3 Wave Propagation in General Media

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3.1 Poynting's Theorem

Despite the apparent complexity of electromagnetic theory, there are in fact merely four ways that electromagnetic energy can be manipulated. Electromagnetic energy can be:

- Transferred; i.e., conveyed by transmission lines or in waves;
- Stored in an electric field (capacitance);
- Stored in a magnetic field (inductance); or
- Dissipated (converted to heat; i.e., resistance).

Poynting's theorem is an expression of conservation of energy that elegantly relates these various possibilities. Once recognized, the theorem has important applications in the analysis and design of electromagnetic systems. Some of these emerge from the derivation of the theorem, as opposed to the unsurprising result. So, let us now derive the theorem.

We begin with a statement of the theorem. Consider a volume $\mathcal{V}$ that may contain materials and structures in any combination. This is crudely depicted in Figure 3.1.

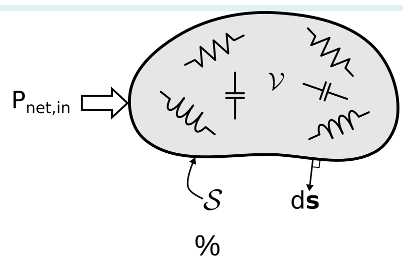

Figure 3.1. *Poynting's theorem describes the fate of power entering a region $\mathcal{V}$ consisting of materials and structures capable of storing and dissipating energy.*

Also recall that power is the time rate of change of energy. Then:

$$P_{net,in} = P_E + P_M + P_\Omega \tag{3.1}$$

where $P_{net,in}$ is the net power flow into $\mathcal{V}$, P_E is the power associated with energy storage in electric fields within $\mathcal{V}$, P_M is the power associated with energy storage in magnetic fields within $\mathcal{V}$, and P_Ω is the power dissipated (converted to heat) in $\mathcal{V}$.

Some preliminary mathematical results. We now derive two mathematical relationships that will be useful later. Let $\mathbf{E}$ be the electric field intensity (SI base units of V/m) and let $\mathbf{D}$ be the electric flux density (SI base units of C/m^2). We require that the constituents of $\mathcal{V}$ be linear and time-invariant; therefore, $\mathbf{D} = \epsilon\mathbf{E}$ where the permittivity ϵ is constant with respect to time, but not necessarily with respect to position. Under these conditions, we find

$$\frac{\partial}{\partial t}\left(\mathbf{E}\cdot\mathbf{D}\right) = \frac{1}{\epsilon}\frac{\partial}{\partial t}\left(\mathbf{D}\cdot\mathbf{D}\right) \tag{3.2}$$

Note that $\mathbf{E}$ and $\mathbf{D}$ point in the same direction. Let $\hat{\mathbf{e}}$ be the unit vector describing this direction. Then, $\mathbf{E} = \hat{\mathbf{e}}E$ and $\mathbf{D} = \hat{\mathbf{e}}D$ where E and D are the scalar components of $\mathbf{E}$ and $\mathbf{D}$, respectively. Subsequently,

$$\begin{aligned}\frac{1}{\epsilon}\frac{\partial}{\partial t}\left(\mathbf{D}\cdot\mathbf{D}\right) &= \frac{1}{\epsilon}\frac{\partial}{\partial t}D^2 \\ &= \frac{1}{\epsilon}2D\frac{\partial}{\partial t}D \\ &= 2\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}\end{aligned} \tag{3.3}$$

Summarizing:

$$\frac{\partial}{\partial t}\left(\mathbf{E}\cdot\mathbf{D}\right) = 2\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} \tag{3.4}$$

which is the expression we seek. It is worth noting that the expressions on both sides of the equation have the same units, namely, those of power.

Using the same reasoning, we find:

$$\frac{\partial}{\partial t}\left(\mathbf{H}\cdot\mathbf{B}\right) = 2\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} \tag{3.5}$$

where $\mathbf{H}$ is the magnetic field intensity (SI base units of A/m) and $\mathbf{B}$ is the magnetic flux density (SI base units of T).

Derivation of the theorem. We begin with the differential form of Ampere’s law:

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \tag{3.6}$$

Taking the dot product with $\mathbf{E}$ on both sides:

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial}{\partial t}\mathbf{D} \tag{3.7}$$

Let's deal with the left side of this equation first. For this, we will employ a vector identity (Equation B.27, Appendix B.3). This identity states that for any two vector fields $\mathbf{F}$ and $\mathbf{G}$,

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \tag{3.8}$$

Substituting $\mathbf{E}$ for $\mathbf{F}$ and $\mathbf{H}$ for $\mathbf{G}$ and rearranging terms, we find

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{3.9}$$

Next, we invoke the Maxwell-Faraday equation:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{3.10}$$

Using this equation to replace the factor in the parentheses in the second term of Equation 3.9, we obtain:

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot \left(-\frac{\partial}{\partial t}\mathbf{B}\right) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{3.11}$$

Substituting this expression for the left side of Equation 3.7, we have

$$-\mathbf{H} \cdot \frac{\partial}{\partial t}\mathbf{B} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial}{\partial t}\mathbf{D} \tag{3.12}$$

Next we invoke the identities of Equations 3.4 and 3.5 to replace the first and last terms:

$$\begin{split} -\frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{H} \cdot \mathbf{B}\right) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \\ = \mathbf{E} \cdot \mathbf{J} + \frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{E} \cdot \mathbf{D}\right) \end{split} \tag{3.13}$$

Now we move the first term to the right-hand side and integrate both sides over the volume $\mathcal{V}$:

$$
\begin{aligned}
-\int_{\mathcal{V}} \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv \\
&= \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J}\ dv \\
&+ \int_{\mathcal{V}} \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D})\, dv \\
&+ \int_{\mathcal{V}} \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{B})\, dv
\end{aligned} \tag{3.14}
$$

The left side may be transformed into a surface integral using the divergence theorem:

$$
\int_{\mathcal{V}} \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv = \oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} \tag{3.15}
$$

where $\mathcal{S}$ is the closed surface that bounds $\mathcal{V}$ and $d\mathbf{s}$ is the outward-facing normal to this surface, as indicated in Figure 3.1. Finally, we exchange the order of time differentiation and volume integration in the last two terms:

$$
\begin{aligned}
-\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} \\
&= \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J}\ dv \\
&+ \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{D}\ dv \\
&+ \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathcal{V}} \mathbf{H} \cdot \mathbf{B}\ dv
\end{aligned} \tag{3.16}
$$

Equation 3.16 is Poynting's theorem. Each of the four terms has the particular physical interpretation identified in Equation 3.1, as we will now demonstrate.

Power dissipated by ohmic loss. The first term of the right side of Equation 3.16 is

$$
\boxed{P_{\Omega} \triangleq \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J}\ dv} \tag{3.17}
$$

Equation 3.17 is *Joule's law*. Joule's law gives the power dissipated due to finite conductivity of material. The role of conductivity (σ, SI base units of S/m) can be made explicit using the relationship $\mathbf{J} = \sigma\mathbf{E}$ (Ohm's law) in Equation 3.17, which yields:

$$
P_{\Omega} = \int_{\mathcal{V}} \mathbf{E} \cdot \sigma\mathbf{E}\ dv = \int_{\mathcal{V}} \sigma \left|\mathbf{E}\right|^2\ dv \tag{3.18}
$$

P_Ω is due to the conversion of the electric field into conduction current, and subsequently into heat. This mechanism is commonly known as *ohmic loss* or *joule heating*. It is worth noting that this expression has the expected units. For example, in Equation 3.17, $\mathbf{E}$ (SI base units of V/m) times $\mathbf{J}$ (SI base units of A/m^2) yields a quantity with units of W/m^3; i.e., power per unit volume, which is power density. Then integration over volume yields units of W; hence, power.

Energy storage in electric and magnetic fields. The second term on the right side of Equation 3.16 is:

$$\boxed{P_E \triangleq \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}} \mathbf{E}\cdot\mathbf{D}\ dv} \tag{3.19}$$

Recall $\mathbf{D} = \epsilon\mathbf{E}$, where ϵ is the permittivity (SI base units of F/m) of the material. Thus, we may rewrite the previous equation as follows:

$$\begin{aligned} P_E &= \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}} \mathbf{E}\cdot\epsilon\mathbf{E}\ dv \\ &= \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}} \epsilon\left|\mathbf{E}\right|^2\ dv \\ &= \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{\mathcal{V}} \epsilon\left|\mathbf{E}\right|^2\ dv\right) \end{aligned} \tag{3.20}$$

The quantity in parentheses is W_e, the energy stored in the electric field within $\mathcal{V}$.

The third term on the right side of Equation 3.16 is:

$$\boxed{P_M \triangleq \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}} \mathbf{H}\cdot\mathbf{B}\ dv} \tag{3.21}$$

Recall $\mathbf{B} = \mu\mathbf{H}$, where μ is the permeability (SI base units of H/m) of the material. Thus, we may rewrite the previous equation as follows:

$$\begin{aligned} P_M &= \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}} \mathbf{H}\cdot\mu\mathbf{H}\ dv \\ &= \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}} \mu\left|\mathbf{H}\right|^2\ dv \\ &= \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{\mathcal{V}} \mu\left|\mathbf{H}\right|^2\ dv\right) \end{aligned} \tag{3.22}$$

The quantity in parentheses is W_m, the energy stored in the magnetic field within $\mathcal{V}$.

Power flux through $\mathcal{S}$. The left side of Equation 3.16 is

$$\boxed{P_{net,in} \triangleq -\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s}} \tag{3.23}$$

The quantity $\mathbf{E} \times \mathbf{H}$ is the *Poynting vector*, which quantifies the spatial power density (SI base units of W/m^2) of an electromagnetic wave and the direction in which it propagates. The reader has likely already encountered this concept. Regardless, we'll confirm this interpretation of the quantity $\mathbf{E} \times \mathbf{H}$ in Section 3.2. For now, observe that integration of the Poynting vector over $\mathcal{S}$ as indicated in Equation 3.23 yields the total power flowing out of $\mathcal{V}$ through $\mathcal{S}$. The negative sign in Equation 3.23 indicates that the combined quantity represents power flow *in* to $\mathcal{V}$ through $\mathcal{S}$. Finally, note the use of a single quantity $P_{net,in}$ does *not* imply that power is entirely inward-directed or outward-directed. Rather, $P_{net,in}$ represents the *net* flux; i.e., the *sum* of the inward- and outward-flowing power.

Summary. Equation 3.16 may now be stated concisely as follows:

$$\boxed{P_{net,in} = P_{\Omega} + \frac{\partial}{\partial t} W_e + \frac{\partial}{\partial t} W_m} \tag{3.24}$$

Poynting's theorem (Equation 3.24, with Equations 3.23, 3.17, 3.19, and 3.21) states that the net electromagnetic power flowing into a region of space may be either dissipated, or used to change the energy stored in electric and magnetic fields within that region.

Since we derived this result directly from the general form of Maxwell's equations, these three possibilities are *all* the possibilities allowed by classical physics, so this is a statement of conservation of power.

Finally, note the theorem also works in reverse; i.e., the net electromagnetic power flowing *out* of a region of space must have originated from some active source (i.e., the reverse of power dissipation) or released from energy stored in electric or magnetic fields.

Additional Reading:

- "Poynting vector (https://en.wikipedia.org/wiki/Poynting_vector)" on Wikipedia.

3.2 Poynting Vector

In this section, we use Poynting's theorem (Section 3.1) to confirm the interpretation of the Poynting vector

$$\boxed{\mathbf{S} \triangleq \mathbf{E} \times \mathbf{H}} \tag{3.25}$$

as the spatial power density (SI base units of W/m^2) and the direction of power flow.

Figure 3.2 shows a uniform plane wave incident on a homogeneous and lossless region $\mathcal{V}$ defined by the closed right cylinder $\mathcal{S}$. The axis of the cylinder is aligned along the direction of propagation $\hat{\mathbf{k}}$ of the plane wave. The ends of the cylinder are planar and perpendicular to $\hat{\mathbf{k}}$.

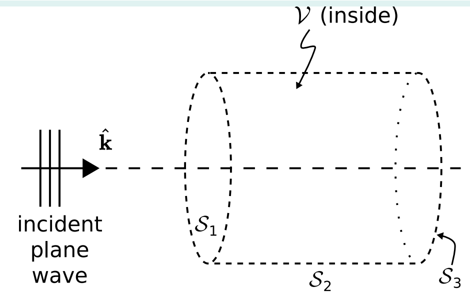

Figure 3.2. *A uniform plane wave incident on a region bounded by the cylindrical surface $\mathcal{S}$.*

Now let us apply Poynting's theorem:

$$P_{net,in} = P_\Omega + \frac{\partial}{\partial t} W_e + \frac{\partial}{\partial t} W_m \tag{3.26}$$

Since the region is lossless, $P_\Omega = 0$. Presuming no other electric or magnetic fields, W_e and W_m must also be zero. Consequently, $P_{net,in} = 0$. However, this does *not* mean that there is no power flowing into $\mathcal{V}$. Instead, $P_{net,in} = 0$ merely indicates that the *net* power flowing into $\mathcal{V}$ is zero. Thus, we might instead express the result for the present scenario as follows:

$$P_{net,in} = P_{in} - P_{out} = 0 \tag{3.27}$$

where P_{in} and P_{out} indicate power flow explicitly into and explicitly out of $\mathcal{V}$ as separate quantities.

Proceeding, let's ignore what we know about power flow in plane waves, and instead see where Poynting's theorem takes us. Here,

$$P_{net,in} \triangleq -\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = 0 \tag{3.28}$$

The surface $\mathcal{S}$ consists of three sides: The two flat ends, and the curved surface that connects them. Let us refer to these sides as $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, from left to right as shown in Figure 3.2. Then,

$$
\begin{aligned}
&-\int_{\mathcal{S}_1} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} \\
&-\int_{\mathcal{S}_2} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} \\
&-\int_{\mathcal{S}_3} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = 0
\end{aligned} \tag{3.29}
$$

On the curved surface $\mathcal{S}_2$, $d\mathbf{s}$ is everywhere perpendicular to $\mathbf{E} \times \mathbf{H}$ (i.e., perpendicular to $\hat{\mathbf{k}}$). Therefore, the second integral is equal to zero. On the left end $\mathcal{S}_1$, the outward-facing differential surface element $d\mathbf{s} = -\hat{\mathbf{k}} ds$. On the right end $\mathcal{S}_3$, $d\mathbf{s} = +\hat{\mathbf{k}} ds$. We are left with:

$$
\begin{aligned}
&\int_{\mathcal{S}_1} (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{k}} ds \\
&-\int_{\mathcal{S}_3} (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{k}} ds = 0
\end{aligned} \tag{3.30}
$$

Compare this to Equation 3.27. Since $\hat{\mathbf{k}}$ is, by definition, the direction in which $\mathbf{E} \times \mathbf{H}$ points, the first integral must be P_{in} and the second integral must be P_{out}. Thus,

$$
P_{in} = \int_{\mathcal{S}_1} (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{k}} ds \tag{3.31}
$$

and it follows that the magnitude and direction of $\mathbf{E} \times \mathbf{H}$, which we recognize as the Poynting vector, are spatial power density and direction of power flow, respectively.

The Poynting vector is named after English physicist J.H. Poynting, one of the co-discoverers of the concept. The fact that this vector points in the direction of power flow, and is therefore also a "pointing vector," is simply a remarkable coincidence.

Additional Reading:

- "Poynting vector (https://en.wikipedia.org/wiki/Poynting_vector)" on Wikipedia.

3.3 Wave Equations for Lossy Regions

The wave equations for electromagnetic propagation in lossless and source-free media, in differential phasor form, are:

$$
\nabla^2 \widetilde{\mathbf{E}} + \omega^2 \mu\epsilon \widetilde{\mathbf{E}} = 0 \tag{3.32}
$$

$$
\nabla^2 \widetilde{\mathbf{H}} + \omega^2 \mu\epsilon \widetilde{\mathbf{H}} = 0 \tag{3.33}
$$

The constant $\omega^2\mu\epsilon$ is labeled β^2, and β turns out to be the phase propagation constant.

Now, we wish to upgrade these equations to account for the possibility of loss. First, let's be clear on what we mean by "loss." Specifically, we mean the possibility of conversion of energy from the propagating wave into current, and subsequently to heat. This mechanism is described by Ohm's law:

$$\widetilde{\mathbf{J}} = \sigma \widetilde{\mathbf{E}} \tag{3.34}$$

where σ is conductivity and $\widetilde{\mathbf{J}}$ is conduction current density (SI base units of A/m^2). In the lossless case, σ is presumed to be zero (or at least negligible), so $\mathbf{J}$ is presumed to be zero. To obtain wave equations for media exhibiting significant loss, we cannot assume $\mathbf{J} = 0$.

To obtain equations that account for the possibility of significant conduction current, hence loss, we return to the phasor forms of Maxwell's equations:

$$\nabla \cdot \widetilde{\mathbf{E}} = \frac{\tilde{\rho}_v}{\epsilon} \tag{3.35}$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}} \tag{3.36}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0 \tag{3.37}$$

$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\epsilon\widetilde{\mathbf{E}} \tag{3.38}$$

where $\tilde{\rho}_v$ is the charge density. If the region of interest is source-free, then $\tilde{\rho}_v = 0$. However, we may not similarly suppress $\widetilde{\mathbf{J}}$ since σ may be non-zero. To make progress, let us identify the possible contributions to $\widetilde{\mathbf{J}}$ as follows:

$$\widetilde{\mathbf{J}} = \widetilde{\mathbf{J}}_{imp} + \widetilde{\mathbf{J}}_{ind} \tag{3.39}$$

where $\widetilde{\mathbf{J}}_{imp}$ represents *impressed* sources of current and $\widetilde{\mathbf{J}}_{ind}$ represents current which is *induced* by loss. An impressed current is one whose behavior is independent of other features, analogous to an independent current source in elementary circuit theory. In the absence of such sources, Equation 3.39 becomes:

$$\widetilde{\mathbf{J}} = 0 + \sigma\widetilde{\mathbf{E}} \tag{3.40}$$

Equation 3.38 may now be rewritten as:

$$\nabla \times \widetilde{\mathbf{H}} = \sigma\widetilde{\mathbf{E}} + j\omega\epsilon\widetilde{\mathbf{E}} \tag{3.41}$$

$$= (\sigma + j\omega\epsilon)\,\widetilde{\mathbf{E}} \tag{3.42}$$

$$= j\omega\epsilon_c\widetilde{\mathbf{E}} \tag{3.43}$$

where we defined the new constant ϵ_c as follows:

$$\epsilon_c \triangleq \epsilon - j\frac{\sigma}{\omega} \tag{3.44}$$

This constant is known as *complex permittivity*. In the lossless case, $\epsilon_c = \epsilon$; i.e., the imaginary part of $\epsilon_c \rightarrow 0$ so there is no difference between the physical permittivity ϵ and ϵ_c. The effect of the material's loss is represented as a non-zero imaginary component of the permittivity.

It is also common to express ϵ_c as follows:

$$\epsilon_c = \epsilon' - j\epsilon'' \tag{3.45}$$

where the real-valued constants ϵ' and ϵ'' are in this case:

$$\epsilon' = \epsilon \tag{3.46}$$
$$\epsilon'' = \frac{\sigma}{\omega} \tag{3.47}$$

This alternative notation is useful for three reasons. First, some authors use the symbol ϵ to refer to *both* physical permittivity *and* complex permittivity. In this case, the "$\epsilon' - j\epsilon''$" notation is helpful in mitigating confusion. Second, it is often more convenient to specify ϵ'' at a frequency than it is to specify σ, which may also be a function of frequency. In fact, in some applications the loss of a material is most conveniently specified using the ratio ϵ''/ϵ' (known as *loss tangent*, for reasons explained elsewhere). Finally, it turns out that *nonlinearity* of permittivity can *also* be accommodated as an imaginary component of the permittivity. The "ϵ''" notation allows us to accommodate both effects – nonlinearity and conductivity – using common notation. In this section, however, we remain focused exclusively on conductivity.

Complex permittivity ϵ_c (SI base units of F/m) describes the combined effects of permittivity and conductivity. Conductivity is represented as an imaginary-valued component of the permittivity.

Returning to Equations 3.35–3.38, we obtain:

$$\nabla \cdot \widetilde{\mathbf{E}} = 0 \tag{3.48}$$
$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}} \tag{3.49}$$
$$\nabla \cdot \widetilde{\mathbf{H}} = 0 \tag{3.50}$$
$$\nabla \times \widetilde{\mathbf{H}} = j\omega\epsilon_c\widetilde{\mathbf{E}} \tag{3.51}$$

These equations are *identical* to the corresponding equations for the lossless case, with the exception that ϵ has been replaced by ϵ_c. Similarly, we may replace the factor $\omega^2\mu\epsilon$ in Equations 3.32 and 3.33, yielding:

$$\nabla^2\widetilde{\mathbf{E}} + \omega^2\mu\epsilon_c\widetilde{\mathbf{E}} = 0 \tag{3.52}$$
$$\nabla^2\widetilde{\mathbf{H}} + \omega^2\mu\epsilon_c\widetilde{\mathbf{H}} = 0 \tag{3.53}$$

In the lossless case, $\omega^2\mu\epsilon_c \to \omega^2\mu\epsilon$, which is β^2 as expected. For the general (i.e., possibly lossy) case, we shall make an analogous definition

$$\gamma^2 \triangleq -\omega^2\mu\epsilon_c \tag{3.54}$$

such that the wave equations may now be written as follows:

$$\boxed{\nabla^2\widetilde{\mathbf{E}} - \gamma^2\widetilde{\mathbf{E}} = 0} \tag{3.55}$$

$$\boxed{\nabla^2\widetilde{\mathbf{H}} - \gamma^2\widetilde{\mathbf{H}} = 0} \tag{3.56}$$

Note the minus sign in Equation 3.54, and the associated changes of signs in Equations 3.55 and 3.56. For the moment, this choice of sign may be viewed as arbitrary – we would have been equally justified in choosing the opposite sign for the definition of γ^2. However, the choice we have made is customary, and yields some notational convenience that will become apparent later.

In the lossless case, β is the phase propagation constant, which determines the rate at which the phase of a wave progresses with increasing distance along the direction of propagation. Given the similarity of Equations 3.55 and 3.56 to Equations 3.32 and 3.33, respectively, the constant γ must play a similar role. So, we are motivated to find an expression for γ. At first glance, this is simple: $\gamma = \sqrt{\gamma^2}$. However, recall that every number has two square roots. When one declares $\beta = \sqrt{\beta^2} = \omega\sqrt{\mu\epsilon}$, there is no concern since β is, by definition, positive; therefore, one knows to take the positive-valued square root. In contrast, γ^2 is complex-valued, and so the two possible values of $\sqrt{\gamma^2}$ are both potentially complex-valued. We are left without clear guidance on which of these values is appropriate and physically-relevant. So we proceed with caution.

First, consider the special case that γ is purely imaginary; e.g., $\gamma = j\gamma''$ where γ'' is a real-valued constant. In this case, $\gamma^2 = -\left(\gamma''\right)^2$ and the wave equations become

$$\nabla^2\widetilde{\mathbf{E}} + \left(\gamma''\right)^2\widetilde{\mathbf{E}} = 0 \tag{3.57}$$
$$\nabla^2\widetilde{\mathbf{H}} + \left(\gamma''\right)^2\widetilde{\mathbf{H}} = 0 \tag{3.58}$$

Comparing to Equations 3.32 and 3.33, we see γ'' plays the exact same role as β; i.e., γ'' is the phase propagation constant for whatever wave we obtain as the solution to the above equations. Therefore, let us make that definition formally:

$$\beta \triangleq \mathrm{Im}\left\{\gamma\right\} \tag{3.59}$$

Be careful: Note that we are *not* claiming that γ'' in the possibly-lossy case is equal to $\omega^2\mu\epsilon$. Instead, we are asserting exactly the opposite; i.e., that there is a phase propagation constant in the general (possibly lossy) case, and we should find that this constant simplifies to $\omega^2\mu\epsilon$ in the lossless case.

Now we make the following definition for the real component of γ:

$$\alpha \triangleq \mathrm{Re}\left\{\gamma\right\} \tag{3.60}$$

Such that

$$\gamma = \alpha + j\beta \tag{3.61}$$

where α and β are real-valued constants.

Now, it is possible to determine γ explicitly in terms of ω and the constitutive properties of the material. First, note:

$$\begin{aligned} \gamma^2 &= \left(\alpha + j\beta\right)^2 \\ &= \alpha^2 - \beta^2 + j2\alpha\beta \end{aligned} \tag{3.62}$$

Expanding Equation 3.54 using Equations 3.45–3.47, we obtain:

$$\begin{aligned} \gamma^2 &= -\omega^2\mu\left(\epsilon - j\frac{\sigma}{\omega}\right) \\ &= -\omega^2\mu\epsilon + j\omega\mu\sigma \end{aligned} \tag{3.63}$$

The real and imaginary parts of Equations 3.62 and 3.63 must be equal. Enforcing this equality yields the following equations:

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon \tag{3.64}$$

$$2\alpha\beta = \omega\mu\sigma \tag{3.65}$$

Equations 3.64 and 3.65 are independent simultaneous equations that may be solved for α and β. Sparing the reader the remaining steps, which are purely mathematical, we find:

$$\boxed{\alpha = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} - 1 \right] \right\}^{1/2}} \tag{3.66}$$

$$\boxed{\beta = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} + 1 \right] \right\}^{1/2}} \tag{3.67}$$

Equations 3.66 and 3.67 can be verified by confirming that Equations 3.64 and 3.65 are satisfied. It is also useful to confirm that the expected results are obtained in the lossless case. In the lossless case, $\sigma = 0$, so $\epsilon'' = 0$. Subsequently, Equation 3.66 yields $\alpha = 0$ and Equation 3.67 yields $\beta = \omega\sqrt{\mu\epsilon}$, as expected.

The electromagnetic wave equations accounting for the possibility of lossy media are Equations 3.55 and 3.56 with $\gamma = \alpha + j\beta$, where α and β are the positive real-valued constants determined by Equations 3.66 and 3.67, respectively.

We conclude this section by pointing out a very useful analogy to transmission line theory. In the section "Wave Propagation on a TEM Transmission Line,"[1] we found that the potential and current along a transverse electromagnetic (TEM) transmission line satisfy the same wave equations that we have developed in this section, having a complex-valued propagation constant $\gamma = \alpha + j\beta$, and the same physical interpretation of β as the phase propagation constant. As is explained in another section, α too has the same interpretation in both applications – that is, as the *attenuation constant*.

Additional Reading:

- "Electromagnetic Wave Equation (https://en.wikipedia.org/wiki/Electromagnetic_wave_equation)" on Wikipedia.

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1. This section may appear in a different volume, depending on the version of this book. ↩

3.4 Complex Permittivity

The relationship between electric field intensity $\mathbf{E}$ (SI base units of V/m) and electric flux density $\mathbf{D}$ (SI base units of C/m^2) is:

$$\mathbf{D} = \epsilon \mathbf{E} \tag{3.68}$$

where ϵ is the permittivity (SI base units of F/m). In simple media, ϵ is a real positive value which does not depend on the time variation of $\mathbf{E}$. That is, the response ($\mathbf{D}$) to a change in $\mathbf{E}$ is observed instantaneously and without delay.

In practical materials, however, the change in $\mathbf{D}$ in response to a change in $\mathbf{E}$ may depend on the manner in which $\mathbf{E}$ changes. The response may not be instantaneous, but rather might take some time to fully manifest. This behavior can be modeled using the following generalization of Equation 3.68:

$$\mathbf{D} = a_0\mathbf{E} + a_1\frac{\partial}{\partial t}\mathbf{E} + a_2\frac{\partial^2}{\partial t^2}\mathbf{E} + a_3\frac{\partial^3}{\partial t^3}\mathbf{E} + ... \tag{3.69}$$

where a_0, a_1, a_2, and so on are real-valued constants, and the number of terms is infinite. In practical materials, the importance of the terms tends to diminish with increasing order. Thus, it is common that only the first few terms are significant. In many applications involving common materials, only the first term is significant; i.e., $a_0 \approx \epsilon$ and $a_n \approx 0$ for $n \geq 1$. As we shall see in a moment, this distinction commonly depends on frequency.

In the phasor domain, differentiation with respect to time becomes multiplication by $j\omega$. Thus, Equation 3.69 becomes

$$\begin{aligned}\widetilde{\mathbf{D}} &= a_0\widetilde{\mathbf{E}} + a_1\left(j\omega\right)\widetilde{\mathbf{E}} + a_2\left(j\omega\right)^2\widetilde{\mathbf{E}} + a_3\left(j\omega\right)^3\widetilde{\mathbf{E}} + ... \\ &= a_0\widetilde{\mathbf{E}} + j\omega a_1\widetilde{\mathbf{E}} - \omega^2 a_2\widetilde{\mathbf{E}} - j\omega^3 a_3\widetilde{\mathbf{E}} + ... \\ &= \left(a_0 + j\omega a_1 - \omega^2 a_2 - j\omega^3 a_3 + ...\right)\widetilde{\mathbf{E}}\end{aligned} \tag{3.70}$$

Note that the factor in parentheses is a complex-valued number which depends on frequency ω and materials parameters a_0, a_1, ... We may summarize this as follows:

$$\widetilde{\mathbf{D}} = \epsilon_c\widetilde{\mathbf{E}} \tag{3.71}$$

where ϵ_c is a complex-valued constant that depends on frequency.

The notation "ϵ_c" is used elsewhere in this book (e.g., Section 3.3) to represent a generalization of simple permittivity that accommodates loss associated with non-zero conductivity σ. In Section 3.3, we defined

$$\epsilon_c \triangleq \epsilon' - j\epsilon'' \tag{3.72}$$

where $\epsilon' \triangleq \epsilon$ (i.e., real-valued, simple permittivity) and $\epsilon'' \triangleq \sigma/\omega$ (i.e., the effect of non-zero conductivity). Similarly, ϵ_c in Equation 3.71 can be expressed as

$$\epsilon_c \triangleq \epsilon' - j\epsilon'' \tag{3.73}$$

but in this case

$$\epsilon' \triangleq a_0 - \omega^2 a_2 + \ldots \tag{3.74}$$

and

$$\epsilon'' \triangleq -\omega a_1 + \omega^3 a_3 - \ldots \tag{3.75}$$

> When expressed as phasors, the temporal relationship between electric flux density and electric field intensity can be expressed as a complex-valued permittivity.

Thus, we now see two possible applications for the concept of complex permittivity: Modeling the delayed response of $\mathbf{D}$ to changing $\mathbf{E}$, as described above; and modeling loss associated with non-zero conductivity. In practical work, it may not always be clear precisely what combination of effects the complex permittivity $\epsilon_c = \epsilon' - j\epsilon''$ is taking into account. For example, if ϵ_c is obtained by measurement, both delayed response and conduction loss may be represented. Therefore, it is not reasonable to assume a value of ϵ'' obtained by measurement represents only conduction loss (i.e., is equal to σ/ω); in fact, the measurement also may include a significant frequency-dependent contribution associated with the delayed response behavior identified in this section.

An example of the complex permittivity of a typical dielectric material is shown in Figure 3.3. Note the frequency dependence is quite simple and slowly-varying at frequencies below 1 GHz or so, but becomes relatively complex at higher frequencies.

Figure 3.3. *The relative contributions of the real and imaginary components of permittivity for a typical dielectric material (in this case, a polymer).*

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Additional Reading:

- "Permittivity (https://en.wikipedia.org/wiki/Permittivity)" on Wikipedia.

3.5 Loss Tangent

In Section 3.3, we found that the effect of loss due to non-zero conductivity σ could be concisely quantified using the ratio

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} \tag{3.76}$$

where ϵ' and ϵ'' are the real and imaginary components of the complex permittivity ϵ_c, and $\epsilon' \triangleq \epsilon$. In this section, we explore this relationship in greater detail.

Recall Ampere's law in differential (but otherwise general) form:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \tag{3.77}$$

The first term on the right is conduction current, whereas the second term on the right is displacement current. In the phasor domain, differentiation with respect to time ($\partial/\partial t$) becomes multiplication by $j\omega$. Therefore, the phasor form of Equation 3.77 is

$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\widetilde{\mathbf{D}} \tag{3.78}$$

Also recall that $\widetilde{\mathbf{J}} = \sigma\widetilde{\mathbf{E}}$ and that $\widetilde{\mathbf{D}} = \epsilon\widetilde{\mathbf{E}}$. Thus,

$$\nabla \times \widetilde{\mathbf{H}} = \sigma\widetilde{\mathbf{E}} + j\omega\epsilon\widetilde{\mathbf{E}} \tag{3.79}$$

Interestingly, the total current is the sum of a real-valued conduction current and an imaginary-valued displacement current. This is shown graphically in Figure 3.4.

Figure 3.4. *In the phasor domain, the total current is the sum of a real-valued conduction current and an imaginary-valued displacement current.*

Note that the angle δ indicated in Figure 3.4 is given by

$$\boxed{\tan\delta \triangleq \frac{\sigma}{\omega\epsilon}} \tag{3.80}$$

The quantity $\tan\delta$ is referred to as the *loss tangent*. Note that loss tangent is zero for a lossless ($\sigma \equiv 0$) material, and increases with increasing loss. Thus, loss tangent provides an alternative way to quantify the effect of loss on the electromagnetic field within a material.

Loss tangent presuming only ohmic (conduction) loss is given by Equation 3.80.

Comparing Equation 3.80 to Equation 3.76, we see loss tangent can equivalently be calculated as

$$\tan\delta = \frac{\epsilon''}{\epsilon} \tag{3.81}$$

and subsequently interpreted as shown in Figure 3.5.

Figure 3.5. *Loss tangent defined in terms of the real and imaginary components of the complex permittivity ϵ_c.*

The discussion in this section has assumed that ϵ_c is complex-valued solely due to ohmic loss. However, it is explained in Section 3.4 that permittivity may also be complex-valued as a way to model delay in the response of $\mathbf{D}$ to changing $\mathbf{E}$. Does the concept of loss tangent apply also in this case? Since the math does not distinguish between permittivity which is complex due to loss and permittivity which is complex due to delay, subsequent mathematically-derived results apply in either case. On the other hand, there may be potentially significant differences in the physical manifestation of these effects. For example, a material having large loss tangent due to ohmic loss might become hot when a large electric field is applied, whereas a material having large loss tangent due to delayed response might not. Summarizing:

> The expression for loss tangent given by Equation 3.81 and Figure 3.5 does not distinguish between ohmic loss and delayed response.

Additional Reading:

- "Dielectric loss (https://en.wikipedia.org/wiki/Dielectric_loss)" on Wikipedia.

3.6 Plane Waves in Lossy Regions

The electromagnetic wave equations for source-free regions consisting of possibly-lossy material are (see Section 3.3):

$$\nabla^2\widetilde{\mathbf{E}} - \gamma^2\widetilde{\mathbf{E}} = 0 \tag{3.82}$$

$$\nabla^2\widetilde{\mathbf{H}} - \gamma^2\widetilde{\mathbf{H}} = 0 \tag{3.83}$$

where

$$\gamma^2 \triangleq -\omega^2\mu\epsilon_c \tag{3.84}$$

We now turn our attention to the question, what are the characteristics of waves that propagate in these conditions? As in the lossless case, these equations permit waves having a variety of geometries including plane waves, cylindrical waves, and spherical waves. In this section, we will consider the important special case of uniform plane waves.

To obtain the general expression for the uniform plane wave solution, we follow precisely the same procedure described in the section “Uniform Plane Waves: Derivation.”[1] Although that section presumed lossless media, the only difference in the present situation is that the real-valued constant $+\beta^2$ is replaced with the complex-valued constant $-\gamma^2$. Thus, we obtain the desired solution through a simple modification of the solution for the lossless case. For a wave exhibiting uniform magnitude and phase in planes of constant z, we find that the electric field is:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y \tag{3.85}$$

where

$$\widetilde{E}_x = E_{x0}^{+}e^{-\gamma z} + E_{x0}^{-}e^{+\gamma z} \tag{3.86}$$
$$\widetilde{E}_y = E_{y0}^{+}e^{-\gamma z} + E_{y0}^{-}e^{+\gamma z} \tag{3.87}$$

where the complex-valued coefficients E_{x0}^{+}, E_{x0}^{-}, E_{y0}^{+}, and E_{y0}^{-} are determined by boundary conditions (possibly including sources) outside the region of interest. This result can be confirmed by verifying that Equations 3.86 and 3.87 each satisfy Equation 3.82. Also, it may be helpful to note that these expressions are identical to those obtained for the voltage and current in lossy transmission lines, as described in the section “Wave Equation for a TEM Transmission Line.”[2]

Let’s consider the special case of an $\hat{\mathbf{x}}$-polarized plane wave propagating in the $+\hat{\mathbf{z}}$ direction:

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}E_{x0}^{+}e^{-\gamma z} \tag{3.88}$$

We established in Section 3.3 that γ may be written explicitly in terms of its real and imaginary components as follows:

$$\gamma = \alpha + j\beta \tag{3.89}$$

where α and β are positive real-valued constants depending on frequency (ω) and constitutive properties of the medium; i.e., permittivity, permeability, and conductivity. Thus:

$$\begin{aligned}\widetilde{\mathbf{E}} &= \hat{\mathbf{x}}E_{x0}^{+}e^{-(\alpha+j\beta)z} \\ &= \hat{\mathbf{x}}E_{x0}^{+}e^{-\alpha z}e^{-j\beta z}\end{aligned} \tag{3.90}$$

Observe that the variation of phase with distance is determined by β through the factor $e^{-j\beta z}$; thus, β is the phase propagation constant and plays precisely the same role as in the lossless case. Observe also that the variation in magnitude is determined by α through the real-valued factor $e^{-\alpha z}$. Specifically, magnitude is reduced inverse-exponentially with increasing distance along the direction of propagation. Thus, α is the attenuation constant and plays precisely the same role as the attenuation constant for a lossy transmission line.

> The presence of loss in material gives rise to a real-valued factor $e^{-\alpha z}$ which describes the attenuation of the wave with propagation (in this case, along z) in the material.

We may continue to exploit the similarity of the potentially-lossy and lossless plane wave results to quickly ascertain the characteristics of the magnetic field. In particular, the plane wave relationships apply exactly as they do in the lossless case. These relationships are:

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\hat{\mathbf{k}} \times \widetilde{\mathbf{E}} \tag{3.91}$$

$$\widetilde{\mathbf{E}} = -\eta\hat{\mathbf{k}} \times \widetilde{\mathbf{H}} \tag{3.92}$$

where $\hat{\mathbf{k}}$ is the direction of propagation and η is the wave impedance. In the lossless case, $\eta = \sqrt{\mu/\epsilon}$; however, in the possibly-lossy case we must replace $\epsilon = \epsilon'$ with $\epsilon_c = \epsilon' - j\epsilon''$. Thus:

$$\begin{aligned} \eta \rightarrow \eta_c &= \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon' - j\epsilon''}} \\ &= \sqrt{\frac{\mu}{\epsilon'}}\sqrt{\frac{1}{1 - j\left(\epsilon''/\epsilon'\right)}} \end{aligned} \tag{3.93}$$

Thus:

$$\boxed{\eta_c = \sqrt{\frac{\mu}{\epsilon'}} \cdot \left[1 - j\frac{\epsilon''}{\epsilon'}\right]^{-1/2}} \tag{3.94}$$

Remarkably, we find that the wave impedance for a lossy material is equal to $\sqrt{\mu/\epsilon}$ – the wave impedance we would calculate if we neglected loss (i.e., assumed $\sigma = 0$) – times a correction factor that accounts for the loss. This correction factor is complex-valued; therefore, $\mathbf{E}$ and $\mathbf{H}$ are not in phase when propagating through lossy material. We now see that in the phasor domain:

$$\widetilde{\mathbf{H}} = \frac{1}{\eta_c}\hat{\mathbf{k}} \times \widetilde{\mathbf{E}} \tag{3.95}$$

$$\widetilde{\mathbf{E}} = -\eta_c\hat{\mathbf{k}} \times \widetilde{\mathbf{H}} \tag{3.96}$$

The plane wave relationships in media which are possibly lossy are given by Equations 3.95 and 3.96, with the complex-valued wave impedance given by Equation 3.94.

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1. Depending on the version of this book, this section may appear in a different volume. ↩
2. Depending on the version of this book, this section may appear in a different volume. ↩

3.7 Wave Power in a Lossy Medium

In this section, we consider the power associated with waves propagating in materials which are potentially lossy; i.e., having conductivity σ significantly greater than zero. This topic has previously been considered in the section "Wave Power in a Lossless Medium" for the case in loss is not significant.[1] A review of that section may be useful before reading this section.

Recall that the Poynting vector

$$\mathbf{S} \triangleq \mathbf{E} \times \mathbf{H} \tag{3.97}$$

indicates the power density (i.e., W/m^2) of a wave and the direction of power flow. This is "instantaneous" power, applicable to waves regardless of the way they vary with time. Often we are interested specifically in waves which vary sinusoidally, and which subsequently may be represented as phasors. In this case, the *time-average* Poynting vector is

$$\mathbf{S}_{ave} \triangleq \frac{1}{2}\text{Re}\left\{\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right\} \tag{3.98}$$

Further, we have already used this expression to find that the time-average power density for a sinusoidally-varying uniform plane wave in a lossless medium is simply

$$S_{ave} = \frac{|E_0|^2}{2\eta} \quad \text{(lossless case)} \tag{3.99}$$

where $|E_0|$ is the peak (as opposed to RMS) magnitude of the electric field intensity phasor, and η is the wave impedance.

Let us now use Equation 3.98 to determine the expression corresponding to Equation 3.99 in the case of possibly-lossy media. We may express the electric and magnetic field intensities of a uniform plane wave as

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}} E_0 e^{-\alpha z} e^{-j\beta z} \tag{3.100}$$

and

$$\widetilde{\mathbf{H}} = \hat{\mathbf{y}} \frac{E_0}{\eta_c} e^{-\alpha z} e^{-j\beta z} \tag{3.101}$$

where α and β are the attenuation constant and phase propagation constant, respectively, and η_c is the complex-valued wave impedance. As written, these expressions describe a wave which is $+\hat{\mathbf{x}}$-polarized and propagates in the $+\hat{\mathbf{z}}$ direction. We make these choices for convenience only – as long as the medium is homogeneous and isotropic, we expect our findings to apply regardless of polarization and direction of propagation. Applying Equation 3.98:

$$\begin{aligned}
\mathbf{S}_{ave} &= \frac{1}{2} \mathrm{Re}\left\{ \widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^* \right\} \\
&= \frac{1}{2}\ \hat{\mathbf{z}}\ \mathrm{Re}\left\{ \frac{|E_0|^2}{\eta_c^*} e^{-2\alpha z} \right\} \\
&= \hat{\mathbf{z}} \frac{|E_0|^2}{2}\ \mathrm{Re}\left\{ \frac{1}{\eta_c^*} \right\} e^{-2\alpha z}
\end{aligned} \tag{3.102}$$

Because η_c is complex-valued when the material is lossy, we must proceed with caution. First, let us write η_c explicitly in terms of its magnitude $|\eta_c|$ and phase ψ_η:

$$\eta \triangleq |\eta|\, e^{j\psi_\eta} \tag{3.103}$$

Then:

$$\eta_c^* = |\eta_c|\, e^{-j\psi_\eta} \tag{3.104}$$
$$(\eta_c^*)^{-1} = |\eta_c|^{-1}\, e^{+j\psi_\eta} \tag{3.105}$$
$$\mathrm{Re}\left\{ (\eta_c^*)^{-1} \right\} = |\eta_c|^{-1} \cos\psi_\eta \tag{3.106}$$

Then Equation 3.102 may be written:

$$\boxed{\mathbf{S}_{ave} = \hat{\mathbf{z}} \frac{|E_0|^2}{2\,|\eta_c|} \; e^{-2\alpha z} \;\; \cos\psi_\eta} \tag{3.107}$$

The time-average power density of the plane wave described by Equation 3.100 in a possibly-lossy material is given by Equation 3.107.

As a check, note that this expression gives the expected result for lossless media; i.e., $\alpha = 0$, $|\eta_c| = \eta$, and $\psi_\eta = 0$. We now see that the effect of loss is that power density is now proportional to $(e^{-\alpha z})^2$, so, as expected, power density is proportional to the square of either $|\mathbf{E}|$ or $|\mathbf{H}|$. The result further indicates a one-time scaling of the power density by a factor of

$$\frac{|\eta|}{|\eta_c|}\cos\psi_\eta < 1 \tag{3.108}$$

relative to a medium without loss.

The reduction in power density due to non-zero conductivity is proportional to a distance-dependent factor $e^{-2\alpha z}$ and an additional factor that depends on the magnitude and phase of η_c.

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1. Depending on the version of this book, this section may appear in another volume. ↩

3.8 Decibel Scale for Power Ratio

In many disciplines within electrical engineering, it is common to evaluate the ratios of powers and power densities that differ by many orders of magnitude. These ratios could be expressed in scientific notation, but it is more common to use the logarithmic *decibel* (dB) scale in such applications.

In the conventional (linear) scale, the ratio of power P_1 to power P_0 is simply

$$G = \frac{P_1}{P_0} \quad \text{(linear units)} \tag{3.109}$$

Here, “G” might be interpreted as “power gain.” Note that $G < 1$ if $P_1 < P_0$ and $G > 1$ if $P_1 > P_0$.

In the decibel scale, the ratio of power P_1 to power P_0 is

$$\boxed{G \triangleq 10 \log_{10} \frac{P_1}{P_0} \quad \text{(dB)}} \tag{3.110}$$

where “dB” denotes a unitless quantity which is expressed in the decibel scale. Note that $G < 0$ dB (i.e., is “negative in dB”) if $P_1 < P_0$ and $G > 0$ dB if $P_1 > P_0$.

The power gain P_1/P_0 in dB is given by Equation 3.110.

Alternatively, one might choose to interpret a power ratio as a loss L with $L \triangleq 1/G$ in linear units, which is $L = -G$ when expressed in dB. Most often, but not always, engineers interpret a power ratio as “gain” if the output power is expected to be greater than input power (e.g., as expected for an amplifier) and as “loss” if output power is expected to be less than input power (e.g., as expected for a lossy transmission line).

Power loss L is the reciprocal of power gain G. Therefore, $L = -G$ when these quantities are expressed in dB.

Example

Example 3.1

Power loss from a long cable.

A 2 W signal is injected into a long cable. The power arriving at the other end of the cable is 10 μW. What is the power loss in dB?

In linear units:

$$G = \frac{10\ \mu\text{W}}{2\ \text{W}} = 5 \times 10^{-6} \quad \text{(linear units)}$$

In dB:

$$\begin{aligned} G &= 10 \log_{10}\left(5 \times 10^{-6}\right) \cong -53.0\ \text{dB} \\ L &= -G \cong \underline{+53.0\ \text{dB}} \end{aligned}$$

The decibel scale is used in precisely the same way to relate ratios of spatial power densities for waves. For example, the loss incurred when the spatial power density is reduced from S_0 (SI base units of W/m^2) to S_1 is

$$L = 10 \log_{10} \frac{S_0}{S_1} \quad \text{(dB)} \tag{3.111}$$

This works because the common units of m^{-2} in the numerator and denominator cancel, leaving a power ratio.

A common point of confusion is the proper use of the decibel scale to represent voltage or current ratios. To avoid confusion, simply refer to the definition expressed in Equation 3.110. For example, let's say $P_1 = V_1^2/R_1$ where V_1 is potential and R_1 is the impedance across which V_1 is defined. Similarly, let us define $P_0 = V_0^2/R_0$ where V_0 is potential and R_0 is the impedance across which V_0 is defined. Applying Equation 3.110:

$$\begin{aligned} G &\triangleq 10 \log_{10} \frac{P_1}{P_0} \quad \text{(dB)} \\ &= 10 \log_{10} \frac{V_1^2/R_1}{V_0^2/R_0} \quad \text{(dB)} \end{aligned} \tag{3.112}$$

Now, if $R_1 = R_0$, then

$$\begin{aligned} G &= 10\log_{10}\frac{V_1^2}{V_0^2} \quad \text{(dB)} \\ &= 10\log_{10}\left(\frac{V_1}{V_0}\right)^2 \quad \text{(dB)} \\ &= 20\log_{10}\frac{V_1}{V_0} \quad \text{(dB)} \end{aligned} \tag{3.113}$$

However, note that this is *not* true if $R_1 \neq R_0$.

A power ratio in dB is equal to $20\log_{10}$ of the voltage ratio only if the associated impedances are equal.

Adding to the potential for confusion on this point is the concept of *voltage gain* G_v:

$$G_v \triangleq 20\log_{10}\frac{V_1}{V_0} \quad \text{(dB)} \tag{3.114}$$

which applies regardless of the associated impedances. Note that $G_v = G$ only if the associated impedances are equal, and that these ratios are different otherwise. Be careful!

The decibel scale simplifies common calculations. Here's an example. Let's say a signal having power P_0 is injected into a transmission line having loss L. Then the output power $P_1 = P_0/L$ in linear units. However, in dB, we find:

$$\begin{aligned} 10\log_{10}P_1 &= 10\log_{10}\frac{P_0}{L} \\ &= 10\log_{10}P_0 - 10\log_{10}L \end{aligned}$$

Division has been transformed into subtraction; i.e.,

$$P_1 = P_0 - L \quad \text{(dB)} \tag{3.115}$$

This form facilitates easier calculation and visualization, and so is typically preferred.

Finally, note that the units of P_1 and P_0 in Equation 3.115 are not dB *per se*, but rather dB with respect to the original power units. For example, if P_1 is in mW, then taking $10\log_{10}$ of this quantity results in a quantity having units of dB relative to 1 mW. A power expressed in dB relative to 1 mW is said to have units of “dBm.” For example, “0 dBm” means 0 dB relative to 1 mW, which is simply 1 mW. Similarly $+10$ dBm is 10 mW, -10 dBm is 0.1 mW, and so on.

Additional Reading:

- "Decibel (https://en.wikipedia.org/wiki/Decibel)" on Wikipedia.
- "Scientific notation (https://en.wikipedia.org/wiki/Scientific_notation)" on Wikipedia.

3.9 Attenuation Rate

Attenuation rate is a convenient way to quantify loss in general media, including transmission lines, using the decibel scale.

Consider a transmission line carrying a wave in the $+z$ direction. Let P_0 be the power at $z=0$. Let P_1 be the power at $z=l$. Then the power at $z=0$ relative to the power at $z=l$ is:

$$\frac{P_0}{P_1} = \frac{e^{-2\alpha\cdot 0}}{e^{-2\alpha\cdot l}} = e^{2\alpha l} \quad \text{(linear units)} \tag{3.116}$$

where α is the attenuation constant; that is, the real part of the propagation constant $\gamma = \alpha + j\beta$. Expressed in this manner, the power ratio is a loss; that is, a number greater than 1 represents attenuation. In the decibel scale, the loss is

$$\begin{aligned} 10\log_{10}\frac{P_0}{P_1} &= 10\log_{10} e^{2\alpha l} \\ &= 20\alpha l \log_{10} e \\ &\cong 8.69\alpha l \quad \text{dB} \end{aligned} \tag{3.117}$$

Attenuation rate is defined as this quantity per unit length. Dividing by l, we obtain:

$$\text{attenuation rate} \cong 8.69\alpha \tag{3.118}$$

This has units of dB/length, where the units of length are the same length units in which α is expressed. For example, if α is expressed in units of m^{-1}, then attenuation rate has units of dB/m.

Attenuation rate $\cong 8.69\alpha$ is the loss in dB, per unit length.

The utility of the attenuation rate concept is that it allows us to quickly calculate loss for any distance of wave travel: This loss is simply attenuation rate (dB/m) times length (m), which yields loss in dB.

Example

Example 3.2

Attenuation rate in a long cable.

A particular coaxial cable has an attenuation constant $\alpha \cong 8.5 \times 10^{-3}$ m^{-1}. What is the attenuation rate and the loss in dB for 100 m of this cable?

The attenuation rate is

$$\cong 8.69\alpha \cong \underline{0.0738 \text{ dB/m}}$$

The loss in 100 m of this cable is

$$\cong (0.0738 \text{ dB/m})\,(100 \text{ m}) \cong \underline{7.4 \text{ dB}}$$

Note that it would be entirely appropriate, and equivalent, to state that the attenuation rate for this cable is 7.4 dB/(100 m).

The concept of attenuation rate is used in precisely the same way to relate ratios of spatial power densities for unguided waves. This works because spatial power density has SI base units of W/m^2, so the common units of m^{-2} in the numerator and denominator cancel in the power density ratio, leaving a simple power ratio.

3.10 Poor Conductors

A *poor conductor* is a material for which conductivity is low, yet sufficient to exhibit significant loss. To be clear, the loss we refer to here is the conversion of the electric field to current through Ohm's law.

The threshold of significance depends on the application. For example, the dielectric spacer separating the conductors in a coaxial cable might be treated as lossless ($\sigma = 0$) for short lengths at low frequencies; whereas the loss of the cable for long lengths and higher frequencies is typically significant, and must be taken into account. In the latter case, the material is said to be a poor conductor because the loss is significant yet the material can still be treated in most other respects as an ideal dielectric.

A quantitative but approximate criterion for identification of a poor conductor can be obtained from the concept of complex permittivity ϵ_c, which has the form:

$$\epsilon_c = \epsilon' - j\epsilon'' \tag{3.119}$$

Recall that ϵ'' quantifies loss, whereas ϵ' exists independently of loss. In fact, $\epsilon_c = \epsilon' = \epsilon$ for a perfectly lossless material. Therefore, we may quantify the lossiness of a material using the ratio ϵ''/ϵ', which is sometimes referred to as *loss tangent* (see Section 3.5). Using this quantity, we define a poor conductor as a material for which ϵ'' is very small relative to ϵ'. Thus,

$$\frac{\epsilon''}{\epsilon'} \ll 1 \quad \text{(poor conductor)} \tag{3.120}$$

A poor conductor is a material having loss tangent much less than 1, such that it behaves in most respects as an ideal dielectric except that ohmic loss may not be negligible.

An example of a poor conductor commonly encountered in electrical engineering includes the popular printed circuit board substrate material FR4 (fiberglass epoxy), which has $\epsilon''/\epsilon' \sim 0.008$ over the frequency range it is most commonly used. Another example, already mentioned, is the dielectric spacer material (for example, polyethylene) typically used in coaxial cables. The loss of these materials may or may not be significant, depending on the particulars of the application.

The imprecise definition of Equation 3.120 is sufficient to derive some characteristics exhibited by all poor conductors. To do so, first recall that the propagation constant γ is given in general as follows:

$$\gamma^2 = -\omega^2 \mu \epsilon_c \tag{3.121}$$

Therefore:

$$\gamma = \sqrt{-\omega^2 \mu \epsilon_c} \tag{3.122}$$

In general a number has two square roots, so some caution is required here. In this case, we may proceed as follows:

$$\begin{aligned} \gamma &= j\omega\sqrt{\mu}\sqrt{\epsilon' - j\epsilon''} \\ &= j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\epsilon''}{\epsilon'}} \end{aligned} \tag{3.123}$$

The requirement that $\epsilon''/\epsilon' \ll 1$ for a poor conductor allows this expression to be “linearized.” For this, we invoke the *binomial series* representation:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... \tag{3.124}$$

where x and n are, for our purposes, any constants; and "..." indicates the remaining terms in this infinite series, with each term containing the factor x^n with $n > 2$. If $x \ll 1$, then all terms containing x^n with $n \geq 2$ will be very small relative to the first two terms of the series. Thus,

$$(1+x)^n \approx 1 + nx \quad \text{for } x \ll 1 \tag{3.125}$$

Applying this to the present problem:

$$\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2} \approx 1 - j\frac{\epsilon''}{2\epsilon'} \tag{3.126}$$

where we have used $n = 1/2$ and $x = -j\epsilon''/\epsilon'$. Applying this approximation to Equation 3.123, we obtain:

$$\begin{aligned} \gamma &\approx j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{2\epsilon'}\right) \\ &\approx j\omega\sqrt{\mu\epsilon'} + \omega\sqrt{\mu\epsilon'}\,\frac{\epsilon''}{2\epsilon'} \end{aligned} \tag{3.127}$$

At this point, we are able to identify an expression for the phase propagation constant:

$$\boxed{\beta \triangleq \text{Im}\{\gamma\} \approx \omega\sqrt{\mu\epsilon'} \quad \text{(poor conductor)}} \tag{3.128}$$

Remarkably, we find that β for a poor conductor is approximately equal to β for an ideal dielectric.

For the attenuation constant, we find

$$\boxed{\alpha \triangleq \text{Re}\{\gamma\} \approx \omega\sqrt{\mu\epsilon'}\,\frac{\epsilon''}{2\epsilon'} \quad \text{(poor conductor)}} \tag{3.129}$$

Alternatively, this expression may be written in the following form:

$$\alpha \approx \frac{1}{2}\beta\frac{\epsilon''}{\epsilon'} \quad \text{(poor conductor)} \tag{3.130}$$

Presuming that ϵ_c is determined entirely by ohmic loss, then

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} \tag{3.131}$$

Under this condition, Equation 3.129 may be rewritten:

$$\alpha \approx \omega\sqrt{\mu\epsilon'}\frac{\sigma}{2\omega\epsilon} \quad \text{(poor conductor)} \tag{3.132}$$

Since $\epsilon' = \epsilon$ under these assumptions, the expression simplifies to

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{1}{2}\sigma\eta \quad \text{(poor conductor)} \tag{3.133}$$

where $\eta \triangleq \sqrt{\mu/\epsilon'}$ is the wave impedance presuming lossless material. This result is remarkable for two reasons: First, factors of ω have been eliminated, so there is no dependence on frequency separate from the frequency dependence of the constitutive parameters σ, μ, and ϵ. These parameters vary slowly with frequency, so the value of α for a poor conductor also varies slowly with frequency. Second, we see α is proportional to σ and η. This makes it quite easy to anticipate how the attenuation constant is affected by changes in conductivity and wave impedance in poor conductors.

Finally, what is the wave impedance in a poor conductor? In contrast to η, η_c is potentially complex-valued and may depend on σ. First, recall:

$$\eta_c = \sqrt{\frac{\mu}{\epsilon'}} \cdot \left[1 - j\frac{\epsilon''}{\epsilon'}\right]^{-1/2} \tag{3.134}$$

Applying the same approximation applied to γ earlier, this may be written

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon'}} \cdot \left[1 - j\frac{\epsilon''}{2\epsilon'}\right] \quad \text{(poor conductor)} \tag{3.135}$$

We see that for a poor conductor, Re$\{\eta_c\} \approx \eta$ and that Im$\{\eta_c\} \ll \mathrm{Re}\,\{\eta_c\}$. The usual approximation in this case is simply

$$\boxed{\eta_c \approx \eta \quad \text{(poor conductor)}} \tag{3.136}$$

Additional Reading:

- "Binomial series (https://en.wikipedia.org/wiki/Binomial_series)" on Wikipedia.

3.11 Good Conductors

A *good conductor* is a material which behaves in most respects as a perfect conductor, yet exhibits significant loss. Now, we have to be very careful: The term "loss" applied to the concept of a "conductor" means something quite different from the term "loss" applied to other types of materials. Let us take a moment to disambiguate this term.

Conductors are materials which are intended to efficiently sustain current, which requires high conductivity σ. In contrast, non-conductors are materials which are intended to efficiently sustain the electric field, which requires low σ. "Loss" for a non-conductor (see in particular "poor conductors," Section 3.10) means the conversion of energy in the electric field into current. In contrast, "loss" for a conductor refers to energy *already* associated with current, which is subsequently dissipated in resistance. Summarizing: A good ("low-loss") conductor is a material with *high* conductivity, such that power dissipated in the resistance of the material is low.

A quantitative criterion for a good conductor can be obtained from the concept of complex permittivity ϵ_c, which has the form:

$$\epsilon_c = \epsilon' - j\epsilon'' \tag{3.137}$$

Recall that ϵ'' is proportional to conductivity (σ) and so ϵ'' is very large for a good conductor. Therefore, we may identify a good conductor using the ratio ϵ''/ϵ', which is sometimes referred to as "loss tangent" (see Section 3.5). Using this quantity we define a good conductor as a material for which:

$$\frac{\epsilon''}{\epsilon'} \gg 1 \quad \text{(good conductor)} \tag{3.138}$$

This condition is met for most materials classified as "metals," and especially for metals exhibiting very high conductivity such as gold, copper, and aluminum.

> A good conductor is a material having loss tangent much greater than 1.

The imprecise definition of Equation 3.138 is sufficient to derive some characteristics that are common to materials over a wide range of conductivity. To derive these characteristics, first recall that the propagation constant γ is given in general as follows:

$$\gamma^2 = -\omega^2 \mu \epsilon_c \tag{3.139}$$

Therefore:

$$\gamma = \sqrt{-\omega^2 \mu \epsilon_c} \tag{3.140}$$

In general, a number has two square roots, so some caution is required here. In this case, we may proceed as follows:

$$\begin{aligned}\gamma &= j\omega\sqrt{\mu}\sqrt{\epsilon' - j\epsilon''} \\ &= j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\epsilon''}{\epsilon'}}\end{aligned} \tag{3.141}$$

Since $\epsilon''/\epsilon' \gg 1$ for a good conductor,

$$\begin{aligned}\gamma &\approx j\omega\sqrt{\mu\epsilon'}\sqrt{-j\frac{\epsilon''}{\epsilon'}} \\ &\approx j\omega\sqrt{\mu\epsilon''}\sqrt{-j}\end{aligned} \tag{3.142}$$

To proceed, we must determine the principal value of $\sqrt{-j}$. The answer is that $\sqrt{-j} = (1-j)/\sqrt{2}$.[1] Continuing:

$$\gamma \approx j\omega\sqrt{\frac{\mu\epsilon''}{2}} + \omega\sqrt{\frac{\mu\epsilon''}{2}} \tag{3.143}$$

We are now able to identify expressions for the attenuation and phase propagation constants:

$$\boxed{\alpha \triangleq \mathrm{Re}\left\{\gamma\right\} \approx \omega\sqrt{\frac{\mu\epsilon''}{2}} \quad \text{(good conductor)}} \tag{3.144}$$

$$\boxed{\beta \triangleq \mathrm{Im}\left\{\gamma\right\} \approx \alpha \quad \text{(good conductor)}} \tag{3.145}$$

Remarkably, we find that $\alpha \approx \beta$ for a good conductor, and neither α nor β depend on ϵ'.

In the special case that ϵ_c is determined entirely by conductivity loss (i.e., $\sigma > 0$) and is not accounting for delayed polarization response (as described in Section 3.4), then

$$\epsilon'' = \frac{\sigma}{\omega} \tag{3.146}$$

Under this condition, Equation 3.144 may be rewritten:

$$\alpha \approx \omega\sqrt{\frac{\mu\sigma}{2\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} \quad \text{(good conductor)} \tag{3.147}$$

Since $\omega = 2\pi f$, another possible form for this expression is

$$\boxed{\alpha \approx \sqrt{\pi f \mu\sigma} \quad \text{(good conductor)}} \tag{3.148}$$

The conductivity of most materials changes very slowly with frequency, so this expression indicates that α (and β) increases approximately in proportion to the square root of frequency for good conductors. This is commonly observed in electrical engineering applications. For example, the attenuation rate of transmission lines increases approximately as $\sqrt{f}$. This is so because the principal contribution to the attenuation is resistance in the conductors comprising the line.

> The attenuation rate for signals conveyed by transmission lines is approximately proportional to the square root of frequency.

Let us now consider the wave impedance η_c in a good conductor. Recall:

$$\eta_c = \sqrt{\frac{\mu}{\epsilon'}} \cdot \left[1 - j\frac{\epsilon''}{\epsilon'}\right]^{-1/2} \tag{3.149}$$

Applying the same approximation applied to γ earlier in this section, the previous expression may be written

$$\begin{aligned} \eta_c &\approx \sqrt{\frac{\mu}{\epsilon'}} \cdot \left[-j\frac{\epsilon''}{\epsilon'}\right]^{-1/2} \quad \text{(good conductor)} \\ &\approx \sqrt{\frac{\mu}{\epsilon''}} \cdot \frac{1}{\sqrt{-j}} \end{aligned} \tag{3.150}$$

We've already established that $\sqrt{-j} = (1-j)/\sqrt{2}$. Applying that result here:

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon''}} \cdot \frac{\sqrt{2}}{1-j} \tag{3.151}$$

Now multiplying numerator and denominator by $1+j$, we obtain

$$\eta_c \approx \sqrt{\frac{\mu}{2\epsilon''}} \cdot (1+j) \tag{3.152}$$

In the special case that ϵ_c is determined entirely by conductivity loss and is not accounting for delayed polarization response, then $\epsilon'' = \sigma/\omega$, and we find:

$$\boxed{\eta_c \approx \sqrt{\frac{\mu\omega}{2\sigma}} \cdot (1+j)} \tag{3.153}$$

There are at least two other ways in which this expression is commonly written. First, we can use $\omega = 2\pi f$ to obtain:

$$\eta_c \approx \sqrt{\frac{\pi f \mu}{\sigma}} \cdot (1+j) \tag{3.154}$$

Second, we can use the fact that $\alpha \approx \sqrt{\pi f \mu \sigma}$ for good conductors to obtain:

$$\eta_c \approx \frac{\alpha}{\sigma} \cdot (1+j) \tag{3.155}$$

In any event, we see that the magnitude of the wave impedance bears little resemblance to the wave impedance for a poor conductor. In particular, there is no dependence on the physical permittivity $\epsilon' = \epsilon$, as we saw also for α and β. In this sense, the concept of permittivity does not apply to good conductors, and especially so for perfect conductors.

Note also that ψ_η, the phase of η_c, is always $\approx \pi/4$ for a good conductor, in contrast to ≈ 0 for a poor conductor. This has two implications that are useful to know. First, since η_c is the ratio of the magnitude of the electric field intensity to the magnitude of the magnetic field intensity, the phase of the magnetic field will be shifted by $\approx \pi/4$ relative to the phase of the electric field in a good conductor. Second, recall from Section 3.7 that the power density for a wave is proportional to $\cos\psi_\eta$. Therefore, the extent to which a good conductor is able to “extinguish” a wave propagating through it is determined entirely by α, and specifically is proportional to $e^{-\alpha l}$ where l is distance traveled through the material. In other words, only a perfect conductor ($\sigma \to \infty$) is able to completely suppress wave propagation, whereas waves are always able to penetrate some distance into any conductor which is merely “good.” A measure of this distance is the *skin depth* of the material. The concept of skin depth is presented in Section 3.12.

The dependence of β on conductivity leads to a particularly surprising result for the phase velocity of the beleaguered waves that do manage to propagate within a good conductor. Recall that for both lossless and low-loss (“poor conductor”) materials, the phase velocity v_p is either exactly or approximately $c/\sqrt{\epsilon_r}$, where $\epsilon_r \triangleq \epsilon'/\epsilon_0$, resulting in typical phase velocities within one order of magnitude of c. For a good conductor, we find instead:

$$v_p = \frac{\omega}{\beta} \approx \frac{\omega}{\sqrt{\pi f \mu \sigma}} \quad \text{(good conductor)} \tag{3.156}$$

and since $\omega = 2\pi f$:

$$v_p \approx \sqrt{\frac{4\pi f}{\mu\sigma}} \quad \text{(good conductor)} \tag{3.157}$$

Note that the phase velocity in a good conductor increases with frequency and decreases with conductivity. In contrast to poor conductors and non-conductors, the phase velocity in good conductors is usually a tiny fraction of c. For example, for a non-magnetic ($\mu \approx \mu_0$) good conductor with typical $\sigma \sim 10^6$ S/m, we find $v_p \sim 100$ km/s at 1 GHz – just ~ 0.03% of the speed of light in free space.

This result also tells us something profound about the nature of signals that are conveyed by transmission lines. Regardless of whether we analyze such signals as voltage and current waves associated with the conductors or in terms of guided waves between the conductors, we find that the phase velocity is within an order of magnitude or so of c. Thus, *the information conveyed by signals propagating along transmission lines travels primarily within the space between the conductors, and not within the conductors*. Information cannot travel primarily in the conductors, as this would then result in apparent phase velocity which is orders of magnitude less than c, as noted previously. Remarkably, classical transmission line theory employing the R', G', C', L' equivalent circuit model[2] gets this right, even though that approach does not explicitly consider the possibility of guided waves traveling between the conductors.

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1. You can confirm this simply by squaring this result. The easiest way to derive this result is to work in polar form, in which $-j$ is 1 at an angle of $-\pi/2$, and the square root operation consists of taking the square root of the magnitude and dividing the phase by 2. ↩
2. Depending on the version of this book, this topic may appear in another volume. ↩

3.12 Skin Depth

The electric and magnetic fields of a wave are diminished as the wave propagates through lossy media. The magnitude of these fields is proportional to

$$e^{-\alpha l}$$

where $\alpha \triangleq \text{Re}\{\gamma\}$ is the attenuation constant (SI base units of m^{-1}), γ is the propagation constant, and l is the distance traveled. Although the rate at which magnitude is reduced is completely described by α, particular values of α typically do not necessarily provide an intuitive sense of this rate.

An alternative way to characterize attenuation is in terms of *skin depth* δ_s, which is defined as the distance at which the magnitude of the electric and magnetic fields is reduced by a factor of $1/e$. In other words:

$$e^{-\alpha\delta_s} = e^{-1} \cong 0.368 \tag{3.158}$$

Skin depth δ_s is the distance over which the magnitude of the electric or magnetic field is reduced by a factor of $1/e \cong 0.368$.

Since power is proportional to the square of the field magnitude, δ_s may also be interpreted as the distance at which the power in the wave is reduced by a factor of $(1/e)^2 \cong 0.135$. In yet other words: δ_s is the distance at which $\cong 86.5$% of the power in the wave is lost.

This definition for skin depth makes δ_s easy to compute: From Equation 3.158, it is simply

$$\boxed{\delta_s = \frac{1}{\alpha}} \tag{3.159}$$

The concept of skin depth is most commonly applied to good conductors. For a good conductor, $\alpha \approx \sqrt{\pi f \mu \sigma}$ (Section 3.11), so

$$\delta_s \approx \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \text{(good conductors)} \tag{3.160}$$

Example

Example 3.3

Skin depth of aluminum.

Aluminum, a good conductor, exhibits $\sigma \approx 3.7 \times 10^7$ S/m and $\mu \approx \mu_0$ over a broad range of radio frequencies. Using Equation 3.160, we find $\delta_s \sim 26\ \mu$m at 10 MHz. Aluminum sheet which is 1/16-in ($\cong 1.59$ mm) thick can also be said to have a thickness of $\sim 61\delta_s$ at 10 MHz. The reduction in the power density of an electromagnetic wave after traveling through this sheet will be

$$\sim \left(e^{-\alpha(61\delta_s)}\right)^2 = \left(e^{-\alpha(61/\alpha)}\right)^2 = e^{-122}$$

which is effectively zero from a practical engineering perspective. Therefore, 1/16-in aluminum sheet provides excellent shielding from electromagnetic waves at 10 MHz.

At 1 kHz, the situation is significantly different. At this frequency, $\delta_s \sim 2.6$ mm, so 1/16-in aluminum is only $\sim 0.6\delta_s$ thick. In this case the power density is reduced by only

$$\sim \left(e^{-\alpha(0.6\delta_s)}\right)^2 = \left(e^{-\alpha(0.6/\alpha)}\right)^2 = e^{-1.2} \approx 0.3$$

This is a reduction of only ~ 70% in power density. Therefore, 1/16-in aluminum sheet provides very little shielding at 1 kHz.

End Matter

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