2 Magnetostatics Redux

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2.1 Lorentz Force

The *Lorentz force* is the force experienced by charge in the presence of electric and magnetic fields.

Consider a particle having charge q. The force $\mathbf{F}_e$ experienced by the particle in the presence of electric field intensity $\mathbf{E}$ is

$$\mathbf{F}_e = q\mathbf{E}$$

The force $\mathbf{F}_m$ experienced by the particle in the presence of magnetic flux density $\mathbf{B}$ is

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{v}$ is the velocity of the particle. The Lorentz force experienced by the particle is simply the sum of these forces; i.e.,

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_e + \mathbf{F}_m \\ &= q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \end{aligned} \tag{2.1}$$

The term “Lorentz force” is simply a concise way to refer to the *combined* contributions of the electric and magnetic fields.

A common application of the Lorentz force concept is in analysis of the motions of charged particles in electromagnetic fields. The Lorentz force causes charged particles to exhibit distinct rotational (“cyclotron”) and translational (“drift”) motions. This is illustrated in Figures 2.1 and 2.2.

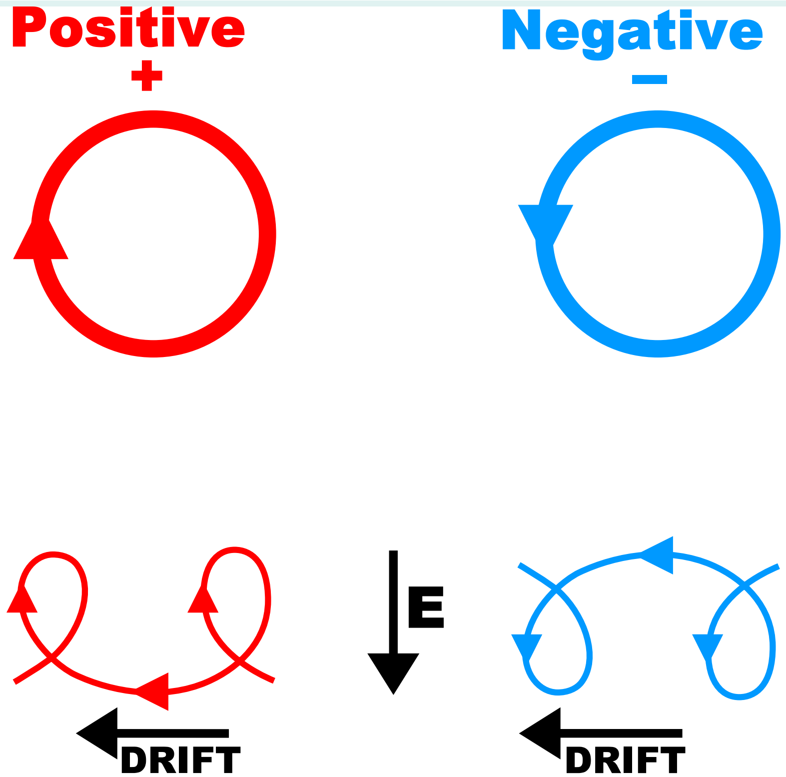

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Figure 2.1. *Motion of a particle bearing (left) positive charge and (right)*

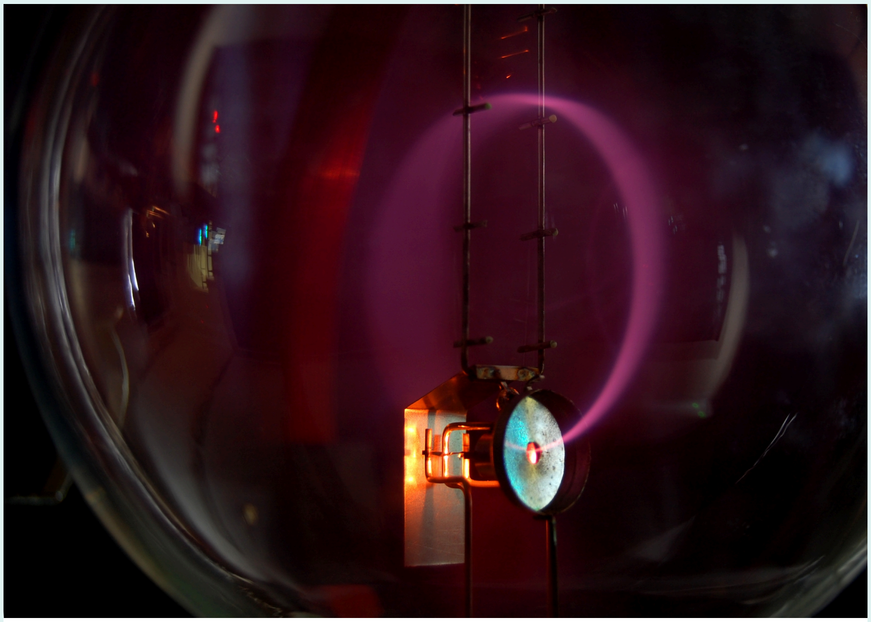

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Figure 2.2. *Electrons moving in a circle in a magnetic field (cyclotron motion). The electrons are produced by an electron gun at bottom, consisting of a hot cathode, a metal plate heated by a filament so it emits electrons, and a metal anode at a high voltage with a hole which accelerates the electrons into a beam. The electrons are normally*

invisible, but enough air has been left in the tube so that the air molecules glow pink when struck by the fast-moving electrons.

Additional Reading:

- "Lorentz force (https://wikipedia.org/wiki/Lorentz_force)" on Wikipedia.

2.2 Magnetic Force on a Current-Carrying Wire

Consider an infinitesimally-thin and perfectly-conducting wire bearing a current I (SI base units of A) in free space. Let $\mathbf{B}\left(\mathbf{r}\right)$ be the impressed magnetic flux density at each point $\mathbf{r}$ in the region of space occupied by the wire. By *impressed*, we mean that the field exists in the absence of the current-carrying wire, as opposed to the field that is induced by this current. Since current consists of charged particles in motion, we expect that $\mathbf{B}(\mathbf{r})$ will exert a force on the current. Since the current is constrained to flow on the wire, we expect this force will also be experienced by the wire. Let us now consider this force.

To begin, recall that the force exerted on a particle bearing charge q having velocity $\mathbf{v}$ is

$$\mathbf{F}_m\left(\mathbf{r}\right) = q\mathbf{v}\left(\mathbf{r}\right) \times \mathbf{B}\left(\mathbf{r}\right) \tag{2.2}$$

Thus, the force exerted on a *differential* amount of charge dq is

$$d\mathbf{F}_m\left(\mathbf{r}\right) = dq\ \mathbf{v}\left(\mathbf{r}\right) \times \mathbf{B}\left(\mathbf{r}\right) \tag{2.3}$$

Let $d\mathbf{l}\left(\mathbf{r}\right)$ represent a differential-length segment of the wire at $\mathbf{r}$, pointing in the direction of current flow. Then

$$dq\ \mathbf{v}\left(\mathbf{r}\right) = I d\mathbf{l}\left(\mathbf{r}\right) \tag{2.4}$$

(If this is not clear, it might help to consider the units: On the left, C·m/s $=$ (C/s)·m $=$ A·m, as on the right.) Subsequently,

$$d\mathbf{F}_m\left(\mathbf{r}\right) = I d\mathbf{l}\left(\mathbf{r}\right) \times \mathbf{B}\left(\mathbf{r}\right) \tag{2.5}$$

There are three important cases of practical interest. First, consider a straight segment $\mathbf{l}$ forming part of a closed loop of current in a spatially-uniform impressed magnetic flux density $\mathbf{B}\left(\mathbf{r}\right) = \mathbf{B}_0$. In this case, the force exerted by the magnetic field on such a segment is given by Equation 2.5 with $d\mathbf{l}$ replaced by $\mathbf{l}$; i.e.:

$$\boxed{\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}_0} \tag{2.6}$$

Summarizing,

The force experienced by a straight segment of current-carrying wire in a spatially-uniform magnetic field is given by Equation 2.6.

The second case of practical interest is a rigid closed loop of current in a spatially-uniform magnetic flux density $\mathbf{B}_0$. If the loop consists of straight sides – e.g., a rectangular loop – then the force applied to the loop is the sum of the forces applied to each side separately, as determined by Equation 2.6. However, we wish to consider loops of arbitrary shape. To accommodate arbitrarily-shaped loops, let $\mathcal{C}$ be the path through space occupied by the loop. Then the force experienced by the loop is

$$\begin{aligned}\mathbf{F} &= \int_{\mathcal{C}} d\mathbf{F}_m(\mathbf{r}) \\ &= \int_{\mathcal{C}} I d\mathbf{l}(\mathbf{r}) \times \mathbf{B}_0\end{aligned} \tag{2.7}$$

Since I and $\mathbf{B}_0$ are constants, they may be extracted from the integral:

$$\mathbf{F} = I\left[\int_{\mathcal{C}} d\mathbf{l}(\mathbf{r})\right] \times \mathbf{B}_0 \tag{2.8}$$

Note the quantity in square brackets is zero. Therefore:

The net force on a current-carrying loop of wire in a uniform magnetic field is zero.

Note that this does not preclude the possibility that the rigid loop *rotates*; for example, the force on opposite sides of the loop may be equal and opposite. What we have found is merely that the force will not lead to a *translational* net force on the loop; e.g., force that would propel the loop away from its current position in space. The possibility of rotation without translation leads to the most rudimentary concept for an electric motor. Practical electric motors use variations on essentially this same idea; see “Additional Reading” for more information.

The third case of practical interest is the force experienced by two parallel infinitesimally-thin wires in free space, as shown in Figure 2.3. Here the wires are infinite in length (we’ll return to that in a moment), lie in the $x = 0$ plane, are separated by distance d, and carry currents I_1 and I_2, respectively. The current in wire 1 gives rise to a magnetic flux density $\mathbf{B}_1$. The force exerted on wire 2 by $\mathbf{B}_1$ is:

$$\mathbf{F}_2 = \int_{\mathcal{C}} \left[I_2 d\mathbf{l}(\mathbf{r}) \times \mathbf{B}_1(\mathbf{r})\right] \tag{2.9}$$

where $\mathcal{C}$ is the path followed by I_2, and $d\mathbf{l}\left(\mathbf{r}\right) = \hat{\mathbf{z}}dz$. A simple way to determine $\mathbf{B}_1$ in this situation is as follows. First, if wire 1 had been aligned along the $x = y = 0$ line, then the magnetic flux density everywhere would be

$$\hat{\phi}\frac{\mu_0 I_1}{2\pi\rho}$$

In the present problem, wire 1 is displaced by $d/2$ in the $-\hat{\mathbf{y}}$ direction. Although this would seem to make the new expression more complicated, note that the only positions where values of $\mathbf{B}_1\left(\mathbf{r}\right)$ are required are those corresponding to $\mathcal{C}$; i.e., points on wire 2. For these points,

$$\mathbf{B}_1\left(\mathbf{r}\right) = -\hat{\mathbf{x}}\frac{\mu_0 I_1}{2\pi d} \quad \text{along } \mathcal{C} \tag{2.10}$$

That is, the relevant distance is d (not ρ), and the direction of $\mathbf{B}_1\left(\mathbf{r}\right)$ for points along $\mathcal{C}$ is $-\hat{\mathbf{x}}$ (not $\hat{\phi}$). Returning to Equation 2.9, we obtain:

$$\begin{aligned} \mathbf{F}_2 &= \int_{\mathcal{C}} \left[I_2 \ \hat{\mathbf{z}}dz \times \left(-\hat{\mathbf{x}}\frac{\mu_0 I_1}{2\pi d} \right) \right] \\ &= -\hat{\mathbf{y}}\frac{\mu_0 I_1 I_2}{2\pi d} \ \int_{\mathcal{C}} dz \end{aligned} \tag{2.11}$$

The remaining integral is simply the length of wire 2 that we wish to consider. Infinitely-long wires will therefore result in infinite force. This is not a very interesting or useful result. However, the force *per unit length* of wire is finite, and is obtained simply by dropping the integral in the previous equation. We obtain:

$$\frac{\mathbf{F}_2}{\Delta l} = -\hat{\mathbf{y}}\frac{\mu_0 I_1 I_2}{2\pi d} \tag{2.12}$$

where Δl is the length of the section of wire 2 being considered. Note that when the currents I_1 and I_2 flow in the same direction (i.e., have the same sign), the magnetic force exerted by the current on wire 1 pulls wire 2 toward wire 1.

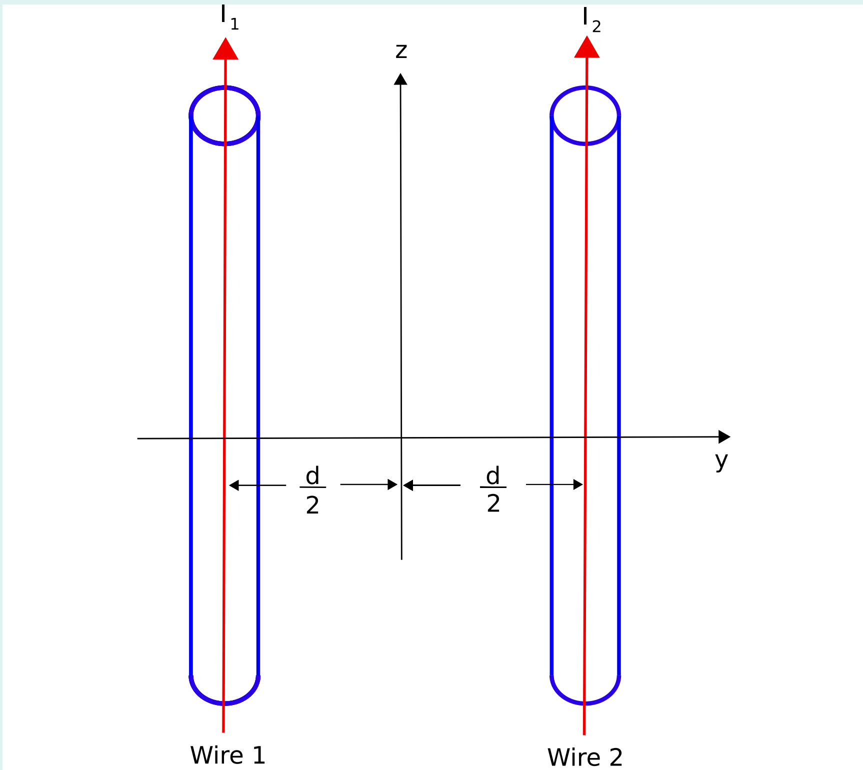

Figure 2.3. *Parallel current-carrying wires.*

The same process can be used to determine the magnetic force $\mathbf{F}_1$ exerted by the current in wire 1 on wire 2. The result is

$$\frac{\mathbf{F}_1}{\Delta l} = +\hat{\mathbf{y}}\frac{\mu_0 I_1 I_2}{2\pi d} \tag{2.13}$$

When the currents I_1 and I_2 flow in the same direction (i.e., when the product $I_1 I_2$ is positive), then the magnetic force exerted by the current on wire 2 pulls wire 1 toward wire 2.

We are now able to summarize the results as follows:

If currents in parallel wires flow in the same direction, then the wires attract; whereas if the currents flow in opposite directions, then the wires repel.

Also:

The magnitude of the associated force is $\mu_0 I_1 I_2 / 2\pi d$ for wires separated by distance d in non-magnetic media.

If the wires are fixed in position and not able to move, these forces represent stored (potential) energy. It is worth noting that this is precisely the energy which is stored by an inductor – for example, the two wire segments here might be interpreted as segments in adjacent windings of a coil-shaped inductor.

Example

Example 2.1

DC power cable.

A power cable connects a 12 V battery to a load exhibiting an impedance of $10\ \Omega$. The conductors are separated by 3 mm by a plastic insulating jacket. Estimate the force between the conductors.

Solution. The current flowing in each conductor is 12 V divided by $10\ \Omega$, which is 1.2 A. In terms of the theory developed in this section, a current $I_1 = +1.2$ A flows from the positive terminal of the battery to the load on one conductor, and a current $I_2 = -1.2$ A returns to the battery on the other conductor. The change in sign indicates that the currents at any given distance from the battery are flowing in opposite directions. Also from the problem statement, $d = 3$ mm and the insulator is presumably non-magnetic. Assuming the conductors are approximately straight, the force between conductors is

$$\approx \frac{\mu_0 I_1 I_2}{2\pi d} \cong -96.0\ \mu\text{N}$$

with the negative sign indicating that the wires repel.

Note in the above example that this force is quite small, which explains why it is not always observed. However, this force becomes significant when the current is large or when many sets of conductors are mechanically bound together (amounting to a larger net current), as in a motor.

Additional Reading:

- “Electric motor (https://wikipedia.org/wiki/Electric_motor)” on Wikipedia.

2.3 Torque Induced by a Magnetic Field

A magnetic field exerts a force on current. This force is exerted in a direction perpendicular to the direction of current flow. For this reason, current-carrying structures in a magnetic field tend to rotate. A convenient description of force associated with rotational motion is *torque*. In this section, we define torque and apply this concept to a closed loop of current. These concepts apply to a wide range of practical devices, including electric motors.

Figure 2.4 illustrates the concept of torque.

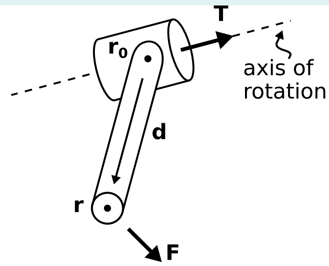

Figure 2.4. *Torque associated with a single lever arm.*

Torque depends on the following:

- A local origin $\mathbf{r}_0$,
- A point **r** which is connected to $\mathbf{r}_0$ by a perfectly-rigid mechanical structure, and
- The force $\mathbf{F}$ applied at $\mathbf{r}$.

In terms of these parameters, the torque $\mathbf{T}$ is:

$$\mathbf{T} \triangleq \mathbf{d} \times \mathbf{F} \tag{2.14}$$

where the *lever arm* $\mathbf{d} \triangleq \mathbf{r} - \mathbf{r}_0$ gives the location of $\mathbf{r}$ relative to $\mathbf{r}_0$. Note that $\mathbf{T}$ is a position-free vector which points in a direction perpendicular to both $\mathbf{d}$ and $\mathbf{F}$.

Note that $\mathbf{T}$ does not point in the direction of rotation. Nevertheless, $\mathbf{T}$ indicates the direction of rotation through a “right hand rule”: If you point the thumb of your right hand in the direction of $\mathbf{T}$, then the curled fingers of your right hand will point in the direction of torque-induced rotation.

Whether rotation actually occurs depends on the geometry of the structure. For example, if $\mathbf{T}$ aligns with the axis of a perfectly-rigid mechanical shaft, then all of the work done by $\mathbf{F}$ will be applied to rotation of the shaft on this axis. Otherwise, torque will tend to rotate the shaft in other directions as well. If the shaft is not free to rotate in these other directions, then the *effective torque* – that is, the torque that contributes to rotation of the shaft – is reduced.

The magnitude of $\mathbf{T}$ has SI base units of N·m and quantifies the energy associated with the rotational force. As you might expect, the magnitude of the torque increases with increasing lever arm magnitude $|\mathbf{d}|$. In other words, the torque resulting from a constant applied force increases with the length of the lever arm.

Torque, like the translational force **F**, satisfies superposition. That is, the torque resulting from forces applied to multiple rigidly-connected lever arms is the sum of the torques applied to the lever arms individually.

Now consider the current loop shown in Figure 2.5. The loop is perfectly rigid and is rigidly attached to a non-conducting shaft. The assembly consisting of the loop and the shaft may rotate without friction around the axis of the shaft. The loop consists of four straight segments that are perfectly-conducting and infinitesimally-thin. A spatially-uniform and static impressed magnetic flux density $\mathbf{B}_0 = \hat{\mathbf{x}}B_0$ exists throughout the domain of the problem. (Recall that an *impressed* field is one that exists in the absence of any other structure in the problem.) What motion, if any, is expected?

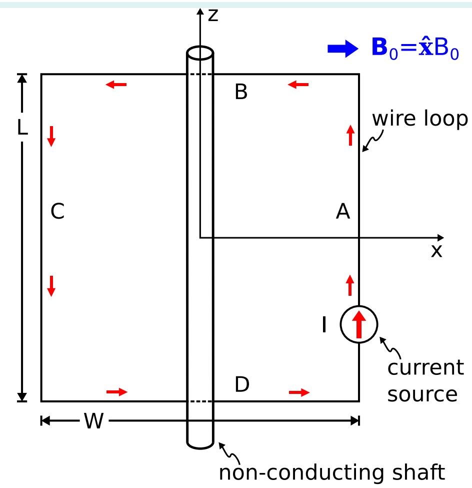

Figure 2.5. *A rudimentary electric motor consisting of a single current loop.*

Recall that the net translational force on a current loop in a spatially-uniform and static magnetic field is zero (Section 2.2). However, this does not preclude the possibility of different translational forces acting on each of the loop segments resulting in a rotation of the shaft. Let us first calculate these forces. The force $\mathbf{F}_A$ on segment A is

$$\mathbf{F}_A = I\mathbf{l}_A \times \mathbf{B}_0 \tag{2.15}$$

where $\mathbf{l}_A$ is a vector whose magnitude is equal to the length of the segment and which points in the direction of the current. Thus,

$$\begin{aligned}\mathbf{F}_A &= I\left(\hat{\mathbf{z}}L\right) \times \left(\hat{\mathbf{x}}B_0\right) \\ &= \hat{\mathbf{y}}ILB_0\end{aligned} \tag{2.16}$$

Similarly, the force $\mathbf{F}_C$ on segment C is

$$\begin{aligned}\mathbf{F}_C &= I\left(-\hat{\mathbf{z}}L\right)\times\left(\hat{\mathbf{x}}B_0\right)\\ &= -\hat{\mathbf{y}}ILB_0\end{aligned} \tag{2.17}$$

The forces $\mathbf{F}_B$ and $\mathbf{F}_D$ on segments B and D, respectively, are:

$$\mathbf{F}_B = I\left(-\hat{\mathbf{x}}L\right)\times\left(\hat{\mathbf{x}}B_0\right) = 0 \tag{2.18}$$

and

$$\mathbf{F}_D = I\left(+\hat{\mathbf{x}}L\right)\times\left(\hat{\mathbf{x}}B_0\right) = 0 \tag{2.19}$$

Thus, the force exerted on the current loop by the impressed magnetic field will lead to rotation in the $+\hat{\phi}$ direction.

We calculate the associated torque $\mathbf{T}$ as

$$\mathbf{T} = \mathbf{T}_A + \mathbf{T}_B + \mathbf{T}_C + \mathbf{T}_D \tag{2.20}$$

where $\mathbf{T}_A$, $\mathbf{T}_B$, $\mathbf{T}_C$, and $\mathbf{T}_D$ are the torques associated with segments A, B, C, and D, respectively. For example, the torque associated with segment A is

$$\begin{aligned}\mathbf{T}_A &= \frac{W}{2}\hat{\mathbf{x}}\times\mathbf{F}_A\\ &= \hat{\mathbf{z}}L\frac{W}{2}IB_0\end{aligned} \tag{2.21}$$

Similarly,

$$\mathbf{T}_B = 0 \text{ since } \mathbf{F}_B = 0 \tag{2.22}$$

$$\mathbf{T}_C = \hat{\mathbf{z}}L\frac{W}{2}IB_0 \tag{2.23}$$

$$\mathbf{T}_D = 0 \text{ since } \mathbf{F}_D = 0 \tag{2.24}$$

Summing these contributions, we find

$$\mathbf{T} = \hat{\mathbf{z}}LWIB_0 \tag{2.25}$$

Note that $\mathbf{T}$ points in the $+\hat{\mathbf{z}}$ direction, indicating rotational force exerted in the $+\hat{\phi}$ direction, as expected. Also note that the torque is proportional to the area LW of the loop, is proportional to the current I, and is proportional to the magnetic field magnitude B_0.

The analysis that we just completed was *static*; that is, it applies only at the instant depicted in Figure 2.5. If the shaft is allowed to turn without friction, then the loop will rotate in the $+\hat{\phi}$ direction. So, what will happen to the forces and torque? First, note that $\mathbf{F}_A$ and $\mathbf{F}_C$ are always in the $+\hat{\mathbf{y}}$ and $-\hat{\mathbf{y}}$ directions, respectively, regardless of the rotation of the loop. Once the loop rotates away from the position shown in Figure 2.5, the forces $\mathbf{F}_B$ and $\mathbf{F}_D$ become non-zero; however, they are always equal and opposite, and so do not affect the rotation. Thus, the loop will rotate one-quarter turn and then come to rest, perhaps with some damped oscillation around the rest position depending on the momentum of the loop. At the rest position, the lever arms for segments A and C are pointing in the same directions as $\mathbf{F}_A$ and $\mathbf{F}_C$, respectively. Therefore, the cross product of the lever arm and translational force for each segment is zero and subsequently $\mathbf{T}_A = \mathbf{T}_C = 0$. Once stopped in this position, both the net translational force and the net torque are zero.

If such a device is to be used as a motor, it is necessary to find a way to sustain the rotation. There are several ways in which this might be accomplished. First, one might make I variable in time. For example, the direction of I could be reversed as the loop passes the quarter-turn position. This reverses $\mathbf{F}_A$ and $\mathbf{F}_C$, propelling the loop toward the half-turn position. The direction of I can be changed again as the loop passes half-turn position, propelling the loop toward the three-quarter-turn position. Continuing this periodic reversal of the current sustains the rotation. Alternatively, one may periodically reverse the direction of the impressed magnetic field to the same effect. These methods can be combined or augmented using multiple current loops or multiple sets of time-varying impressed magnetic fields. Using an appropriate combination of current loops, magnetic fields, and waveforms for each, it is possible to achieve sustained torque throughout the rotation. An example is shown in Figure 2.6.

Figure 2.6. *This DC electric motor uses brushes (here, the motionless leads labeled "+" and "–") combined with the motion of the shaft to periodically alternate the direction of current between two coils, thereby creating nearly constant torque.*

Additional Reading:

- "Torque (https://wikipedia.org/wiki/Torque)" on Wikipedia.
- "Electric motor (https://wikipedia.org/wiki/Electric_motor)" on Wikipedia.

2.4 The Biot-Savart Law

The Biot-Savart law (BSL) provides a method to calculate the magnetic field due to any distribution of steady (DC) current. In magnetostatics, the general solution to this problem employs Ampere's law; i.e.,

$$\int_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl} \tag{2.26}$$

in integral form or

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2.27}$$

in differential form. The integral form is relatively simple when the problem exhibits a high degree of symmetry, facilitating a simple description in a particular coordinate system. An example is the magnetic field due to a straight and infinitely-long current filament, which is easily determined by solving the integral equation in cylindrical coordinates. However, many problems of practical interest do not exhibit the necessary symmetry. A commonly-encountered example is the magnetic field due to a single loop of current, which will be addressed in Example 2.2. For such problems, the differential form of Ampere's law is needed.

BSL is the solution to the differential form of Ampere's law for a differential-length current element, illustrated in Figure 2.7. The current element is $I\ d\mathbf{l}$, where I is the magnitude of the current (SI base units of A) and $d\mathbf{l}$ is a differential-length vector indicating the direction of the current at the "source point" $\mathbf{r}'$. The resulting contribution to the magnetic field intensity at the "field point" $\mathbf{r}$ is

$$\boxed{d\mathbf{H}(\mathbf{r}) = I\ d\mathbf{l}\ \frac{1}{4\pi R^2} \times \hat{\mathbf{R}}} \tag{2.28}$$

where

$$\mathbf{R} = \hat{\mathbf{R}} R \triangleq \mathbf{r} - \mathbf{r}' \tag{2.29}$$

In other words, $\mathbf{R}$ is the vector pointing from the source point to the field point, and $d\mathbf{H}$ at the field point is given by Equation 2.28. The magnetic field due to a current-carrying wire of any shape may be obtained by integrating over the length of the wire:

$$\mathbf{H}(\mathbf{r}) = \int_{\mathcal{C}} d\mathbf{H}(\mathbf{r}) = \frac{I}{4\pi} \int_{\mathcal{C}} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \tag{2.30}$$

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__Figure 2.7.__ Use of the Biot-Savart law to calculate the magnetic field due to a line current.

In addition to obviating the need to solve a differential equation, BSL provides some useful insight into the behavior of magnetic fields. In particular, Equation 2.28 indicates that magnetic fields follow the *inverse square law* – that is, the magnitude of the magnetic field due to a differential current element decreases in proportion to the inverse square of distance (R^{-2}). Also, Equation 2.28 indicates that the direction of the magnetic field due to a differential current element is perpendicular to both the direction of current flow $\hat{\mathbf{l}}$ and the vector $\hat{\mathbf{R}}$ pointing from the source point to field point. This observation is quite useful in anticipating the direction of magnetic field vectors in complex problems.

It may be helpful to note that BSL is analogous to Coulomb's law for electric fields, which is a solution to the differential form of Gauss' law, $\nabla \cdot \mathbf{D} = \rho_v$. However, BSL applies only under magnetostatic conditions. If the variation in currents or magnetic fields over time is significant, then the problem becomes significantly more complicated. See "Jefimenko's Equations" in "Additional Reading" for more information.

Example

Example 2.2

Magnetic field along the axis of a circular loop of current.

Consider a ring of radius a in the $z = 0$ plane, centered on the origin, as shown in Figure 2.8. As indicated in the figure, the current I flows in the $\hat{\phi}$ direction. Find the magnetic field intensity along the z axis.

Solution. The source current position is given in cylindrical coordinates as

$$\mathbf{r}' = \hat{\rho}a \tag{2.31}$$

The position of a field point along the z axis is

$$\mathbf{r} = \hat{\mathbf{z}}z \tag{2.32}$$

Thus,

$$\hat{\mathbf{R}}R \triangleq \mathbf{r} - \mathbf{r}' = -\hat{\rho}a + \hat{\mathbf{z}}z \tag{2.33}$$

and

$$R \triangleq |\mathbf{r} - \mathbf{r}'| = \sqrt{a^2 + z^2} \tag{2.34}$$

Equation 2.28 becomes:

$$\begin{aligned} d\mathbf{H}(\hat{\mathbf{z}}z) &= \frac{I\ \hat{\phi}\ ad\phi}{4\pi\left[a^2+z^2\right]} \times \frac{\hat{\mathbf{z}}z - \hat{\rho}a}{\sqrt{a^2+z^2}} \\ &= \frac{Ia}{4\pi}\ \frac{\hat{\mathbf{z}}a - \hat{\rho}z}{\left[a^2+z^2\right]^{3/2}}\ d\phi \end{aligned} \tag{2.35}$$

Now integrating over the current:

$$\mathbf{H}(\hat{\mathbf{z}}z) = \int_0^{2\pi} \frac{Ia}{4\pi}\ \frac{\hat{\mathbf{z}}a - \hat{\rho}z}{\left[a^2+z^2\right]^{3/2}}\ d\phi \tag{2.36}$$

$$= \frac{Ia}{4\pi \left[a^2+z^2\right]^{3/2}} \int_0^{2\pi} \left(\hat{\mathbf{z}}a - \hat{\rho}z\right) \ d\phi \tag{2.37}$$

$$= \frac{Ia}{4\pi \left[a^2+z^2\right]^{3/2}} \left(\hat{\mathbf{z}}a \int_0^{2\pi} d\phi - z \int_0^{2\pi} \hat{\rho} \ d\phi\right) \tag{2.38}$$

The second integral is equal to zero. To see this, note that the integral is simply summing values of $\hat{\rho}$ for all possible values of ϕ. Since $\hat{\rho}(\phi+\pi) = -\hat{\rho}(\phi)$, the integrand for any given value of ϕ is equal and opposite the integrand π radians later. (This is one example of a *symmetry* argument.)

The first integral in the previous equation is equal to 2π. Thus, we obtain

$$\mathbf{H}(\hat{\mathbf{z}}z) = \hat{\mathbf{z}}\frac{Ia^2}{2\left[a^2+z^2\right]^{3/2}} \tag{2.39}$$

Note that the result is consistent with the associated "right hand rule" of magnetostatics: That is, the direction of the magnetic field is in the direction of the curled fingers of the right hand when the thumb of the right hand is aligned with the location and direction of current. It is a good exercise to confirm that this result is also dimensionally correct.

Figure 2.8. *Calculation of the magnetic field along the z axis due to a circular loop of current centered in the $z=0$ plane.*

Equation 2.28 extends straightforwardly to other distributions of current. For example, the magnetic field due to surface current $\mathbf{J}_s$ (SI base units of A/m) can be calculated using Equation 2.28 with $I \ d\mathbf{l}$ replaced by

$$\mathbf{J}_s \ ds$$

where ds is the differential element of surface area. This can be confirmed by dimensional analysis: $I \ d\mathbf{l}$ has SI base units of A·m, as does $\mathbf{J}_S \ ds$. Similarly, the magnetic field due to volume current $\mathbf{J}$ (SI base units of A/m^2) can be calculated using Equation 2.28 with $I \ d\mathbf{l}$ replaced by

$$\mathbf{J} \ dv$$

where dv is the differential element of volume. For a single particle with charge q (SI base units of C) and velocity $\mathbf{v}$ (SI base units of m/s), the relevant quantity is

$$q\mathbf{v}$$

since C·m/s $=$ (C/s)·m = A·m. In all of these cases, Equation 2.28 applies with the appropriate replacement for $I\ d\mathbf{l}$.

Note that the quantities $q\mathbf{v}$, $I\ d\mathbf{l}$, $\mathbf{J}_S\ ds$, and $\mathbf{J}\ dv$, all having the same units of A·m, seem to be referring to the same physical quantity. This physical quantity is known as *current moment*. Thus, the "input" to BSL can be interpreted as current moment, regardless of whether the current of interest is distributed as a line current, a surface current, a volumetric current, or simply as moving charged particles. See "Additional Reading" at the end of this section for additional information on the concept of "moment" in classical physics.

Additional Reading:

- "Biot-Savart Law (https://wikipedia.org/wiki/Biot-Savart_law)" on Wikipedia.
- "Jefimenko's Equations (https://wikipedia.org/wiki/Jefimenko's_equations)" on Wikipedia.
- "Moment (physics) (https://en.wikipedia.org/wiki/Moment_(physics))" on Wikipedia.

2.5 Force, Energy, and Potential Difference in a Magnetic Field

The force $\mathbf{F}_m$ experienced by a particle at location $\mathbf{r}$ bearing charge q due to a magnetic field is

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}(\mathbf{r}) \tag{2.40}$$

where $\mathbf{v}$ is the velocity (magnitude and direction) of the particle, and $\mathbf{B}(\mathbf{r})$ is the magnetic flux density at $\mathbf{r}$. Now we must be careful: In this description, the motion of the particle is $\backslash\mathrm{emph}not$ due to $\mathbf{F}_m$. In fact the cross product in Equation 2.40 clearly indicates that $\mathbf{F}_m$ and $\mathbf{v}$ must be in perpendicular directions. Instead, the reverse is true: i.e., it is the motion of the particle that is giving rise to the force. The motion described by $\mathbf{v}$ may be due to the presence of an electric field, or it may simply be that that charge is contained within a structure that is itself in motion.

Nevertheless, the force $\mathbf{F}_m$ has an associated potential energy. Furthermore, this potential energy may change as the particle moves. This change in potential energy may give rise to an electrical potential difference (i.e., a "voltage"), as we shall now demonstrate.

The change in potential energy can be quantified using the concept of *work*, W. The incremental work ΔW done by moving the particle a short distance Δl, over which we assume the change in $\mathbf{F}_m$ is negligible, is

$$\Delta W \approx \mathbf{F}_m \cdot \hat{\mathbf{l}} \Delta l \tag{2.41}$$

where in this case $\hat{\mathbf{l}}$ is the unit vector in the direction of the motion; i.e., the direction of $\mathbf{v}$. Note that the purpose of the dot product in Equation 2.41 is to ensure that only the component of $\mathbf{F}_m$ parallel to the direction of motion is included in the energy tally. Any component of $\mathbf{v}$ which is due to $\mathbf{F}_m$ (i.e., ultimately due to $\mathbf{B}$) must be perpendicular to $\mathbf{F}_m$, so ΔW for such a contribution must be, from Equation 2.41, equal to zero. In other words: In the absence of a mechanical force or an electric field, the potential energy of a charged particle remains constant regardless of how it is moved by $\mathbf{F}_m$. This surprising result may be summarized as follows:

> The magnetic field does no work.

Instead, the change of potential energy associated with the magnetic field must be completely due to a change in position resulting from other forces, such as a mechanical force or the Coulomb force. The presence of a magnetic field merely increases or decreases this potential difference once the particle has moved, and it is this change in the potential difference that we wish to determine.

We can make the relationship between potential difference and the magnetic field explicit by substituting the right side of Equation 2.40 into Equation 2.41, yielding

$$\Delta W \approx q \left[\mathbf{v} \times \mathbf{B}(\mathbf{r})\right] \cdot \hat{\mathbf{l}} \Delta l \tag{2.42}$$

Equation 2.42 gives the work only for a short distance around $\mathbf{r}$. Now let us try to generalize this result. If we wish to know the work done over a larger distance, then we must account for the possibility that $\mathbf{v} \times \mathbf{B}$ varies along the path taken. To do this, we may sum contributions from points along the path traced out by the particle, i.e.,

$$W \approx \sum_{n=1}^{N} \Delta W(\mathbf{r}_n) \tag{2.43}$$

where $\mathbf{r}_n$ are positions defining the path. Substituting the right side of Equation 2.42, we have

$$W \approx q \sum_{n=1}^{N} \left[\mathbf{v} \times \mathbf{B}(\mathbf{r}_n)\right] \cdot \hat{\mathbf{l}}(\mathbf{r}_n) \Delta l \tag{2.44}$$

Taking the limit as $\Delta l \to 0$, we obtain

$$W = q \int_{\mathcal{C}} [\mathbf{v} \times \mathbf{B}(\mathbf{r})] \cdot \hat{\mathbf{l}}(\mathbf{r}) dl \tag{2.45}$$

where $\mathcal{C}$ is the path (previously, the sequence of $\mathbf{r}_n$'s) followed by the particle. Now omitting the explicit dependence on $\mathbf{r}$ in the integrand for clarity:

$$W = q \int_{\mathcal{C}} [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l} \tag{2.46}$$

where $d\mathbf{l} = \hat{\mathbf{l}} dl$ as usual. Now, we are able to determine the change in potential energy for a charged particle moving along any path in space, given the magnetic field.

At this point, it is convenient to introduce the electric *potential difference* V_{21} between the start point (1) and end point (2) of $\mathcal{C}$. V_{21} is defined as the work done by traversing $\mathcal{C}$, per unit of charge; i.e.,

$$V_{21} \triangleq \frac{W}{q} \tag{2.47}$$

This has units of J/C, which is volts (V). Substituting Equation 2.46, we obtain:

$$\boxed{V_{21} = \int_{\mathcal{C}} [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l}} \tag{2.48}$$

Equation 2.48 is electrical potential induced by charge traversing a magnetic field.

Figure 2.9 shows a simple scenario that illustrates this concept.

%

Figure 2.9. *A straight wire moving through a magnetic field.*

Here, a straight perfectly-conducting wire of length l is parallel to the y axis and moves at speed v in the $+z$ direction through a magnetic field $\mathbf{B} = \hat{\mathbf{x}} B$. Thus,

$$\begin{aligned} \mathbf{v} \times \mathbf{B} &= \hat{\mathbf{z}} v \times \hat{\mathbf{x}} B \\ &= \hat{\mathbf{y}} B v \end{aligned} \tag{2.49}$$

Taking endpoints 1 and 2 of the wire to be at $y = y_0$ and $y = y_0 + l$, respectively, we obtain

$$\begin{aligned} V_{21} &= \int_{y_0}^{y_0+l} [\hat{\mathbf{y}}Bv] \cdot \hat{\mathbf{y}} dy \\ &= Bvl \end{aligned} \tag{2.50}$$

Thus, we see that endpoint 2 is at an electrical potential of Bvl greater than that of endpoint 1. This “voltage” exists even though the wire is perfectly-conducting, and therefore cannot be attributed to the electric field. This voltage exists even though the force required for movement must be the same on both endpoints, or could even be zero, and therefore cannot be attributed to mechanical forces. Instead, this change in potential is due entirely to the magnetic field.

Because the wire does not form a closed loop, no current flows in the wire. Therefore, this scenario has limited application in practice. To accomplish something useful with this concept we must at least form a closed loop, so that current may flow. For a closed loop, Equation 2.48 becomes:

$$V = \oint_{\mathcal{C}} [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l} \tag{2.51}$$

Examination of this equation indicates one additional requirement: $\mathbf{v} \times \mathbf{B}$ must somehow vary over $\mathcal{C}$. This is because if $\mathbf{v} \times \mathbf{B}$ does not vary over $\mathcal{C}$, the result will be

$$[\mathbf{v} \times \mathbf{B}] \cdot \oint_{\mathcal{C}} d\mathbf{l}$$

which is zero because the integral is zero. The following example demonstrates a practical application of this idea.

Example

Example 2.3

Potential induced in a time-varying loop.

Figure 2.10 shows a modification to the problem originally considered in Figure 2.9. Now, we have created a closed loop using perfectly-conducting and motionless wire to form three sides of a rectangle, and assigned the origin to the lower left corner. An infinitesimally-small gap has been inserted in the left ($z = 0$) side of the loop and closed with an ideal resistor of value R. As before, $\mathbf{B} = \hat{\mathbf{x}}B$ (spatially uniform and time invariant) and $\mathbf{v} = \hat{\mathbf{z}}v$ (constant). What is the voltage V_T across the resistor and what is the current in the loop?

Solution. Since the gap containing the resistor is infinitesimally small,

$$V_T = \oint_{\mathcal{C}} [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l} \tag{2.52}$$

where $\mathcal{C}$ is the perimeter formed by the loop, beginning at the "$-$" terminal of V_T and returning to the "$+$" terminal of V_T. Only the shorting bar is in motion, so $\mathbf{v} = 0$ for the other three sides of the loop. Therefore, only the portion of $\mathcal{C}$ traversing the shorting bar contributes to V_T. Thus, we find

$$V_T = \int_{y=0}^{l} [\hat{\mathbf{z}}v \times \hat{\mathbf{x}}B] \cdot \hat{\mathbf{y}}dy = Bvl \tag{2.53}$$

This potential gives rise to a current Bvl/R, which flows in the counter-clockwise direction.

%

Figure 2.10. *A time-varying loop created by moving a "shorting bar" along rails comprising two adjacent sides of the loop.*

Note in the previous example that the magnetic field has induced V_T, not the current. The current is simply a response to the existence of the potential, regardless of the source. In other words, the same potential V_T would exist even if the gap was not closed by a resistor.

Astute readers will notice that this analysis seems to have a lot in common with Faraday's law,

$$V = -\frac{\partial}{\partial t}\Phi \tag{2.54}$$

which says the potential induced in a single closed loop is proportional to the time rate of change of magnetic flux Φ, where

$$\Phi = \int_{S} \mathbf{B} \cdot d\mathbf{s} \tag{2.55}$$

and where $\mathcal{S}$ is the surface through which the flux is calculated. From this perspective, we see that Equation 2.51 is simply a special case of Faraday's law, pertaining specifically to "motional emf." Thus, the preceding example can also be solved by Faraday's law, taking $\mathcal{S}$ to be the time-varying surface bounded by $\mathcal{C}$.

End Matter

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