4 Current Flow in Imperfect Conductors

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4.1 AC Current Flow in a Good Conductor

In this section, we consider the distribution of current in a conductor which is imperfect (i.e., a “good conductor”) and at frequencies greater than DC.

To establish context, consider the simple DC circuit shown in Figure 4.1. In this circuit, the current source provides a steady current which flows through a cylinder-shaped wire. As long as the conductivity σ (SI base units of S/m) of the wire is uniform throughout the wire, the current density $\mathbf{J}$ (SI base units of A/m^2) is uniform throughout the wire.

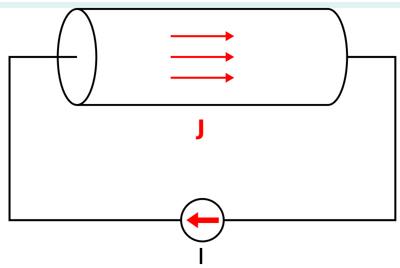

Figure 4.1. *Current flow in cylinder at DC.*

Now let us consider the AC case. Whereas the electric field intensity $\mathbf{E}$ is constant in the DC case, $\mathbf{E}$ exists as a wave in the AC case. In a good conductor, the magnitude of $\mathbf{E}$ decreases in proportion to $e^{-\alpha d}$ where α is the attenuation constant and d is distance traversed by the wave. The attenuation constant increases with increasing σ, so the rate of decrease of the magnitude of $\mathbf{E}$ increases with increasing σ.

In the limiting case of a perfect conductor, $\alpha \to \infty$ and so $\mathbf{E} \to 0$ everywhere inside the material. Any current within the wire must be the result of either an impressed source or it must be a response to $\mathbf{E}$. Without either of these, we conclude that in the AC case, $\mathbf{J} \to 0$ everywhere inside a perfect conductor.[1] But if $\mathbf{J} = 0$ in the material, then how does current pass through the wire? We are forced to conclude that the current must exist as a *surface current*; i.e., entirely outside the wire, yet bound to the surface of the wire. Thus:

> In the AC case, the current passed by a perfectly-conducting material lies entirely on the surface of the material.

The perfectly-conducting case is unobtainable in practice, but the result gives us a foothold from which we may determine what happens when σ is not infinite. If σ is merely finite, then α is also finite and subsequently wave magnitude may be non-zero over finite distances.

Let us now consider the direction in which this putative wave propagates. The two principal directions in the present problem are parallel to the axis of the wire and perpendicular to the axis of the wire. Waves propagating in any other direction may be expressed as a linear combination of waves traveling in the principal directions, so we need only consider the principal directions to obtain a complete picture.

Consider waves propagating in the perpendicular direction first. In this case, we presume a boundary condition in the form of non-zero surface current, which we infer from the perfectly-conducting case considered previously. We also note that $\mathbf{E}$ deep inside the wire must be weaker than $\mathbf{E}$ closer to the surface, since a wave deep inside the wire must have traversed a larger quantity of material than a wave measured closer to the surface. Applying Ohm's law ($\mathbf{J} = \sigma\mathbf{E}$), the current deep inside the wire must similarly diminish for a good conductor. We conclude that a wave traveling in the perpendicular direction exists and propagates toward the center of the wire, decreasing in magnitude with increasing distance from the surface.

We are unable to infer the presence of a wave traveling in the other principal direction – i.e., along the axis of the wire – since there is no apparent boundary condition to be satisfied on either end of the wire. Furthermore, the presence of such a wave would mean that different cross-sections of the wire exhibit different radial distributions of current. This is not consistent with physical observations.

We conclude that the only relevant wave is one which travels from the surface of the wire inward. Since the current density is proportional to the electric field magnitude, we conclude:

> In the AC case, the current passed by a wire comprised of a good conductor is distributed with maximum current density on the surface of the wire, and the current density decays exponentially with increasing distance from the surface.

This phenomenon is known as the *skin effect*, referring to the notion of current forming a skin-like layer below the surface of the wire. The effect is illustrated in Figure 4.2.

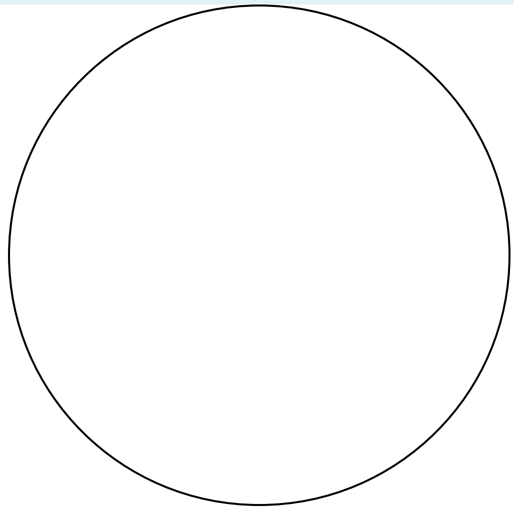

Figure 4.2. *Distribution of AC current in a wire of circular cross-section. Shading indicates current density.*

by Biezl (modified) (public domain)

Since α increases with increasing frequency, we see that the specific distribution of current within the wire depends on frequency. In particular, the current will be concentrated close to the surface at high frequencies, uniformly distributed throughout the wire at DC, and in an intermediate state for intermediate frequencies.

Additional Reading:

- "Skin effect (https://en.wikipedia.org/wiki/Skin_effect)" on Wikipedia.

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1. You might be tempted to invoke Ohm's law ($\mathbf{J} = \sigma\mathbf{E}$) to argue against this conclusion. However, Ohm's law provides no useful information about the current in this case, since $\sigma \to \infty$ at the same time $\mathbf{E} \to 0$. What Ohm's law is really saying in this case is that $\mathbf{E} = \mathbf{J}/\sigma \to 0$ because $\mathbf{J}$ must be finite and $\sigma \to \infty$. ↩

4.2 Impedance of a Wire

The goal of this section is to determine the impedance – the ratio of potential to current – of a wire. The answer to this question is relatively simple in the DC ("steady current") case: The impedance is found to be equal to the resistance of the wire, which is given by

$$R = \frac{l}{\sigma A} \qquad \text{(DC)} \tag{4.1}$$

where l is the length of the wire and A is the cross-sectional area of the wire. Also, the impedance of a wire comprised of a perfect conductor at *any* frequency is simply zero, since there is no mechanism in the wire that can dissipate or store energy in this case.

However, all practical wires are comprised of good – not perfect – conductors, and of course many practical signals are time-varying, so the two cases above do not address a broad category of practical interest. The more general case of non-steady currents in imperfect conductors is complicated by the fact that the current in an imperfect conductor is not uniformly distributed in the wire, but rather is concentrated near the surface and decays exponentially with increasing distance from the surface (this is determined in Section 4.1).

We are now ready to consider the AC case for a wire comprised of a good but imperfect conductor. What is the impedance of the wire if the current source is sinusoidally-varying? Equation 4.1 for the DC case was determined by first obtaining expressions for the potential (V) and net current (I) for a length-l section of wire in terms of the electric field intensity $\mathbf{E}$ in the wire. To follow that same approach here, we require an expression for I in terms of $\mathbf{E}$ that accounts for the non-uniform distribution of current. This is quite difficult to determine in the case of a cylindrical wire. However, we may develop an approximate solution by considering a surface which is not cylindrical, but rather planar. Here we go:

Consider the experiment described in Figure 4.3.

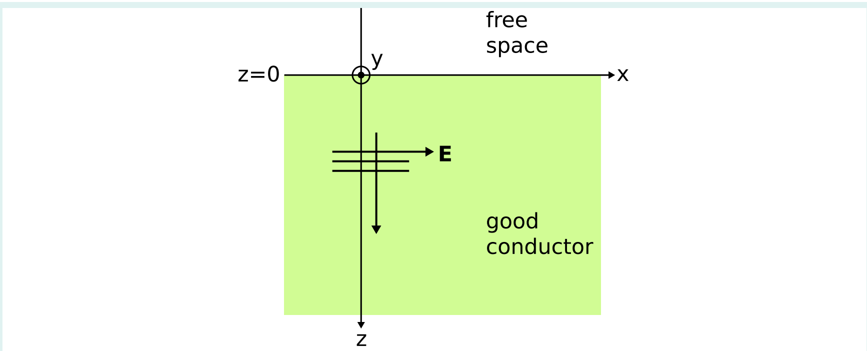

Figure 4.3. *Experiment used to determine current density and resistance in the AC case.*

Here, a semi-infinite region filled with a homogeneous good conductor meets a semi-infinite region of free space along a planar interface at $z = 0$, with increasing z corresponding to increasing depth into the material. A plane wave propagates in the $+\hat{\mathbf{z}}$ direction, beginning from just inside the structure's surface. The justification for presuming the existence of this wave was presented in Section 4.1. The electric field intensity is given by

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}} E_0 e^{-\alpha z} e^{-j\beta z} \tag{4.2}$$

where E_0 is an arbitrary complex-valued constant. The current density is given by Ohm's law of electromagnetics:[1]

$$\widetilde{\mathbf{J}} = \sigma\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\sigma E_0 e^{-\alpha z} e^{-j\beta z} \tag{4.3}$$

Recall that $\alpha = \delta_s^{-1}$ where δ_s is skin depth (see Section 3.12). Also, for a good conductor, we know that $\beta \approx \alpha$ (Section 3.11). Using these relationships, we may rewrite Equation 4.3 as follows:

$$\begin{aligned}\widetilde{\mathbf{J}} &\approx \hat{\mathbf{x}}\sigma E_0 e^{-z/\delta_s} e^{-jz/\delta_s} \\ &= \hat{\mathbf{x}}\sigma E_0 e^{-(1+j)z/\delta_s}\end{aligned} \tag{4.4}$$

The net current $\tilde{I}$ is obtained by integrating $\widetilde{\mathbf{J}}$ over any cross-section $\mathcal{S}$ through which all the current flows; i.e.,

$$\tilde{I} = \int_{\mathcal{S}} \widetilde{\mathbf{J}} \cdot d\mathbf{s} \tag{4.5}$$

Here, the simplest solution is obtained by choosing $\mathcal{S}$ to be a rectangular surface that is perpendicular to the direction of $\widetilde{\mathbf{J}}$ at $x = 0$. This is shown in Figure 4.4.

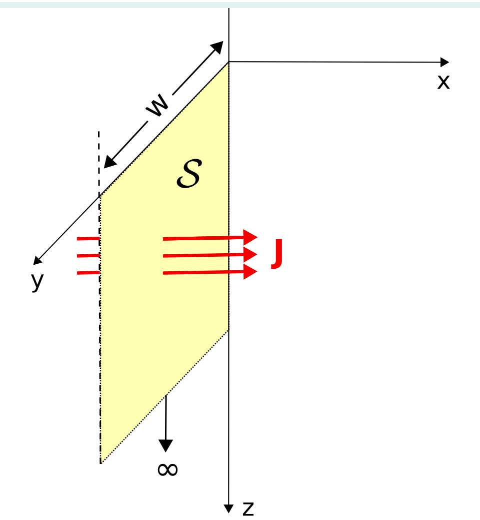

Figure 4.4. *Choice of $\mathcal{S}$ for calculating net current I.*

The dimensions of $\mathcal{S}$ are width W in the y dimension and extending to infinity in the z direction. Then we have

$$\begin{aligned}\tilde{I} &\approx \int_{y=0}^{W}\int_{z=0}^{\infty} \left(\hat{\mathbf{x}}\sigma E_0 e^{-(1+j)z/\delta_s}\right) \cdot \left(\hat{\mathbf{x}}\ dy\ dz\right) \\ &= \sigma E_0 W \int_{z=0}^{\infty} e^{-(1+j)z/\delta_s} dz\end{aligned} \tag{4.6}$$

For convenience, let us define the constant $K \triangleq (1+j)/\delta_s$. Since K is constant with respect to z, the remaining integral is straightforward to evaluate:

$$\begin{aligned}\int_0^{\infty} e^{-Kz} dz &= -\frac{1}{K} e^{-Kz}\Big|_0^{\infty} \\ &= +\frac{1}{K}\end{aligned} \tag{4.7}$$

Incorporating this result into Equation 4.6, we obtain:

$$\tilde{I} \approx \sigma E_0 W \frac{\delta_s}{1+j} \tag{4.8}$$

We calculate $\widetilde{V}$ for a length l of the wire as follows:

$$\widetilde{V} = -\int_{x=l}^{0} \widetilde{\mathbf{E}} \cdot d\mathbf{l} \tag{4.9}$$

where we have determined that $x=0$ corresponds to the "+" terminal and $x=l$ corresponds to the "−" terminal.[2] The path of integration can be *any* path that begins and ends at the proscribed terminals. The simplest path to use is one along the surface, parallel to the x axis. Along this path, $z=0$ and thus $\widetilde{\mathbf{E}} = \hat{\mathbf{x}} E_0$. For this path:

$$\widetilde{V} = -\int_{x=l}^{0} (\hat{\mathbf{x}} E_0) \cdot (\hat{\mathbf{x}} dx) = E_0 l \tag{4.10}$$

The impedance Z measured across terminals at $x=0$ and $x=l$ is now determined to be:

$$Z \triangleq \frac{\widetilde{V}}{\tilde{I}} \approx \frac{1+j}{\sigma \delta_s} \cdot \frac{l}{W} \tag{4.11}$$

The resistance is simply the real part, so we obtain

$$R \approx \frac{l}{\sigma(\delta_s W)} \quad \text{(AC case)} \tag{4.12}$$

The quantity R in this case is referred to specifically as the *ohmic resistance*, since it is due entirely to the limited conductivity of the material as quantified by Ohm's law.[3]

Note the resemblance to Equation 4.1 (the solution for the DC case): In the AC case, the product $\delta_s W$, having units of area, plays the role of the physical cross-section S. Thus, we see an interesting new interpretation of the skin depth δ_s: It is the depth to which a uniform (DC) current would need to flow in order to produce a resistance equal to the observed (AC) resistance.

Equations 4.11 and 4.12 were obtained for a good conductor filling an infinite half-space, having a flat surface. How well do these results describe a cylindrical wire? The answer depends on the radius of the wire, a. For $\delta_s \ll a$, Equations 4.11 and 4.12 are excellent approximations, since $\delta_s \ll a$ implies that most of the current lies in a thin shell close to the surface of the wire. In this case, the model used to develop the equations is a good approximation for any given radial slice through the wire, and we are justified in replacing W with the circumference $2\pi a$. Thus, we obtain the following expressions:

$$\boxed{Z \approx \frac{1+j}{\sigma\delta_s} \cdot \frac{l}{2\pi a} \quad (\delta_s \ll a)} \tag{4.13}$$

and so

$$\boxed{R \approx \frac{l}{\sigma(\delta_s 2\pi a)} \quad (\delta_s \ll a)} \tag{4.14}$$

The impedance of a wire of length l and radius $a \gg \delta_s$ is given by Equation 4.13. The resistance of such a wire is given by Equation 4.14.

If, on the other hand, $a < \delta_s$ or merely $\sim \delta_s$, then current density is significant throughout the wire, including along the axis of the wire. In this case, we cannot assume that the current density decays smoothly to zero with increasing distance from the surface, and so the model leading to Equation 4.13 is a poor approximation. The frequency required for validity of Equation 4.13 can be determined by noting that $\delta_s \approx 1/\sqrt{\pi f \mu \sigma}$ for a good conductor; therefore, we require

$$\frac{1}{\sqrt{\pi f \mu \sigma}} \ll a \tag{4.15}$$

for the derived expressions to be valid. Solving for f, we find:

$$f \gg \frac{1}{\pi \mu \sigma a^2} \tag{4.16}$$

For commonly-encountered wires comprised of typical good conductors, this condition applies at frequencies in the MHz regime and above.

These results lead us to one additional interesting finding about the AC resistance of wires. Since $\delta_s \approx 1/\sqrt{\pi f \mu \sigma}$ for a good conductor, Equation 4.14 may be rewritten in the following form:

$$\boxed{R \approx \frac{1}{2}\sqrt{\frac{\mu f}{\pi \sigma}} \cdot \frac{l}{a}} \tag{4.17}$$

We have found that R is approximately proportional to $\sqrt{f}$. For example, increasing frequency by a factor of 4 increases resistance by a factor of 2. This frequency dependence is evident in all kinds of practical wires and transmission lines. Summarizing:

> The resistance of a wire comprised of a good but imperfect conductor is proportional to the square root of frequency.

At this point, we have determined that resistance is given approximately by Equation 4.17 for $\delta_s \ll a$, corresponding to frequencies in the MHz regime and above, and by Equation 4.1 for $\delta_s \gg a$, typically corresponding to frequencies in the kHz regime and below. We have also found that resistance changes slowly with frequency; i.e., in proportion to $\sqrt{f}$. Thus, it is often possible to roughly estimate resistance at frequencies between these two frequency regimes by comparing the DC resistance from Equation 4.1 to the AC resistance from Equation 4.17. An example follows.

Example

Example 4.1

Resistance of the inner conductor of RG-59.

Elsewhere we have considered RG-59 coaxial cable (see the section “Coaxial Line,” which may appear in another volume depending on the version of this book). We noted that it was not possible to determine the AC resistance per unit length R' for RG-59 from purely electrostatic and magnetostatic considerations. We are now able to consider the resistance per unit length of the inner conductor, which is a solid wire of the type considered in this section. Let us refer to this quantity as R'_{ic}. Note that

$$R' = R'_{ic} + R'_{oc} \tag{4.18}$$

where R'_{oc} is the resistance per unit length of the outer conductor. R'_{oc} remains a bit too complicated to address here. However, R'_{ic} is typically much greater than R'_{oc}, so $R' \sim R'_{ic}$. That is, we get a pretty good idea of R' for RG-59 by considering the inner conductor alone.

The relevant parameters of the inner conductor are $\mu \approx \mu_0$, $\sigma \cong 2.28 \times 10^7$ S/m, and $a \cong 0.292$ mm. Using Equation 4.17, we find:

$$\begin{aligned} R'_{ic} &\triangleq \frac{R_{ic}}{l} = \frac{1}{2}\sqrt{\frac{\mu f}{\pi\sigma}} \cdot \frac{1}{a} \\ &\cong \left(227\ \mu\Omega \cdot \mathrm{m}^{-1} \cdot \mathrm{Hz}^{-1/2}\right)\sqrt{f} \end{aligned} \tag{4.19}$$

Using Expression 4.16, we find this is valid only for $f \gg 130$ kHz. So, for example, we may be confident that $R'_{ic} \approx 0.82\ \Omega$/m at 13 MHz.

At the other extreme ($f \ll 130$ kHz), Equation 4.1 (the DC resistance) is a better estimate. In this low frequency case, we estimate that $R'_{ic} \approx 0.16\ \Omega$/m and is approximately constant with frequency.

We now have a complete picture: As frequency is increased from DC to 13 MHz, we expect that R'_{ic} will increase monotonically from $\approx 0.16\ \Omega$/m to $\approx 0.82\ \Omega$/m, and will continue to increase in proportion to $\sqrt{f}$ from that value.

Returning to Equation 4.11, we see that resistance is not the whole story here. The impedance $Z = R + jX$ also has a reactive component X equal to the resistance R; i.e.,

$$X \approx R \approx \frac{l}{\sigma(\delta_s 2\pi a)} \tag{4.20}$$

This is unique to good conductors at AC; that is, we see no such reactance at DC. Because this reactance is positive, it is often referred to as an inductance. However, this is misleading since inductance refers to the ability of a structure to store energy in a magnetic field, and energy storage is decidedly *not* what is happening here. The similarity to inductance is simply that this reactance is positive, as is the reactance associated with inductance. As long as we keep this in mind, it is reasonable to model the reactance of the wire as an *equivalent* inductance:

$$L_{eq} \approx \frac{1}{2\pi f} \cdot \frac{l}{\sigma(\delta_s 2\pi a)} \tag{4.21}$$

Now substituting an expression for skin depth:

$$\begin{aligned} L_{eq} &\approx \frac{1}{2\pi f} \cdot \sqrt{\frac{\pi f \mu}{\sigma}} \cdot \frac{l}{2\pi a} \\ &= \frac{1}{4\pi^{3/2}} \sqrt{\frac{\mu}{\sigma f}} \cdot \frac{l}{a} \end{aligned} \tag{4.22}$$

for a wire having a circular cross-section with $\delta_s \ll a$. The utility of this description is that it facilitates the modeling of wire reactance as an inductance in an equivalent circuit. Summarizing:

A practical wire may be modeled using an equivalent circuit consisting of an ideal resistor (Equation 4.17) in series with an ideal inductor (Equation 4.22). Whereas resistance increases with the square root of frequency, inductance *decreases* with the square root of frequency.

If the positive reactance of a wire is not due to physical inductance, then to what physical mechanism shall we attribute this effect? A wire has reactance because there is a phase shift between potential and current. This is apparent by comparing Equation 4.6 to Equation 4.10. This is the same phase shift that was found to exist between the electric and magnetic fields propagating in a good conductor, as explained in Section 3.11.

Example

Example 4.2

Equivalent inductance of the inner conductor of RG-59.

Elsewhere in the book we worked out that the inductance per unit length L' of RG-59 coaxial cable was about 370 nH/m. We calculated this from magnetostatic considerations, so the reactance associated with skin effect is not included in this estimate. Let's see how L' is affected by skin effect for the inner conductor. Using Equation 4.22 with $\mu = \mu_0$, $\sigma \cong 2.28 \times 10^7$ S/m, and $a \cong 0.292$ mm, we find

$$L_{eq} \approx \left(3.61 \times 10^{-5} \text{ H}{\cdot}\text{m}^{-1}{\cdot}\text{Hz}^{1/2}\right) \frac{l}{\sqrt{f}} \tag{4.23}$$

Per unit length:

$$L'_{eq} \triangleq \frac{L_{eq}}{l} \approx \frac{3.61 \times 10^{-5} \text{ H}{\cdot}\text{Hz}^{1/2}}{\sqrt{f}} \tag{4.24}$$

This equals the magnetostatic inductance per unit length (≈ 370 nH/m) at $f \approx 9.52$ kHz, and decreases with increasing frequency.

Summarizing: The equivalent inductance associated with skin effect is as important as the magnetostatic inductance in the kHz regime, and becomes gradually less important with increasing frequency.

Recall that the phase velocity in a low-loss transmission line is approximately $1/\sqrt{L'C'}$. This means that skin effect causes the phase velocity in such lines to decrease with decreasing frequency. In other words:

Skin effect in the conductors comprising common transmission lines leads to a form of dispersion in which higher frequencies travel faster than lower frequencies.

This phenomenon is known as *chromatic dispersion*, or simply "dispersion," and leads to significant distortion for signals having large bandwidths.

Additional Reading:

- "Skin effect (https://en.wikipedia.org/wiki/Skin_effect)" on Wikipedia.

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1. To be clear, this is the "point form" of Ohm's law, as opposed to the circuit theory form ($V = IR$). ↩
2. If this is not clear, recall that the electric field vector must point away from positive charge (thus, the $+$ terminal). ↩
3. This is in contrast to other ways that voltage and current can be related; for example, the non-linear V-I characteristic of a diode, which is not governed by Ohm's law. ↩

4.3 Surface Impedance

In Section 4.2, we derived the following expression for the impedance Z of a good conductor having width W, length l, and which is infinitely deep:

$$Z \approx \frac{1+j}{\sigma\delta_s} \cdot \frac{l}{W} \quad \text{(AC case)} \tag{4.25}$$

where σ is conductivity (SI base units of S/m) and δ_s is skin depth. Note that δ_s and σ are constitutive parameters of material, and do not depend on geometry; whereas l and W describe geometry. With this in mind, we define the *surface impedance* Z_S as follows:

$$\boxed{Z_S \triangleq \frac{1+j}{\sigma\delta_s}} \tag{4.26}$$

so that

$$Z \approx Z_S \frac{l}{W} \tag{4.27}$$

Unlike the terminal impedance Z, Z_S is strictly a materials property. In this way, it is like the intrinsic or "wave" impedance η, which is also a materials property. Although the units of Z_S are those of impedance (i.e., ohms), surface impedance is usually indicated as having units of "$\Omega/\Box$" ("ohms per square") to prevent confusion with the terminal impedance. Summarizing:

> Surface impedance Z_S (Equation 4.26) is a materials property having units of $\Omega/\Box$, and which characterizes the AC impedance of a material independently of the length and width of the material.

Surface impedance is often used to specify sheet materials used in the manufacture of electronic and semiconductor devices, where the real part of the surface impedance is more commonly known as the *surface resistance* or *sheet resistance*.

Additional Reading:

- "Sheet resistance (http://en.wikipedia.org/wiki/Sheet_resistance)" on Wikipedia.

End Matter

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