valued constant to be determined and

$$\mathbf{k}^t = \hat{\mathbf{k}}^t \beta_2 \tag{5.159}$$

where $\hat{\mathbf{k}}^t$ is the unit vector indicating the direction of propagation and $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$ is the phase propagation constant in Region 2.

At this point, the unknowns in this problem are the constants B and C, as well as the unknown directions $\hat{\mathbf{k}}^r$ and $\hat{\mathbf{k}}^t$. We may establish a relationship between H^i_{TM}, B, and C by application of boundary conditions at $z = 0$. First, we presume no impressed current at the boundary. Thus, the tangential component of the total magnetic field intensity must be continuous across the boundary. To apply this boundary condition, let us define $\widetilde{\mathbf{H}}_1$ and $\widetilde{\mathbf{H}}_2$ to be the total magnetic fields in Regions 1 and 2, respectively. The total field in Region 1 is the sum of incident and reflected fields, so

$$\widetilde{\mathbf{H}}_1(\mathbf{r}) = \widetilde{\mathbf{H}}^i_{TM}(\mathbf{r}) + \widetilde{\mathbf{H}}^r(\mathbf{r}) \tag{5.160}$$

The field in Region 2 is simply

$$\widetilde{\mathbf{H}}_2(\mathbf{r}) = \widetilde{\mathbf{H}}^t(\mathbf{r}) \tag{5.161}$$

Also, we note that all magnetic field components are already tangent to the boundary. Thus, continuity of the tangential component of the magnetic field across the boundary requires $\widetilde{\mathbf{H}}_1(\mathbf{r}_0) = \widetilde{\mathbf{H}}_2(\mathbf{r}_0)$, where $\mathbf{r}_0 \triangleq \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ since $z = 0$ on the boundary. Therefore,

$$\widetilde{\mathbf{H}}^i_{TM}(\mathbf{r}_0) + \widetilde{\mathbf{H}}^r(\mathbf{r}_0) = \widetilde{\mathbf{H}}^t(\mathbf{r}_0) \tag{5.162}$$

Now employing Equations 5.154, 5.156, and 5.158, we obtain:

$$\hat{\mathbf{y}} H^i_{TM} e^{-j\mathbf{k}^i \cdot \mathbf{r}_0} - \hat{\mathbf{y}} B e^{-j\mathbf{k}^r \cdot \mathbf{r}_0} = \hat{\mathbf{y}} C e^{-j\mathbf{k}^t \cdot \mathbf{r}_0} \tag{5.163}$$

Dropping the vector ($\hat{\mathbf{y}}$) since it is the same in each term, we obtain:

$$H^i_{TM} e^{-j\mathbf{k}^i \cdot \mathbf{r}_0} - B e^{-j\mathbf{k}^r \cdot \mathbf{r}_0} = C e^{-j\mathbf{k}^t \cdot \mathbf{r}_0} \tag{5.164}$$

For this to be true at every point $\mathbf{r}_0$ on the boundary, it must be true that

$$\mathbf{k}^i \cdot \mathbf{r}_0 = \mathbf{k}^r \cdot \mathbf{r}_0 = \mathbf{k}^t \cdot \mathbf{r}_0 \tag{5.165}$$

Essentially, we are requiring the phases of each field in Regions 1 and 2 to be matched at every point along the boundary. Any other choice will result in a violation of boundary conditions at some point along the boundary. This expression allows us to solve for the directions of propagation of the reflected and transmitted fields, which we shall do later. Our priority for now shall be to solve for the coefficients B and C.

Enforcing Equation 5.165, we observe that Equation 5.164 reduces to:

$$H^i_{TM} - B = C \tag{5.166}$$

A second equation is needed since we currently have only one equation (Equation 5.166) and two unknowns (B and C). The second equation is obtained by applying the appropriate boundary conditions to the electric field. The electric field associated with each of the magnetic field components is identified in Figure 5.12. Note the orientations of the electric field vectors may be confirmed using the *plane wave relationships*: Specifically, the cross product of the electric and magnetic fields should point in the direction of propagation. Expressions for each of the electric field components is determined formally below.

From the plane wave relationships, we determine that the incident electric field intensity is

$$\widetilde{\mathbf{E}}^i(\mathbf{r}) = -\eta_1 \hat{\mathbf{k}}^i \times \widetilde{\mathbf{H}}^i_{TM} \tag{5.167}$$

where $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ is the wave impedance in Region 1. To make progress requires that we express $\hat{\mathbf{k}}^i$ in the global fixed coordinate system. Here it is:

$$\hat{\mathbf{k}}^i = \hat{\mathbf{x}} \sin\psi^i + \hat{\mathbf{z}} \cos\psi^i \tag{5.168}$$

Thus:

$$\widetilde{\mathbf{E}}^i(\mathbf{r}) = \left(\hat{\mathbf{x}} \cos\psi^i - \hat{\mathbf{z}} \sin\psi^i\right) \eta_1 H^i_{TM} e^{-j\mathbf{k}^i\cdot\mathbf{r}} \tag{5.169}$$

Similarly we determine that the reflected electric field has the form:

$$\widetilde{\mathbf{E}}^r(\mathbf{r}) = -\eta_1 \hat{\mathbf{k}}^r \times \widetilde{\mathbf{H}}^r \tag{5.170}$$

In the global coordinate system:

$$\hat{\mathbf{k}}^r = \hat{\mathbf{x}} \sin\psi^r - \hat{\mathbf{z}} \cos\psi^r \tag{5.171}$$

Thus:

$$\widetilde{\mathbf{E}}^r(\mathbf{r}) = \left(\hat{\mathbf{x}} \cos\psi^r + \hat{\mathbf{z}} \sin\psi^r\right) \eta_1 B e^{-j\mathbf{k}^r\cdot\mathbf{r}} \tag{5.172}$$

The transmitted magnetic field has the form:

$$\widetilde{\mathbf{E}}^t(\mathbf{r}) = -\eta_2 \hat{\mathbf{k}}^t \times \widetilde{\mathbf{H}}^t \tag{5.173}$$

In the global coordinate system:

$$\hat{\mathbf{k}}^t = \hat{\mathbf{x}} \sin\psi^t + \hat{\mathbf{z}} \cos\psi^t \tag{5.174}$$

Thus:

$$\widetilde{\mathbf{E}}^t(\mathbf{r}) = \left(\hat{\mathbf{x}} \cos\psi^t - \hat{\mathbf{z}} \sin\psi^t\right) \eta_2 C e^{-j\mathbf{k}^t\cdot\mathbf{r}} \tag{5.175}$$

The total electric field in Region 1 is the sum of incident and reflected fields, so

$$\widetilde{\mathbf{E}}_1(\mathbf{r}) = \widetilde{\mathbf{E}}^i(\mathbf{r}) + \widetilde{\mathbf{E}}^r(\mathbf{r}) \tag{5.176}$$

The electric field in Region 2 is simply

$$\widetilde{\mathbf{E}}_2(\mathbf{r}) = \widetilde{\mathbf{E}}^t(\mathbf{r}) \tag{5.177}$$

The tangential component of the total electric field intensity must be continuous across the boundary. Expressed in terms of the quantities already established, this boundary condition requires:

$$\hat{\mathbf{x}} \cdot \widetilde{\mathbf{E}}^i(\mathbf{r}_0) + \hat{\mathbf{x}} \cdot \widetilde{\mathbf{E}}^r(\mathbf{r}_0) = \hat{\mathbf{x}} \cdot \widetilde{\mathbf{E}}^t(\mathbf{r}_0) \tag{5.178}$$

where "$\hat{\mathbf{x}}\cdot$" selects the component of the electric field that is tangent to the boundary. Evaluating this expression, we obtain:

$$\begin{aligned} &+ \left(\cos\psi^i\right)\eta_1 H^i_{TM} e^{-j\mathbf{k}^i\cdot\mathbf{r}_0} \\ &+ \left(\cos\psi^r\right)\eta_1 B e^{-j\mathbf{k}^r\cdot\mathbf{r}_0} \\ =&+ \left(\cos\psi^t\right)\eta_2 C e^{-j\mathbf{k}^t\cdot\mathbf{r}_0} \end{aligned} \tag{5.179}$$

Now employing the "phase matching" condition expressed in Equation 5.165, we find:

$$\begin{aligned} &+ \left(\cos\psi^i\right)\eta_1 H^i_{TM} \\ &+ \left(\cos\psi^r\right)\eta_1 B \\ =&+ \left(\cos\psi^t\right)\eta_2 C \end{aligned} \tag{5.180}$$

Equations 5.166 and 5.180 comprise a linear system of equations with unknowns B and C. This system of equations is easily solved for B as follows. First, use Equation 5.166 to eliminate C in Equation 5.180. The result is:

$$\begin{aligned} &+ \left(\cos\psi^i\right)\eta_1 H^i_{TM} \\ &+ \left(\cos\psi^r\right)\eta_1 B \\ =&+ \left(\cos\psi^t\right)\eta_2 \left(H^i_{TM} - B\right) \end{aligned} \tag{5.181}$$

Solving this equation for B, we obtain:

$$B = \frac{-\eta_1\cos\psi^i + \eta_2\cos\psi^t}{+\eta_1\cos\psi^r + \eta_2\cos\psi^t}\ H^i_{TM} \tag{5.182}$$

We can express this result as follows:

$$B = \Gamma_{TM} H^i_{TM} \tag{5.183}$$

where we have made the definition

$$\Gamma_{TM} \triangleq \frac{-\eta_1\cos\psi^i + \eta_2\cos\psi^t}{+\eta_1\cos\psi^r + \eta_2\cos\psi^t} \tag{5.184}$$

We are now able to express the complete solution in terms of the electric field intensity. First we make the substitution $E^i_{TM} \triangleq \eta_1 H^i_{TM}$ in Equation 5.169, yielding:

$$\widetilde{\mathbf{E}}^i(\mathbf{r}) = \left(\hat{\mathbf{x}}\cos\psi^i - \hat{\mathbf{z}}\sin\psi^i\right) E^i_{TM} e^{-j\mathbf{k}^i\cdot\mathbf{r}} \tag{5.185}$$

The factor $\eta_1 B$ in Equation 5.172 becomes $\Gamma_{TM} E^i_{TM}$, so we obtain:

$$\begin{aligned}\widetilde{\mathbf{E}}^r(\mathbf{r}) =& \left(\hat{\mathbf{x}}\cos\psi^r + \hat{\mathbf{z}}\sin\psi^r\right) \\ & \cdot \Gamma_{TM} E^i_{TM} e^{-j\mathbf{k}^r\cdot\mathbf{r}}\end{aligned} \tag{5.186}$$

Thus, we see Γ_{TM} is the reflection coefficient for the electric field intensity.

Returning to Equation 5.166, we now find

$$\begin{aligned}C &= H^i_{TM} - B \\ &= H^i_{TM} - \Gamma_{TM} H^i_{TM} \\ &= \left(1 - \Gamma_{TM}\right) H^i_{TM} \\ &= \left(1 - \Gamma_{TM}\right) E^i_{TM}/\eta_1\end{aligned} \tag{5.187}$$

Subsequently, Equation 5.175 becomes

$$\begin{aligned}\widetilde{\mathbf{E}}^t(\mathbf{r}) =& \left(\hat{\mathbf{x}}\cos\psi^t - \hat{\mathbf{z}}\sin\psi^t\right) \\ & \cdot \left(1 - \Gamma_{TM}\right) \frac{\eta_2}{\eta_1} E^i_{TM} e^{-j\mathbf{k}^t\cdot\mathbf{r}}\end{aligned} \tag{5.188}$$

This solution is complete except that we have not yet determined $\hat{\mathbf{k}}^r$, which is now completely determined by ψ^r via Equation 5.171, and $\hat{\mathbf{k}}^t$, which is now completely determined by ψ^t via Equation 5.174. In other words, we have not yet determined the directions of propagation ψ^r for the reflected wave and ψ^t for the transmitted wave. However, ψ^r and ψ^i can be found using Equation 5.165. Here we shall simply state the result, and in Section 5.8 we shall perform this part of the derivation in detail and with greater attention to the implications. One finds:

$$\psi^r = \psi^i \tag{5.189}$$

i.e., angle of reflection equals angle of incidence. Also,

$$\psi^t = \arcsin\left(\frac{\beta_1}{\beta_2}\sin\psi^i\right) \tag{5.190}$$

Astute readers may notice that there is something fishy about Equation 5.190. Namely, it seems possible for the argument of $\arcsin$ to be greater than one. This oddity is addressed in Section 5.8.

Now let us return to the following question, raised near the beginning of this section: Why choose $-\hat{\mathbf{y}}$, as opposed to $+\hat{\mathbf{y}}$, as the reference polarization for $\mathbf{H}^r$, as shown in Figure 5.12? To answer this question, first note that Γ_{TM} (Equation 5.184) becomes the reflection coefficient for normal (TEM) incidence when $\psi^i = \psi^t = 0$. If we had chosen $+\hat{\mathbf{y}}$ as the reference polarization for $\mathbf{H}^r$, we would have instead obtained an expression for Γ_{TM} that has the opposite sign for TEM incidence.[1] There is nothing wrong with this answer, but it is awkward to have different values of the reflection coefficient for the same physical scenario. By choosing $-\hat{\mathbf{y}}$, the reflection coefficient for the oblique incidence case computed for $\psi^i = 0$ converges to the reflection coefficient previously computed for the normal-incidence case. It is important to be aware of this issue, as one occasionally encounters work in which the opposite ("$+\hat{\mathbf{y}}$") reference polarization has been employed.

Finally, note that Equation 5.189 allows us to eliminate ψ^r from Equation 5.184, yielding:

$$\boxed{\Gamma_{TM} = \frac{-\eta_1 \cos\psi^i + \eta_2 \cos\psi^t}{+\eta_1 \cos\psi^i + \eta_2 \cos\psi^t}} \tag{5.191}$$

Thus, we obtain what is perhaps the most important finding of this section:

> The electric field reflection coefficient for oblique TM incidence, Γ_{TM}, is given by Equation 5.191.

The following example demonstrates the utility of this result.

Example

Example 5.8

Power transmission at an air-to-glass interface (TM case).

Figure 5.13 illustrates a TM plane wave incident from air onto the planar boundary with a glass region. The glass exhibits relative permittivity of 2.1. Determine the power reflected and transmitted relative to power incident on the boundary.

Solution. The power reflected relative to power incident is $|\Gamma_{TM}|^2$ whereas the power transmitted relative to power incident is $1 - |\Gamma_{TM}|^2$. Γ_{TM} may be calculated using Equation 5.191. Calculating the quantities that enter into this expression:

$$\eta_1 \approx \eta_0 \cong 376.7\ \Omega \quad \text{(air)} \tag{5.192}$$

$$\eta_2 \approx \frac{\eta_0}{\sqrt{2.1}} \cong 260.0\ \Omega \quad \text{(glass)} \tag{5.193}$$

$$\psi^i = 30^\circ \tag{5.194}$$

Note

$$\frac{\beta_1}{\beta_2} \approx \frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\sqrt{\mu_0 \cdot 2.1\epsilon_0}} \cong 0.690 \tag{5.195}$$

so

$$\psi^t = \arcsin\left(\frac{\beta_1}{\beta_2}\sin\psi^i\right) \cong 20.2^\circ \tag{5.196}$$

Now substituting these values into Equation 5.191, we obtain

$$\Gamma_{TM} \cong -0.1442 \tag{5.197}$$

(Did you get an answer closer to -0.1323? If so, you probably did not use sufficient precision to represent intermediate results. This is a good example of a problem in which three significant figures for results that are used in subsequent calculations is not sufficient.)

The fraction of power reflected relative to power incident is now determined to be $|\Gamma_{TM}|^2 \cong 0.021$; i.e., about 2.1%. $1 - |\Gamma_{TM}|^2 \cong$ 97.9% of the power is transmitted into the glass.

Figure 5.13. *A TM uniform plane wave incident from air to glass.*

Note that the result obtained in the preceding example is different from the result for a TE wave incident from the same direction (Example 5.7). In other words:

The fraction of power reflected and transmitted from the planar boundary between dissimilar media depends on the polarization of the incident wave relative to the boundary, as well as the angle of incidence.

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1. Obtaining this result is an excellent way for the student to confirm their understanding of the derivation presented in this section. ↩

5.8 Angles of Reflection and Refraction

Consider the situation shown in Figure 5.14:

Figure 5.14. *A uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions.*

A uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions. Let a point on the boundary be represented as the position vector

$$\mathbf{r}_0 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y \tag{5.198}$$

In both Sections 5.6 ("Plane Waves at Oblique Incidence on a Planar Boundary: TE Case") and 5.7 ("Plane Waves at Oblique Incidence on a Planar Boundary: TM Case"), it is found that

$$\mathbf{k}^i \cdot \mathbf{r}_0 = \mathbf{k}^r \cdot \mathbf{r}_0 = \mathbf{k}^t \cdot \mathbf{r}_0 \tag{5.199}$$

In this expression,

$$\mathbf{k}^i = \beta_1 \hat{\mathbf{k}}^i \tag{5.200}$$
$$\mathbf{k}^r = \beta_1 \hat{\mathbf{k}}^r \tag{5.201}$$
$$\mathbf{k}^t = \beta_2 \hat{\mathbf{k}}^t \tag{5.202}$$

where $\hat{\mathbf{k}}^i$, $\hat{\mathbf{k}}^r$, and $\hat{\mathbf{k}}^t$ are unit vectors in the direction of incidence, reflection, and transmission, respectively; and β_1 and β_2 are the phase propagation constants in Region 1 (from which the wave is incident) and Region 2, respectively. Equation 5.199 is essentially a boundary condition that enforces continuity of the phase of the electric and magnetic fields across the boundary, and is sometimes referred to as the "phase matching" requirement. Since the same requirement emerges independently in the TE and TM cases, and since any plane wave may be decomposed into TE and TM components, the requirement must apply to any incident plane wave regardless of polarization.

Equation 5.199 is the key to finding the direction of reflection ψ^r and direction of transmission ψ^t. First, observe:

$$\hat{\mathbf{k}}^i = \hat{\mathbf{x}} \sin\psi^i + \hat{\mathbf{z}} \cos\psi^i \tag{5.203}$$
$$\hat{\mathbf{k}}^r = \hat{\mathbf{x}} \sin\psi^r - \hat{\mathbf{z}} \cos\psi^r \tag{5.204}$$
$$\hat{\mathbf{k}}^t = \hat{\mathbf{x}} \sin\psi^t + \hat{\mathbf{z}} \cos\psi^t \tag{5.205}$$

Therefore, we may express Equation 5.199 in the following form

$$\begin{aligned} &\beta_1\left(\hat{\mathbf{x}}\sin\psi^i+\hat{\mathbf{z}}\cos\psi^i\right)\cdot\left(\hat{\mathbf{x}}x+\hat{\mathbf{y}}y\right) \\ &=\beta_1\left(\hat{\mathbf{x}}\sin\psi^r-\hat{\mathbf{z}}\cos\psi^r\right)\cdot\left(\hat{\mathbf{x}}x+\hat{\mathbf{y}}y\right) \\ &=\beta_2\left(\hat{\mathbf{x}}\sin\psi^t+\hat{\mathbf{z}}\cos\psi^t\right)\cdot\left(\hat{\mathbf{x}}x+\hat{\mathbf{y}}y\right) \end{aligned} \tag{5.206}$$

which reduces to

$$\beta_1\sin\psi^i=\beta_1\sin\psi^r=\beta_2\sin\psi^t \tag{5.207}$$

Examining the first and second terms of Equation 5.207, and noting that ψ^i and ψ^r are both limited to the range $-\pi/2$ to $+\pi/2$, we find that:

$$\boxed{\psi^r=\psi^i} \tag{5.208}$$

In plain English:

Angle of reflection equals angle of incidence.

Examining the first and third terms of Equation 5.207, we find that

$$\beta_1\sin\psi^i=\beta_2\sin\psi^t \tag{5.209}$$

Since $\beta_1=\omega\sqrt{\mu_1\epsilon_1}$ and $\beta_2=\omega\sqrt{\mu_2\epsilon_2}$, Equation 5.209 expressed explicitly in terms of the constitutive parameters is:

$$\sqrt{\mu_1\epsilon_1}\sin\psi^i=\sqrt{\mu_2\epsilon_2}\sin\psi^t \tag{5.210}$$

Thus, we see that ψ^t does not depend on frequency, except to the extent that the constitutive parameters might. We may also express this relationship in terms of the *relative* values of constitutive parameters; i.e., $\mu_1=\mu_{r1}\mu_0$, $\epsilon_1=\epsilon_{r1}\epsilon_0$, $\mu_2=\mu_{r2}\mu_0$, and $\epsilon_2=\epsilon_{r2}\epsilon_0$. In terms of the relative parameters:

$$\boxed{\sqrt{\mu_{r1}\epsilon_{r1}}\sin\psi^i=\sqrt{\mu_{r2}\epsilon_{r2}}\sin\psi^t} \tag{5.211}$$

This is known as *Snell's law* or the *law of refraction*. Refraction is simply transmission with the result that the direction of propagation is changed.

Snell's law (Equation 5.211) determines the angle of refraction (transmission).

The associated formula for ψ^t explicitly is:

$$\psi^t=\arcsin\left(\sqrt{\frac{\mu_{r1}\epsilon_{r1}}{\mu_{r2}\epsilon_{r2}}}\sin\psi^i\right) \tag{5.212}$$

For the common special case of non-magnetic media, one assumes $\mu_{r1}=\mu_{r2}=1$. In this case, Snell's law simplifies to:

$$\sqrt{\epsilon_{r1}}\sin\psi^i = \sqrt{\epsilon_{r2}}\sin\psi^t \tag{5.213}$$

In optics, it is common to express permittivities in terms of indices of refraction; e.g., $n_1 \triangleq \sqrt{\epsilon_{r1}}$ and $n_2 \triangleq \sqrt{\epsilon_{r2}}$. Thus, Snell's law in optics is often expressed as:

$$n_1 \sin\psi^i = n_2 \sin\psi^t \tag{5.214}$$

When both media are non-magnetic, Equation 5.212 simplifies to

$$\psi^t = \arcsin\left(\sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}}\sin\psi^i\right) \tag{5.215}$$

When $\epsilon_{r2} > \epsilon_{r1}$, we observe that $\psi^t < \psi^i$. In other words, the transmitted wave travels in a direction that is closer to the surface normal than the angle of incidence. This scenario is demonstrated in the following example.

Example

Example 5.9

Refraction of light at an air-glass boundary.

Figure 5.15 shows a narrow beam of light that is incident from air to glass at an angle $\psi^i = 60^\circ$. As expected, the angle of reflection ψ^r is observed to be equal to ψ^i. The angle of refraction ψ^t is observed to be 35°. What is the relative permittivity of the glass?

Solution. Since the permittivity of glass is greater than that of air, we observe the expected result $\psi^t < \psi^i$. Since glass is non-magnetic, the expected relationship between these angles is given by Equation 5.213. Solving that equation for the relative permittivity of the glass, we obtain:

$$\epsilon_{r2} = \epsilon_{r1}\left(\frac{\sin\psi^i}{\sin\psi^t}\right)^2 \cong \underline{2.28} \tag{5.216}$$

Figure 5.15. *Angles of reflection and refraction for a light wave incident from air onto glass.*

When a wave travels in the reverse direction – i.e., from a non-magnetic medium of higher permittivity to a medium of lower permittivity – one finds $\psi^t > \psi^r$. In other words, the refraction is *away* from the surface normal. Figure 5.16 shows an example from common experience.

Figure 5.16. *Refraction accounts for the apparent displacement of an underwater object from the perspective of an observer above the water.*

In non-magnetic media, when $\epsilon_{r1} < \epsilon_{r2}$, $\psi_t < \psi_i$ (refraction toward the surface normal). When $\epsilon_{r1} > \epsilon_{r2}$, $\psi_t > \psi_i$ (refraction away from the surface normal).

Under certain conditions, the $\epsilon_{r2} < \epsilon_{r1}$ case leads to the following surprising observation: When calculating ψ^t using, for example, Equation 5.212, one finds that $\sqrt{\epsilon_{r1}/\epsilon_{r2}} \sin \psi^i$ can be greater than 1. Since the $\sin$ function yields values between -1 and $+1$, the result of the $\arcsin$ function is undefined. This odd situation is addressed in Section 5.11. For now, we will simply note that this condition leads to the phenomenon of *total internal reflection*. For now, all we can say is that when $\epsilon_{r2} < \epsilon_{r1}$, ψ^t is able to reach $\pi/2$ radians, which corresponds to propagation parallel to the boundary. Beyond that threshold, we must account for the unique physical considerations associated with total internal reflection.

We conclude this section with a description of the common waveguiding device known as the *prism*, shown in Figure 5.17. This particular device uses refraction to change the direction of light waves (similar devices can be used to manipulate radio waves as well).

Figure 5.17. *A typical triangular prism.*

Many readers are familiar with the use of prisms to separate white light into its constituent colors (frequencies), as shown in Figure 5.18. The separation of colors is due to frequency dependence of the material comprising the prism. Specifically, the permittivity of the material is a function of frequency, and therefore the angle of refraction is a function of frequency. Thus, each frequency is refracted by a different amount. Conversely, a prism comprised of a material whose permittivity exhibits negligible variation with frequency will not separate incident white light into its constituent colors since each color will be refracted by the same amount.

Figure 5.18. *A color-separating prism.*

Additional Reading:

- "Prism (https://en.wikipedia.org/wiki/Prism)" on Wikipedia.
- "Refraction (https://en.wikipedia.org/wiki/Refraction)" on Wikipedia.
- "Refractive index (https://en.wikipedia.org/wiki/Refractive_index)" on Wikipedia.
- "Snell's law (https://en.wikipedia.org/wiki/Snell's_law)" on Wikipedia.
- "Total internal reflection (https://en.wikipedia.org/wiki/Total_internal_reflection)" on Wikipedia.

5.9 TE Reflection in Non-magnetic Media

Figure 5.19 shows a TE uniform plane wave incident on the planar boundary between two semi-infinite material regions.

Figure 5.19. *A TE uniform plane wave obliquely incident on the planar boundary between two semi-infinite regions of lossless non-magnetic material.*

In this case, the reflection coefficient is given by:

$$\Gamma_{TE} = \frac{\eta_2 \cos\psi^i - \eta_1 \cos\psi^t}{\eta_2 \cos\psi^i + \eta_1 \cos\psi^t} \tag{5.217}$$

where ψ^i and ψ^t are the angles of incidence and transmission (refraction), respectively; and η_1 and η_2 are the wave impedances in Regions 1 and 2, respectively. Many materials of practical interest are non-magnetic; that is, they have permeability that is not significantly different from the permeability of free space. In this section, we consider the behavior of the reflection coefficient for this class of materials.

To begin, recall the general form of Snell's law:

$$\sin\psi^t = \frac{\beta_1}{\beta_2}\sin\psi^i \tag{5.218}$$

In non-magnetic media, the permeabilities μ_1 and μ_2 are assumed equal to μ_0. Thus:

$$\frac{\beta_1}{\beta_2} = \frac{\omega\sqrt{\mu_1\epsilon_1}}{\omega\sqrt{\mu_2\epsilon_2}} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{5.219}$$

Since permittivity ϵ can be expressed as ϵ_0 times the relative permittivity ϵ_r, we may reduce further to:

$$\frac{\beta_1}{\beta_2} = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} \tag{5.220}$$

Now Equation 5.218 reduces to:

$$\sin\psi^t = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}}\sin\psi^i \tag{5.221}$$

Next, note that for any value ψ, one may write cosine in terms of sine as follows:

$$\cos\psi = \sqrt{1-\sin^2\psi} \tag{5.222}$$

Therefore,

$$\cos\psi^t = \sqrt{1-\frac{\epsilon_{r1}}{\epsilon_{r2}}\sin^2\psi^i} \tag{5.223}$$

Also we note that in non-magnetic media

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_0}{\epsilon_{r1}\epsilon_0}} = \frac{\eta_0}{\sqrt{\epsilon_{r1}}} \tag{5.224}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{\mu_0}{\epsilon_{r2}\epsilon_0}} = \frac{\eta_0}{\sqrt{\epsilon_{r2}}} \tag{5.225}$$

where η_0 is the wave impedance in free space. Making substitutions into Equation 5.217, we obtain:

$$\Gamma_{TE} = \frac{(\eta_0/\sqrt{\epsilon_{r2}})\cos\psi^i - (\eta_0/\sqrt{\epsilon_{r1}})\cos\psi^t}{(\eta_0/\sqrt{\epsilon_{r2}})\cos\psi^i + (\eta_0/\sqrt{\epsilon_{r1}})\cos\psi^t} \tag{5.226}$$

Multiplying numerator and denominator by $\sqrt{\epsilon_{r2}}/\eta_0$, we obtain:

$$\Gamma_{TE} = \frac{\cos\psi^i - \sqrt{\epsilon_{r2}/\epsilon_{r1}}\cos\psi^t}{\cos\psi^i + \sqrt{\epsilon_{r2}/\epsilon_{r1}}\cos\psi^t} \tag{5.227}$$

Finally, by substituting Equation 5.223, we obtain:

$$\boxed{\Gamma_{TE} = \frac{\cos\psi^i - \sqrt{\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i}}{\cos\psi^i + \sqrt{\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i}}} \tag{5.228}$$

This expression has the advantage that it is now entirely in terms of ψ^i, with no need to first calculate ψ^t.

Using Equation 5.228, we can see how different combinations of material affect the reflection coefficient. First, we note that when $\epsilon_{r1} = \epsilon_{r2}$ (i.e., same media on both sides of the boundary), $\Gamma_{TE} = 0$ as expected. When $\epsilon_{r1} > \epsilon_{r2}$ (e.g., wave traveling in glass toward air), we see that it is possible for $\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i$ to be negative, which makes Γ_{TE} complex-valued. This results in *total internal reflection*, and is addressed elsewhere in another section. When $\epsilon_{r1} < \epsilon_{r2}$ (e.g., wave traveling in air toward glass), we see that $\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i$ is always positive, so Γ_{TE} is always real-valued.

Let us continue with the $\epsilon_{r1} < \epsilon_{r2}$ condition. Figure 5.20 shows Γ_{TE} plotted for various combinations of media over all possible angles of incidence from 0 (normal incidence) to $\pi/2$ (grazing incidence).

Figure 5.20. *The reflection coefficient Γ_{TE} as a function of angle of incidence ψ^i for various media combinations. Curves are labeled with the ratio $\epsilon_{r2}/\epsilon_{r1}$.*

We observe:

> In non-magnetic media with $\epsilon_{r1} < \epsilon_{r2}$, Γ_{TE} is real-valued, negative, and decreases to -1 as ψ^i approaches grazing incidence.

Also note that at any particular angle of incidence, Γ_{TE} trends toward -1 as $\epsilon_{r2}/\epsilon_{r1}$ increases. Thus, we observe that as $\epsilon_{r2}/\epsilon_{r1} \to \infty$, the result is increasingly similar to the result we would obtain for a perfect conductor in Region 2.

5.10 TM Reflection in Non-magnetic Media

Figure 5.21 shows a TM uniform plane wave incident on the planar boundary between two semi-infinite material regions.

Figure 5.21. *A transverse magnetic uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions.*

In this case, the reflection coefficient is given by:

$$\Gamma_{TM} = \frac{-\eta_1 \cos\psi^i + \eta_2 \cos\psi^t}{+\eta_1 \cos\psi^i + \eta_2 \cos\psi^t} \tag{5.229}$$

where ψ^i and ψ^t are the angles of incidence and transmission (refraction), respectively; and η_1 and η_2 are the wave impedances in Regions 1 and 2, respectively. Many materials of practical interest are non-magnetic; that is, they have permeability that is not significantly different from the permeability of free space. In this section, we consider the behavior of the reflection coefficient for this class of materials.

To begin, recall the general form of Snell's law:

$$\sin\psi^t = \frac{\beta_1}{\beta_2}\sin\psi^i \tag{5.230}$$

In non-magnetic media, the permeabilities μ_1 and μ_2 are assumed equal to μ_0. Thus:

$$\frac{\beta_1}{\beta_2} = \frac{\omega\sqrt{\mu_1\epsilon_1}}{\omega\sqrt{\mu_2\epsilon_2}} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{5.231}$$

Since permittivity ϵ can be expressed as ϵ_0 times the relative permittivity ϵ_r, we may reduce further to:

$$\frac{\beta_1}{\beta_2} = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} \tag{5.232}$$

Now Equation 5.230 reduces to:

$$\sin\psi^t = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}}\sin\psi^i \tag{5.233}$$

Next, note that for any value ψ, one may write cosine in terms of sine as follows:

$$\cos\psi = \sqrt{1-\sin^2\psi} \tag{5.234}$$

Therefore,

$$\cos\psi^t = \sqrt{1-\frac{\epsilon_{r1}}{\epsilon_{r2}}\sin^2\psi^i} \tag{5.235}$$

Also we note that in non-magnetic media

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_0}{\epsilon_{r1}\epsilon_0}} = \frac{\eta_0}{\sqrt{\epsilon_{r1}}} \tag{5.236}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{\mu_0}{\epsilon_{r2}\epsilon_0}} = \frac{\eta_0}{\sqrt{\epsilon_{r2}}} \tag{5.237}$$

where η_0 is the wave impedance in free space. Making substitutions into Equation 5.229, we obtain:

$$\Gamma_{TM} = \frac{-\left(\eta_0/\sqrt{\epsilon_{r1}}\right)\cos\psi^i + \left(\eta_0/\sqrt{\epsilon_{r2}}\right)\cos\psi^t}{+\left(\eta_0/\sqrt{\epsilon_{r1}}\right)\cos\psi^i + \left(\eta_0/\sqrt{\epsilon_{r2}}\right)\cos\psi^t} \tag{5.238}$$

Multiplying numerator and denominator by $\sqrt{\epsilon_{r2}}/\eta_0$, we obtain:

$$\Gamma_{TM} = \frac{-\sqrt{\epsilon_{r2}/\epsilon_{r1}}\cos\psi^i + \cos\psi^t}{+\sqrt{\epsilon_{r2}/\epsilon_{r1}}\cos\psi^i + \cos\psi^t} \tag{5.239}$$

Substituting Equation 5.235, we obtain:

$$\Gamma_{TM} = \frac{-\sqrt{\epsilon_{r2}/\epsilon_{r1}}\cos\psi^i + \sqrt{1-\left(\epsilon_{r1}/\epsilon_{r2}\right)\sin^2\psi^i}}{+\sqrt{\epsilon_{r2}/\epsilon_{r1}}\cos\psi^i + \sqrt{1-\left(\epsilon_{r1}/\epsilon_{r2}\right)\sin^2\psi^i}} \tag{5.240}$$

This expression has the advantage that it is now entirely in terms of ψ^i, with no need to first calculate ψ^t.

Finally, multiplying numerator and denominator by $\sqrt{\epsilon_{r2}/\epsilon_{r1}}$, we obtain:

$$\boxed{\Gamma_{TM} = \frac{-\left(\epsilon_{r2}/\epsilon_{r1}\right)\cos\psi^i + \sqrt{\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i}}{+\left(\epsilon_{r2}/\epsilon_{r1}\right)\cos\psi^i + \sqrt{\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i}}} \tag{5.241}$$

Using Equation 5.241, we can easily see how different combinations of material affect the reflection coefficient. First, we note that when $\epsilon_{r1} = \epsilon_{r2}$ (i.e., same media on both sides of the boundary), $\Gamma_{TM} = 0$ as expected. When $\epsilon_{r1} > \epsilon_{r2}$ (e.g., wave traveling in glass toward air), we see that it is possible for $\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i$ to be negative, which makes Γ_{TM} complex-valued. This results in *total internal reflection*, and is addressed in another section. When $\epsilon_{r1} < \epsilon_{r2}$ (e.g., wave traveling in air toward glass), we see that $\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i$ is always positive, so Γ_{TM} is always real-valued.

Let us continue with the $\epsilon_{r1} < \epsilon_{r2}$ condition. Figure 5.22 shows Γ_{TM} plotted for various combinations of media over all possible angles of incidence from 0 (normal incidence) to $\pi/2$ (grazing incidence).

Figure 5.22. *The reflection coefficient Γ_{TM} as a function of angle of incidence ψ^i for various media combinations, parameterized as $\epsilon_{r2}/\epsilon_{r1}$.*

We observe:

In non-magnetic media with $\epsilon_{r1} < \epsilon_{r2}$, Γ_{TM} is real-valued and increases from a negative value for normal incidence to $+1$ as ψ^i approaches grazing incidence.

Note that at any particular angle of incidence, Γ_{TM} trends toward -1 as $\epsilon_{r2}/\epsilon_{r1} \to \infty$. In this respect, the behavior of the TM component is similar to that of the TE component. In other words: As $\epsilon_{r2}/\epsilon_{r1} \to \infty$, the result for both the TE and TM components are increasingly similar to the result we would obtain for a perfect conductor in Region 2.

Also note that when $\epsilon_{r1} < \epsilon_{r2}$, Γ_{TM} changes sign from negative to positive as angle of incidence increases from 0 to $\pi/2$. This behavior is quite different from that of the TE component, which is always negative for $\epsilon_{r1} < \epsilon_{r2}$. The angle of incidence at which $\Gamma_{TM} = 0$ is referred to as *Brewster's angle*, which we assign the symbol ψ_B^i. Thus:

$$\boxed{\psi_B^i \triangleq \psi^i \text{ at which } \Gamma_{TM} = 0} \tag{5.242}$$

In the discussion that follows, here is the key point to keep in mind:

Brewster's angle ψ_B^i is the angle of incidence at which $\Gamma_{TM} = 0$.

Brewster's angle is also referred to as the *polarizing angle*. The motivation for the term "polarizing angle" is demonstrated in Figure 5.23.

%

Figure 5.23. *Reflection of a plane wave with angle of incidence ψ^i equal to the polarizing angle ψ_B^i. Here the media are non-magnetic with $\epsilon_{r1} < \epsilon_{r2}$.*

In this figure, a plane wave is incident with $\psi^i = \psi_B^i$. The wave may contain TE and TM components in any combination. Applying the principle of superposition, we may consider these components separately. The TE component of the incident wave will scatter as reflected and transmitted waves which are also TE. However, $\Gamma_{TM} = 0$ when $\psi^i = \psi_B^i$, so the TM component of the transmitted wave will be TM, but the TM component of the reflected wave will be zero. Thus, the total (TE+TM) reflected wave will be purely TE, regardless of the TM component of the incident wave. This principle can be exploited to suppress the TM component of a wave having both TE and TM components. This method can be used to isolate the TE and TM components of a wave.

Derivation of a formula for Brewster's angle. Brewster's angle for any particular combination of non-magnetic media may be determined as follows. $\Gamma_{TM} = 0$ when the numerator of Equation 5.241 equals zero, so:

$$-R\cos\psi_B^i + \sqrt{R - \sin^2\psi_B^i} = 0 \tag{5.243}$$

where we have made the substitution $R \triangleq \epsilon_{r2}/\epsilon_{r1}$ to improve clarity. Moving the second term to the right side of the equation and squaring both sides, we obtain:

$$R^2\cos^2\psi_B^i = R - \sin^2\psi_B^i \tag{5.244}$$

Now employing a trigonometric identity on the left side of the equation, we obtain:

$$R^2\left(1 - \sin^2\psi_B^i\right) = R - \sin^2\psi_B^i \tag{5.245}$$

$$R^2 - R^2\sin^2\psi_B^i = R - \sin^2\psi_B^i \tag{5.246}$$

$$\left(1 - R^2\right)\sin^2\psi_B^i = R - R^2 \tag{5.247}$$

and finally

$$\sin\psi_B^i = \sqrt{\frac{R-R^2}{1-R^2}} \tag{5.248}$$

Although this equation gets the job done, it is possible to simplify further. Note that Equation 5.248 can be interpreted as a description of the right triangle shown in Figure 5.24.

%

Figure 5.24. *Derivation of a simple formula for Brewster's angle for non-magnetic media.*

In the figure, we have identified the length h of the vertical side and the length d of the hypotenuse as:

$$h \triangleq \sqrt{R-R^2} \tag{5.249}$$
$$d \triangleq \sqrt{1-R^2} \tag{5.250}$$

The length b of the horizontal side is therefore

$$b = \sqrt{d^2-h^2} = \sqrt{1-R} \tag{5.251}$$

Subsequently, we observe

$$\tan\psi_B^i = \frac{h}{b} = \frac{\sqrt{R-R^2}}{\sqrt{1-R}} = \sqrt{R} \tag{5.252}$$

Thus, we have found

$$\boxed{\tan\psi_B^i = \sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}} \tag{5.253}$$

Example

Example 5.10

Polarizing angle for an air-to-glass interface.

A plane wave is incident from air onto the planar boundary with a glass region. The glass exhibits relative permittivity of 2.1. The incident wave contains both TE and TM components. At what angle of incidence ψ^i will the reflected wave be purely TE?

Solution. Using Equation 5.253:

$$\tan \psi_B^i = \sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}} = \sqrt{\frac{2.1}{1}} \cong 1.449 \tag{5.254}$$

Therefore, Brewster's angle is $\psi_B^i \cong 55.4^\circ$. This is the angle ψ^i at which $\Gamma_{TM} = 0$. Therefore, when $\psi^i = \psi_B^i$, the reflected wave contains no TM component and therefore must be purely TE.

Additional Reading:

- "Brewster's angle (https://en.wikipedia.org/wiki/Brewster's_angle)" on Wikipedia.

5.11 Total Internal Reflection

Total internal reflection refers to a particular condition resulting in the complete reflection of a wave at the boundary between two media, with no power transmitted into the second region. One way to achieve complete reflection with zero transmission is simply to require the second material to be a perfect conductor. However, total *internal* reflection is a distinct phenomenon in which neither of the two media are perfect conductors. Total internal reflection has a number of practical applications; notably, it is the enabling principle of fiber optics.

Consider the situation shown in Figure 5.25:

Figure 5.25. *A uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions. Here $\mu_{r1}\epsilon_{r1} > \mu_{r2}\epsilon_{r2}$, so $\psi^t > \psi^i$.*

A uniform plane wave is obliquely incident on the planar boundary between two semi-infinite material regions. In Section 5.8, it is found that

$$\psi^r = \psi^i \tag{5.255}$$

i.e., angle of reflection equals angle of incidence. Also, from Snell's law:

$$\sqrt{\mu_{r1}\epsilon_{r1}} \sin \psi^i = \sqrt{\mu_{r2}\epsilon_{r2}} \sin \psi^t \tag{5.256}$$

where "r" in the subscripts indicates the relative (unitless) quantities. The associated formula for ψ^t explicitly is:

$$\psi^t = \arcsin\left(\sqrt{\frac{\mu_{r1}\epsilon_{r1}}{\mu_{r2}\epsilon_{r2}}}\sin\psi^i\right) \tag{5.257}$$

From Equation 5.257, it is apparent that when $\mu_{r1}\epsilon_{r1} > \mu_{r2}\epsilon_{r2}$, $\psi^t > \psi^r$; i.e., the transmitted wave appears to bend away from the surface normal, as shown in Figure 5.25. In fact, ψ^t can be as large as $\pi/2$ (corresponding to propagation parallel to the boundary) for angles of incidence which are less than $\pi/2$. What happens if the angle of incidence is further increased? When calculating ψ^t using Equation 5.257, one finds that the argument of the arcsine function becomes greater than 1. Since the possible values of the sine function are between -1 and $+1$, the arcsine function is undefined. Clearly our analysis is inadequate in this situation.

To make sense of this, let us begin by identifying the threshold angle of incidence ψ_c^i at which the trouble begins. From the analysis in the previous paragraph,

$$\sqrt{\frac{\mu_{r1}\epsilon_{r1}}{\mu_{r2}\epsilon_{r2}}}\sin\psi_c^i = 1 \tag{5.258}$$

therefore,

$$\boxed{\psi_c^i = \arcsin\sqrt{\frac{\mu_{r2}\epsilon_{r2}}{\mu_{r1}\epsilon_{r1}}}} \tag{5.259}$$

This is known as the *critical angle*. When $\psi^i < \psi_c^i$, our existing theory applies. When $\psi^i \geq \psi_c^i$, the situation is, at present, unclear. For non-magnetic materials, Equation 5.259 simplifies to

$$\psi_c^i = \arcsin\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}} \tag{5.260}$$

Now let us examine the behavior of the reflection coefficient. For example, the reflection coefficient for TE component and non-magnetic materials is

$$\Gamma_{TE} = \frac{\cos\psi^i - \sqrt{\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i}}{\cos\psi^i + \sqrt{\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i}} \tag{5.261}$$

From Equation 5.260, we see that

$$\sin^2\psi_c^i = \frac{\epsilon_{r2}}{\epsilon_{r1}} \tag{5.262}$$

So, when $\psi^i > \psi_c^i$, we see that

$$\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i < 0 \quad (\psi^i > \psi_c^i) \tag{5.263}$$

and therefore

$$\sqrt{\epsilon_{r2}/\epsilon_{r1} - \sin^2\psi^i} = jB \quad (\psi^i > \psi_c^i) \tag{5.264}$$

where B is a positive real-valued number. Now we may write Equation 5.261 as follows:

$$\Gamma_{TE} = \frac{A - jB}{A + jB} \quad (\psi^i > \psi_c^i) \tag{5.265}$$

where $A \triangleq \cos\psi^i$ is also a positive real-valued number. Note that the numerator and denominator in Equation 5.265 have equal magnitude. Therefore, the magnitude of $|\Gamma_{TE}| = 1$ and

$$\Gamma_{TE} = e^{j\zeta} \quad (\psi^i > \psi_c^i) \tag{5.266}$$

where ζ is a real-valued number indicating the phase of Γ_{TE}. In other words,

When the angle of incidence ψ^i exceeds the critical angle ψ_c^i, the magnitude of the reflection coefficient is 1. In this case, all power is reflected, and no power is transmitted into the second medium. This is *total internal reflection*.

Although we have obtained this result for TE component, the identical conclusion is obtained for TM component as well. This is left as an exercise for the student.

Example

Example 5.11

Total internal reflection in glass.

Figure 5.15 (Section 5.8) shows a demonstration of refraction of a beam of light incident from air onto a planar boundary with glass. Analysis of that demonstration revealed that the relative permittivity of the glass was $\cong 2.28$. Figure 5.26 shows a modification of the demonstration: In this case, a beam of light incident from glass onto a planar boundary with air. Confirm that total internal reflection is the expected result in this demonstration.

Solution. Assuming the glass is non-magnetic, the critical angle is given by Equation 5.260. In the present example, $\epsilon_{r1} \cong 2.28$ (glass), $\epsilon_{r2} \cong 1$ (air). Therefore, the critical angle $\psi_c^i \cong 41.5^\circ$. The angle of incidence in Figure 5.26 is seen to be about $\cong 50^\circ$, which is greater than the critical angle. Therefore, total internal reflection is expected.

Figure 5.26. *Total internal reflection of a light wave incident on a planar boundary between glass and air.*

Note also that the phase ζ of the reflection coefficient is precisely zero unless $\psi^i > \psi_c^i$, at which point it is both non-zero and varies with ψ^i. This is known as the *Goos-Hänchen effect*. This leads to the startling phenomenon shown in Figure 5.27. In this figure, the Goos-Hänchen phase shift is apparent as a displacement between the point of incidence and the point of reflection.

Figure 5.27. *Laser light in a dielectric rod exhibiting the Goos-Hänchen effect.*

The presence of an imaginary component in the reflection coefficient is odd for two reasons. First, we are not accustomed to seeing a complex-valued reflection coefficient emerge when the wave impedances of the associated media are real-valued. Second, the *total* reflection of the incident wave seems to contradict the boundary condition that requires the tangential components of the electric and magnetic fields to be continuous across the boundary. That is, how can these components of the fields be continuous across the boundary if no power is transmitted across the boundary? These considerations suggest that there *is* a field on the opposite side of the boundary, but – somehow – it must have zero power. This is exactly the case, as is explained in Section 5.12.

Additional Reading:

- "Goos-Hänchen effect (https://en.wikipedia.org/wiki/Goos%E2%80%93H%C3%A4nchen_effect)" on Wikipedia.
- "Snell's law (https://en.wikipedia.org/wiki/Snell's_law)" on Wikipedia.
- "Total internal reflection (https://en.wikipedia.org/wiki/Total_internal_reflection)" on Wikipedia.

5.12 Evanescent Waves

Consider the situation shown in Figure 5.28:

Figure 5.28. *A uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions. Here $\mu_{r1}\epsilon_{r1} > \mu_{r2}\epsilon_{r2}$ and $\psi^i > \psi^i_c$.*

A uniform plane wave obliquely incident on the planar boundary between two semi-infinite material regions, and total internal reflection occurs because the angle of incidence ψ^i is greater than the critical angle

$$\psi^i_c = \arcsin\sqrt{\frac{\mu_{r2}\epsilon_{r2}}{\mu_{r1}\epsilon_{r1}}} \tag{5.267}$$

Therefore, the reflection coefficient is complex-valued with magnitude equal to 1 and phase that depends on polarization and the constitutive parameters of the media.

The total reflection of the incident wave seems to contradict the boundary conditions that require the tangential components of the electric and magnetic fields to be continuous across the boundary. How can these components of the fields be continuous across the boundary if no power is transmitted across the boundary? There must be a field on the opposite side of the boundary, but – somehow – it must have zero power. To make sense of this, let us attempt to find a solution for the transmitted field.

We begin by postulating a complex-valued angle of transmission ψ^{tc}. Although the concept of a complex-valued angle may seem counterintuitive, there is mathematical support for this concept. For example, consider the well-known trigonometric identities:

$$\sin\theta = \frac{1}{j2}\left(e^{j\theta} - e^{-j\theta}\right) \tag{5.268}$$

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \tag{5.269}$$

These identities allow us to compute values for sine and cosine even when θ is complex-valued. One may conclude that the sine and cosine of a complex-valued angle exist, although the results may also be complex-valued.

Based on the evidence established so far, we presume that ψ^{tc} behaves as follows:

$$\psi^{tc} \triangleq \psi^{t} \ , \qquad \psi^{i} < \psi^{i}_{c} \tag{5.270}$$
$$\psi^{tc} \triangleq \pi/2 \ , \qquad \psi^{i} = \psi^{i}_{c} \tag{5.271}$$
$$\psi^{tc} \triangleq \pi/2 + j\psi'' \ , \quad \psi^{i} > \psi^{i}_{c} \tag{5.272}$$

In other words, ψ^{tc} is identical to ψ^{t} for $\psi^{i} \leq \psi^{i}_{c}$, but when total internal reflection occurs, we presume the real part of ψ^{tc} remains fixed (parallel to the boundary) and that an imaginary component $j\psi''$ emerges to satisfy the boundary conditions.

For clarity, let us assign the variable ψ' to represent the real part of ψ^{tc} in Equations 5.270–5.272. Then we may refer to all three cases using a single expression as follows:

$$\psi^{tc} = \psi' + j\psi'' \tag{5.273}$$

Now we use a well-known trigonometric identity as follows:

$$\sin\psi^{tc} = \sin\left(\psi' + j\psi''\right) \tag{5.274}$$
$$= \sin\psi' \cos j\psi'' + \cos\psi' \sin j\psi'' \tag{5.275}$$

Using Equation 5.269, we find:

$$\cos j\psi'' = \frac{1}{2}\left(e^{j(j\psi'')} + e^{-j(j\psi'')}\right) \tag{5.276}$$
$$= \frac{1}{2}\left(e^{-\psi''} + e^{+\psi''}\right) \tag{5.277}$$
$$= \cosh\psi'' \tag{5.278}$$

In other words, the cosine of $j\psi''$ is simply the hyperbolic cosine ("$\cosh$") of ψ''. Interestingly, $\cosh$ of a real-valued argument is real-valued, so $\cos j\psi''$ is real-valued.

Using Equation 5.268, we find:

$$\sin j\psi'' = \frac{1}{j2}\left(e^{j(j\psi'')} - e^{-j(j\psi'')}\right) \tag{5.279}$$
$$= \frac{1}{j2}\left(e^{-\psi''} - e^{+\psi''}\right) \tag{5.280}$$
$$= j\frac{1}{2}\left(e^{+\psi''} - e^{-\psi''}\right) \tag{5.281}$$
$$= j\sinh\psi'' \tag{5.282}$$

In other words, the sine of $j\psi''$ is j times hyperbolic sine ("$\sinh$") of ψ''. Now note that $\sinh$ of a real-valued argument is real-valued, so $\sin j\psi''$ is imaginary-valued.

Using these results, we find Equation 5.275 may be written as follows:

$$\sin\psi^{tc} = \sin\psi' \cosh\psi'' + j\cos\psi' \sinh\psi'' \tag{5.283}$$

Using precisely the same approach, we find:

$$\cos\psi^{tc} = \cos\psi' \cosh\psi'' - j \sin\psi' \sinh\psi'' \tag{5.284}$$

Before proceeding, let's make sure Equations 5.283 and 5.284 exhibit the expected behavior before the onset of total internal reflection. For $\psi^i < \psi^i_c$, $\psi' = \psi^t$ and $\psi'' = 0$. In this case, $\sinh\psi'' = 0$, $\cosh\psi'' = 1$, and Equations 5.283 and 5.284 yield

$$\sin\psi^{tc} = \sin\psi^t \tag{5.285}$$
$$\cos\psi^{tc} = \cos\psi^t \tag{5.286}$$

as expected.

When total internal reflection is in effect, $\psi^i > \psi^i_c$, so $\psi' = \pi/2$. In this case, Equations 5.283 and 5.284 yield

$$\sin\psi^{tc} = \cosh\psi'' \tag{5.287}$$
$$\cos\psi^{tc} = -j\sinh\psi'' \tag{5.288}$$

Let us now consider what this means for the field in Region 2. According to the formalism adopted in previous sections, the propagation of wave components in this region is described by the factor $e^{-j\mathbf{k}^t\cdot\mathbf{r}}$ where

$$\begin{aligned}\mathbf{k}^t &= \beta_2\hat{\mathbf{k}}^t \\ &= \beta_2\left(\hat{\mathbf{x}}\sin\psi^{tc} + \hat{\mathbf{z}}\cos\psi^{tc}\right)\end{aligned} \tag{5.289}$$

and

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z \tag{5.290}$$

so

$$\begin{aligned}\mathbf{k}^t\cdot\mathbf{r} &= \left(\beta_2\sin\psi^{tc}\right)x + \left(\beta_2\cos\psi^{tc}\right)z \\ &= \left(\beta_2\cosh\psi''\right)x + \left(-j\beta_2\sinh\psi''\right)z\end{aligned} \tag{5.291}$$

Therefore, the wave in Region 2 propagates according to

$$e^{-j\mathbf{k}^t\cdot\mathbf{r}} = e^{-j(\beta_2\cosh\psi'')x}e^{-(\beta_2\sinh\psi'')z} \tag{5.292}$$

Note that the constants $\beta_2\cosh\psi''$ and $\beta_2\sinh\psi''$ are both real-valued and positive. Therefore, Equation 5.292 describes a wave which propagates in the $+\hat{\mathrm{x}}$ direction, but which is not uniform. Specifically, the magnitude of the transmitted field decreases exponentially with increasing z; i.e., maximum at the boundary, asymptotically approaching zero with increasing distance from the boundary. This wave is unlike the incident or reflected waves (both uniform plane waves), and is unlike the transmitted wave in the $\psi^i < \psi^i_c$ case (also a uniform plane wave). The transmitted wave that we have derived in the $\psi^i > \psi^i_c$ case gives the impression of being somehow attached to the boundary, and so may be described as a *surface wave*. However, in this case we have a particular kind of surface wave, known as an *evanescent wave*. Summarizing:

When total internal reflection occurs, the transmitted field is an evanescent wave; i.e., a surface wave which conveys no power and whose magnitude decays exponentially with increasing distance into Region 2.

At this point, we could enforce the “phase matching” condition at the boundary, which would lead us to a new version of Snell’s law that would allow us to solve for ψ'' in terms of ψ^i and the constitutive properties of the media comprising Regions 1 and 2. We could subsequently determine values for the phase propagation and attenuation constants for the evanescent wave. It suffices to say that the magnitude of the evanescent field becomes negligible beyond a few wavelengths of the boundary.

Finally, we return to the strangest characteristic of this field: It acts like a wave, but conveys no power. This field exists solely to enforce the electromagnetic boundary conditions at the boundary, and does not exist independently of the incident and reflected field. The following thought experiment may provide some additional insight.

In this experiment, a laser illuminates a planar boundary between two material regions, with conditions such that total internal reflection occurs. Thus, all incident power is reflected, and an evanescent wave exists on the opposite side of the boundary. Next, the laser is turned off. All the light incident on the boundary reflects from the boundary and continues to propagate to infinity, even after light is no longer incident on the boundary. In contrast, the evanescent wave vanishes at the moment laser light ceases to illuminate the boundary. In other words, the evanescent field does not continue to propagate along the boundary to infinity. The reason for this is simply that there is no power – hence, no energy – in the evanescent wave.

Additional Reading:

- “Evanescent field (https://en.wikipedia.org/wiki/Evanescent_field)” on Wikipedia.
- “Hyperbolic function (https://en.wikipedia.org/wiki/Hyperbolic_function)” on Wikipedia.
- “Total internal reflection (https://en.wikipedia.org/wiki/Total_internal_reflection)” on Wikipedia.

End Matter

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- **Fig.**
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- **Fig.**
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- **Fig.**
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- **Fig.**
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  https://commons.wikimedia.org/wiki/File:Reflection_of_plane_wave_incidence_angle_equals_polarizing_angle.svg (https://commons.wikimedia.org/wiki/File:Reflection_of_plane_wave_incidence_angle_equals_polarizing_angle.svg),
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- **Fig.**
  5.24: © Sevenchw (C. Wang), https://commons.wikimedia.org/wiki/File:Brewster%E2%80%99s_angle_for_non-magnetic_media.svg (https://commons.wikimedia.org/wiki/File:Brewster%E2%80%99s_angle_for_non-magnetic_media.svg),
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- **Fig.**
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- **Fig.**
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- **Fig.**
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