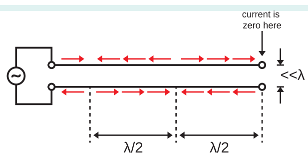
10 Antennas

10 Antennas

10.1 How Antennas Radiate

An antenna is a *transducer*; that is, a device which converts signals in one form into another form. In the case of an antenna, these two forms are (1) conductor-bound voltage and current signals and (2) electromagnetic waves. Traditional passive antennas are capable of this conversion in either direction. In this section, we consider the transmit case, in which a conductor-bound signal is converted into a radiating electromagnetic wave. Radiation from an antenna is due to the time-varying current that is excited by the bound electrical signal applied to the antenna terminals.

Why ideal transmission lines *don't* radiate. To describe the process that allows an antenna to transmit, it is useful to first consider the scenario depicted in Figure 10.1.



%

Figure 10.1. *Twin lead transmission line terminated into an open circuit, giving rise to a standing wave.*

Here a sinusoidal source is applied to the input of an ideal twin lead transmission line. The spacing between conductors is much less than a wavelength, and the output of the transmission line is terminated into an open circuit.

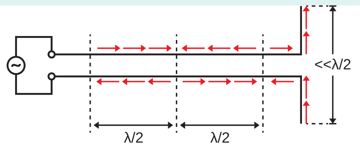
Without any additional information, we already know two things about the current on this transmission line. First, we know the current must be identically zero at the end of the transmission line. Second, we know the current on the two conductors comprising the transmission line must be related as indicated in Figure 10.1. That is, at any given position on the transmission line, the current on each conductor is equal in magnitude and flows in opposite directions.

Further note that Figure 10.1 depicts only the situation over an interval of one-half of the period of the sinusoidal source. For the other half-period, the direction of current will be in the direction *opposite* that depicted in Figure 10.1. In other words: The source is varying periodically, so the sign of the current is changing every half-period. Generally, time-varying currents give rise to radiation. Therefore, we should expect the currents depicted in Figure 10.1 might radiate. We can estimate the radiation from the transmission line by interpreting the current as a collection of Hertzian dipoles. For a review of Hertzian dipoles, see Section 9.4; however, all we need to know to address the present problem is that superposition applies. That is, the total radiation is the sum of the radiation from the individual Hertzian dipoles. Once again looking back at Figure 10.1, note that each Hertzian dipole representing current on one conductor has an associated Hertzian dipole representing the current on the other conductor, and is only a tiny fraction of a wavelength distant. Furthermore, these pairs of Hertzian dipoles are identical in magnitude but opposite in sign. Therefore, the radiated field from any such pair of Hertzian dipoles is approximately zero at distances sufficiently far from the transmission line. Continuing to sum all such pairs of Hertzian dipoles, the radiated field remains approximately zero at distances sufficiently far from the transmission line. We come to the following remarkable conclusion:

> The radiation from an ideal twin lead transmission line with open circuit termination is negligible at distances much greater than the separation between the conductors.

Before proceeding, let us be clear about one thing: The situation at distances which are *not* large relative to separation between the conductors is quite different. This is because the Hertzian dipole pairs do not appear to be quite so precisely collocated at field points close to the transmission line. Subsequently the cancellation of fields from Hertzian dipole pairs is less precise. The resulting sum fields are not negligible and depend on the separation between the conductors. For the present discussion, it suffices to restrict our attention to the simpler "far field" case.

A simple antenna formed by modifying twin lead transmission line. Let us now consider a modification to the previous scenario, shown in Figure 10.2.



%

Figure 10.2. *End of the twin lead transmission line fashioned into an electrically-short dipole (ESD).*

In the new scenario, the ends of the twin lead are bent into right angles. This new section has an overall length which is much less than one-half wavelength. To determine the current distribution on the modified section, we first note that the current must still be precisely zero at the ends of the conductors, but not necessarily at the point at which they are bent. Since the modified section is much shorter than one-half wavelength, the current distribution on this section must be very simple. In fact, the current distribution must be that of the electrically-short dipole (ESD), exhibiting magnitude which is maximum at the center and decreasing approximately linearly to zero at the ends. (See Section 9.5 for a review of the ESD.) Finally, we note that the current should be continuous at the junction between the unmodified and modified sections.

Having established that the modified section exhibits the current distribution of an ESD, and having previously determined that the unmodified section of transmission line does not radiate, we conclude that this system radiates precisely as an ESD does. Note also that we need not interpret the ESD portion of the transmission line as a modification of the transmission line: Instead, we may view this system as an unmodified transmission line attached to an antenna, which in this case in an ESD.

The general case. Although we developed this insight for the ESD specifically, the principle applies generally. That is, any antenna – not just dipoles – can be viewed as a structure that can support a current distribution that radiates. Elaborating:

A transmitting antenna is a device that, when driven by an appropriate source, supports a time-varying distribution of current resulting in an electromagnetic wave that radiates away from the device.

Although this may seem to be simply a restatement of the definition at the beginning of this section – and it is – we now see *how* this can happen, and also why transmission lines typically aren't also antennas.

10.2 Power Radiated by an Electrically-Short Dipole

In this section, we determine the total power radiated by an electrically-short dipole (ESD) antenna in response to a sinusoidally-varying current applied to the antenna terminals. This result is both useful on its own and necessary as an intermediate result in determining the impedance of the ESD. The ESD is introduced in Section 9.5, and a review of that section is suggested before attempting this section.

In Section 9.5, it is shown that the electric field intensity in the far field of a $\hat{\mathbf{z}}$-oriented ESD located at the origin is

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\eta \frac{I_0 \cdot \beta L}{8\pi} \ (\sin\theta) \ \frac{e^{-j\beta r}}{r} \tag{10.1}$$

where $\mathbf{r}$ is the field point, I_0 is a complex number representing the peak magnitude and phase of the sinusoidally-varying terminal current, L is the length of the ESD, and β is the phase propagation constant $2\pi/\lambda$ where λ is wavelength. Note that $L \ll \lambda$ since this is an ESD. Also note that Equation 10.1 is valid only for the far-field conditions $r \gg L$ and $r \gg \lambda$, and presumes propagation in simple (linear, homogeneous, time-invariant, isotropic) media with negligible loss.

Given that we have already limited scope to the far field, it is reasonable to approximate the electromagnetic field at each field point $\mathbf{r}$ as a plane wave propagating radially away from the antenna; i.e., in the $\hat{\mathbf{r}}$ direction. Under this assumption, the time-average power density is

$$\mathbf{S}(\mathbf{r}) = \hat{\mathbf{r}} \frac{\left|\widetilde{\mathbf{E}}(\mathbf{r})\right|^2}{2\eta} \tag{10.2}$$

where η is the wave impedance of the medium. The total power P_{rad} radiated by the antenna is simply $\mathbf{S}(\mathbf{r})$ integrated over any closed surface $\mathcal{S}$ that encloses the antenna. Thus:

$$P_{rad} = \oint_{\mathcal{S}} \mathbf{S}(\mathbf{r}) \cdot d\mathbf{s} \tag{10.3}$$

where $d\mathbf{s}$ is the outward-facing differential element of surface area. In other words, power density (W/m^2) integrated over an area (m^2) gives power (W). Anticipating that this problem will be addressed in spherical coordinates, we note that

$$d\mathbf{s} = \hat{\mathbf{r}} r^2 \ \sin\theta \ d\theta \ d\phi \tag{10.4}$$

and subsequently:

$$\begin{aligned} P_{rad} &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(\hat{\mathbf{r}} \frac{\left|\widetilde{\mathbf{E}}(\mathbf{r})\right|^2}{2\eta} \right) \cdot \left(\hat{\mathbf{r}} r^2 \ \sin\theta \ d\theta \ d\phi \right) \\ &= \frac{1}{2\eta} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left|\widetilde{\mathbf{E}}(\mathbf{r})\right|^2 r^2 \sin\theta \ d\theta \ d\phi \end{aligned} \tag{10.5}$$

Returning to Equation 10.1, we note:

$$\left|\widetilde{\mathbf{E}}(\mathbf{r})\right|^2 \approx \eta^2 \frac{\left|I_0\right|^2 (\beta L)^2}{64\pi^2} \ \ (\sin\theta)^2 \ \ \frac{1}{r^2} \tag{10.6}$$

Substitution into Equation 10.5 yields:

$$P_{rad} \approx \eta \frac{\left|I_0\right|^2 (\beta L)^2}{128\pi^2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin^3\theta \ d\theta \ d\phi \tag{10.7}$$

Note that this may be factored into separate integrals over θ and ϕ. The integral over ϕ is simply 2π, leaving:

$$P_{rad} \approx \eta \frac{\left|I_0\right|^2 (\beta L)^2}{64\pi} \int_{\theta=0}^{\pi} \sin^3\theta \ d\theta \tag{10.8}$$

The remaining integral is equal to $4/3$, leaving:

$$\boxed{P_{rad} \approx \eta \frac{\left|I_0\right|^2 (\beta L)^2}{48\pi}} \tag{10.9}$$

This completes the derivation, but it is useful to check units. Recall that β has SI base units of rad/m, so βL has units of radians. This leaves $\eta \left|I_0\right|^2$, which has SI base units of $\Omega \cdot$ A^2 $=$ W, as expected.

> The power radiated by an ESD in response to the current I_0 applied at the terminals is given by Equation 10.9.

Finally, it is useful to consider how various parameters affect the radiated power. First, note that the radiated power is proportional to the square of the terminal current. Second, note that the product $\beta L = 2\pi L/\lambda$ is the electrical length of the antenna; that is, the length of the antenna expressed in radians, where 2π radians is one wavelength. Thus, we see that the power radiated by the antenna increases as the square of electrical length.

Example

Example 10.1

Power radiated by an ESD.

A dipole is 10 cm in length and is surrounded by free space. A sinusoidal current having frequency 30 MHz and peak magnitude 100 mA is applied to the antenna terminals. What is the power radiated by the antenna?

Solution. If no power is dissipated within the antenna, then all power is radiated. The wavelength $\lambda = c/f \cong 10$ m, so $L \cong 0.01\lambda$. This certainly qualifies as electrically-short, so we may use Equation 10.9. In the present problem, $\eta \cong 376.7\ \Omega$ (the free space wave impedance), $I_0 = 100$ mA, and $\beta = 2\pi/\lambda \cong 0.628$ rad/m. Thus, we find that the radiated power is $\approx$ 98.6 μW.

10.3 Power Dissipated by an Electrically-Short Dipole

The power delivered to an antenna by a source connected to the terminals is nominally radiated. However, it is possible that some fraction of the power delivered by the source will be dissipated within the antenna. In this section, we consider the dissipation due to the finite conductivity of materials comprising the antenna. Specifically, we determine the total power dissipated by an electrically-short dipole (ESD) antenna in response to a sinusoidally-varying current applied to the antenna terminals. (The ESD is introduced in Section 9.5, and a review of that section is suggested before attempting this section.) The result allows us to determine *radiation efficiency* and is a step toward determining the impedance of the ESD.

Consider an ESD centered at the origin and aligned along the z axis. In Section 9.5, it is shown that the current distribution is:

$$\tilde{I}(z) \approx I_0 \left(1 - \frac{2}{L}\,|z|\right) \tag{10.10}$$

where I_0 (SI base units of A) is a complex-valued constant indicating the maximum current magnitude and phase, and L is the length of the ESD. This current distribution may be interpreted as a set of discrete very-short segments of constant-magnitude current (sometimes referred to as “Hertzian dipoles”). In this interpretation, the n^{th} current segment is located at $z = z_n$ and has magnitude $\tilde{I}(z_n)$.

Now let us assume that the material comprising the ESD is a homogeneous good conductor. Then each segment exhibits the same resistance R_{seg}. Subsequently, the power dissipated in the n^{th} segment is

$$P_{seg}(z_n) = \frac{1}{2}\left|\tilde{I}(z_n)\right|^2 R_{seg} \tag{10.11}$$

The segment resistance can be determined as follows. Let us assume that the wire comprising the ESD has circular cross-section of radius a. Since the wire is a good conductor, the segment resistance may be calculated using Equation 4.17 (Section 4.2, “Impedance of a Wire”):

$$R_{seg} \approx \frac{1}{2}\sqrt{\frac{\mu f}{\pi\sigma}} \cdot \frac{\Delta l}{a} \tag{10.12}$$

where μ is permeability, f is frequency, σ is conductivity, and Δl is the length of the segment. Substitution into Equation 10.11 yields:

$$P_{seg}(z_n) \approx \frac{1}{4a}\sqrt{\frac{\mu f}{\pi\sigma}} \left|\tilde{I}(z_n)\right|^2 \Delta l \tag{10.13}$$

Now the total power dissipated in the antenna, P_{loss}, can be expressed as the sum of the power dissipated in each segment:

$$P_{loss} \approx \sum_{n=1}^{N} \left[\frac{1}{4a}\sqrt{\frac{\mu f}{\pi\sigma}} \left|\tilde{I}(z_n)\right|^2 \Delta l\right] \tag{10.14}$$

where N is the number of segments. This may be rewritten as follows:

$$P_{loss} \approx \frac{1}{4a}\sqrt{\frac{\mu f}{\pi\sigma}} \sum_{n=1}^{N} \left|\tilde{I}(z_n)\right|^2 \Delta l \tag{10.15}$$

Reducing Δl to the differential length dz', we may write this in the following integral form:

$$P_{loss} \approx \frac{1}{4a}\sqrt{\frac{\mu f}{\pi\sigma}} \int_{z'=-L/2}^{+L/2} \left|\tilde{I}(z')\right|^2 dz' \tag{10.16}$$

It is worth noting that the above expression applies to *any* straight wire antenna of length L. For the ESD specifically, the current distribution is given by Equation 10.10. Making the substitution:

$$\begin{aligned} P_{loss} &\approx \frac{1}{4a}\sqrt{\frac{\mu f}{\pi\sigma}} \int_{z'=-L/2}^{+L/2} \left|I_0\left(1-\frac{2}{L}\left|z'\right|\right)\right|^2 dz' \\ &\approx \frac{1}{4a}\sqrt{\frac{\mu f}{\pi\sigma}} \left|I_0\right|^2 \int_{z'=-L/2}^{+L/2} \left|1-\frac{2}{L}\left|z'\right|\right|^2 dz' \end{aligned} \tag{10.17}$$

The integral is straightforward to solve, albeit a bit tedious. The integral is found to be equal to $L/3$, so we obtain:

$$\begin{aligned} P_{loss} &\approx \frac{1}{4a}\sqrt{\frac{\mu f}{\pi\sigma}} \left|I_0\right|^2 \cdot \frac{L}{3} \\ &\approx \frac{L}{12a}\sqrt{\frac{\mu f}{\pi\sigma}} \left|I_0\right|^2 \end{aligned} \tag{10.18}$$

Now let us return to interpretation of the ESD as a circuit component. We have found that applying the current I_0 to the terminals results in the dissipated power indicated in Equation 10.18. A current source driving the ESD does not perceive the current distribution of the ESD nor does it perceive the varying power over the length of the ESD. Instead, the current source perceives only a net resistance R_{loss} such that

$$P_{loss} = \frac{1}{2}|I_0|^2 R_{loss} \tag{10.19}$$

Comparing Equations 10.18 and 10.19, we find

$$\boxed{R_{loss} \approx \frac{L}{6a}\sqrt{\frac{\mu f}{\pi\sigma}}} \tag{10.20}$$

The power dissipated within an ESD in response to a sinusoidal current I_0 applied at the terminals is $\frac{1}{2}|I_0|^2 R_{loss}$ where R_{loss} (Equation 10.20) is the resistance perceived by a source applied to the ESD's terminals.

Note that R_{loss} is *not* the impedance of the ESD. R_{loss} is merely the contribution of internal loss to the impedance of the ESD. The impedance of the ESD must also account for contributions from radiation resistance (an additional real-valued contribution) and energy storage (perceived as a reactance). These quantities are addressed in other sections of this book.

Example

Example 10.2

Power dissipated within an ESD.

A dipole is 10 cm in length, 1 mm in radius, and is surrounded by free space. The antenna is comprised of aluminum having conductivity $\approx 3.7 \times 10^7$ S/m and $\mu \approx \mu_0$. A sinusoidal current having frequency 30 MHz and peak magnitude 100 mA is applied to the antenna terminals. What is the power dissipated within this antenna?

Solution. The wavelength $\lambda = c/f \cong 10$ m, so $L = 10 \text{ cm} \cong 0.01\lambda$. This certainly qualifies as electrically-short, so we may use Equation 10.20. In the present problem, $a = 1$ mm and $\sigma \approx 3.7 \times 10^7$ S/m. Thus, we find that the loss resistance $R_{loss} \approx 9.49 \text{ m}\Omega$. Subsequently, the power dissipated within this antenna is

$$P_{loss} = \frac{1}{2} |I_0|^2 R_{loss} \approx \underline{47.5\ \mu\text{W}} \tag{10.21}$$

We conclude this section with one additional caveat: Whereas this section focuses on the limited conductivity of wire, other physical mechanisms may contribute to the loss resistance of the ESD. In particular, materials used to coat the antenna or to provide mechanical support near the terminals may potentially absorb and dissipate power that might otherwise be radiated.

10.4 Reactance of the Electrically-Short Dipole

For any given time-varying voltage appearing across the terminals of an electrically-short antenna, there will be a time-varying current that flows in response. The ratio of voltage to current is impedance. Just as a resistor, capacitor, or inductor may be characterized in terms of an impedance, any antenna may be characterized in terms of this impedance. The real-valued component of this impedance accounts for power which is radiated away from the antenna (Section 10.2) and dissipated within the antenna (Section 10.3). The imaginary component of this impedance – i.e., the reactance – typically represents energy storage within the antenna, in the same way that the reactance of a capacitor or inductor represents storage of electrical or magnetic energy, respectively. In this section, we determine the reactance of the electrically-short dipole (ESD).

Reactance of a zero-length dipole. We begin with what might seem initially to be an absurd notion: A dipole antenna having zero length. However, such a dipole is certainly electrically-short, and in fact serves as an extreme condition from which we can deduce some useful information.

Consider Figure 10.3, which shows a zero-length dipole being driven by a source via a transmission line.

%

Figure 10.3. *Transmission line terminated into a zero-length dipole, which is equivalent to an open circuit.*

It is immediately apparent that such a dipole is equivalent to an open circuit. So, the impedance of a zero-length dipole is equal to that of an open circuit.

What is the impedance of an open circuit? One might be tempted to say “infinite,” since the current is zero independently of the voltage. However, we must now be careful to properly recognize the real and imaginary components of this infinite number, and signs of these components. In fact, the impedance of an open circuit is $0 - j\infty$. One way to confirm this is via basic transmission line theory; i.e., the input impedance of transmission line stubs. A more direct justification is as follows: The real part of the impedance is zero because there is no transfer of power into the termination. The magnitude of the imaginary part of the impedance must be infinite because the current is zero. The sign of the imaginary component is negative because the reflected current experiences a sign change (required to make the total current zero), whereas the reflected voltage does not experience a sign change (and so is not canceled at the terminals). Thus, the impedance of an open circuit, and the zero-length dipole, is $0 - j\infty$.

Reactance of a nearly-zero-length dipole. Let us now consider what happens as the length of the dipole increases from zero. Since such a dipole is short compared to a wavelength, the variation with length must be very simple. The reactance remains large and negative, but can only increase (become less negative) monotonically with increasing electrical length. This trend continues until the dipole is no longer electrically short. Thus, we conclude that the reactance of an ESD is always large and negative.

The reactance of an ESD is very large and negative, approaching the reactance of an open circuit termination as length decreases to zero.

Note that this behavior is similar to that of a capacitor, whose impedance is also negative and increases toward zero with increasing frequency. Thus, the reactance of an ESD is sometimes represented in circuit diagrams as a capacitor. However, the specific dependence of reactance on frequency is different from that of a capacitor (as we shall see in a moment), so this model is generally valid only for analysis at one frequency at time.

An approximate expression for the reactance of an ESD. Sometimes it is useful to be able to estimate the reactance X_A of an ESD. Although a derivation is beyond the scope of this text, a suitable expression is (see, e.g., Johnson (1993) in “Additional References” at the end of this section):

$$X_A \approx -\frac{120\ \Omega}{\pi L/\lambda}\left[\ln\left(\frac{L}{2a}\right) - 1\right] \tag{10.22}$$

where L is length and $a \ll L$ is the radius of the wire comprising the ESD. Note that this expression yields the expected behavior; i.e., $X_A \to -\infty$ as $L \to 0$, and increases monotonically as L increases from zero.

Example

Example 10.3

Reactance of an ESD.

A dipole is 10 cm in length, 1 mm in radius, and is surrounded by free space. What is the reactance of this antenna at 30 MHz?

Solution. The wavelength $\lambda = c/f \cong 10$ m, so $L = 10 \text{ cm} \cong 0.01\lambda$. This certainly qualifies as electrically-short, so we may use Equation 10.22. Given $a = 1$ mm, we find the reactance $X_A \approx \underline{-11.1 \text{ k}\Omega}$. For what it's worth, this antenna exhibits approximately the same reactance as a 0.47 pF capacitor at the same frequency.

The reactance of the ESD is typically orders of magnitude larger than the real part of the impedance of the ESD. The reactance of the ESD is also typically very large compared to the characteristic impedance of typical transmission lines (usually 10s to 100s of ohms). This makes ESDs quite difficult to use in practical transmit applications.

Additional Reading:

- R.C. Johnson (Ed.), *Antenna Systems Handbook* (Ch. 4), McGraw-Hill, 1993.

10.5 Equivalent Circuit Model for Transmission; Radiation Efficiency

A radio transmitter consists of a source which generates the electrical signal intended for transmission, and an antenna which converts this signal into a propagating electromagnetic wave. Since the transmitter is an electrical system, it is useful to be able to model the antenna as an equivalent circuit. From a circuit analysis point of view, it should be possible to describe the antenna as a passive one-port circuit that presents an impedance to the source. Thus, we have the following question: What is the equivalent circuit for an antenna which is transmitting?

We begin by emphasizing that the antenna is passive. That is, the antenna does not add power. Invoking the principle of conservation of power, there are only three possible things that *can* happen to power that is delivered to the antenna by the transmitter:[1]

- Power can be converted to a propagating electromagnetic wave. (The desired outcome.)
- Power can be dissipated within the antenna.
- Energy can be stored by the antenna, analogous to the storage of energy in a capacitor or inductor.

We also note that these outcomes can occur in any combination. Taking this into account, we model the antenna using the equivalent circuit shown in Figure 10.4.

%

Figure 10.4. *Equivalent circuit for an antenna which is transmitting.*

Since the antenna is passive, it is reasonable to describe it as an impedance Z_A which is (by definition) the ratio of voltage $\widetilde{V}_A$ to current $\tilde{I}_A$ at the terminals; i.e.,

$$Z_A \triangleq \frac{\widetilde{V}_A}{\tilde{I}_A} \tag{10.23}$$

In the phasor domain, Z_A is a complex-valued quantity and therefore has, in general, a real-valued component and an imaginary component. We may identify those components using the power conservation argument made previously: Since the real-valued component must represent power transfer and the imaginary component must represent energy storage, we infer:

$$Z_A \triangleq R_A + jX_A \tag{10.24}$$

where R_A represents power transferred to the antenna, and X_A represents energy stored by the antenna. Note that the energy stored by the antenna is being addressed in precisely the same manner that we address energy storage in a capacitor or an inductor; in all cases, as reactance. Further, we note that R_A consists of components R_{rad} and R_{loss} as follows:

$$Z_A = R_{rad} + R_{loss} + jX_A \tag{10.25}$$

where R_{rad} represents power transferred to the antenna and subsequently radiated, and R_{loss} represents power transferred to the antenna and subsequently dissipated.

To confirm that this model works as expected, consider what happens when a voltage is applied across the antenna terminals. The current $\tilde{I}_A$ flows and the time-average power P_A transferred to the antenna is

$$P_A = \frac{1}{2}\mathrm{Re}\left\{\widetilde{V}_A \tilde{I}_A^*\right\} \tag{10.26}$$

where we have assumed peak (as opposed to root mean squared) units for voltage and current. Since $\widetilde{V}_A = Z_A\tilde{I}_A$, we have:

$$P_A = \frac{1}{2}\mathrm{Re}\left\{\left(R_{rad} + R_{loss} + jX_A\right)\tilde{I}_A \tilde{I}_A^*\right\} \tag{10.27}$$

which reduces to:

$$P_A = \frac{1}{2}\left|\tilde{I}_A\right|^2 R_{rad} + \frac{1}{2}\left|\tilde{I}_A\right|^2 R_{loss} \tag{10.28}$$

As expected, the power transferred to the antenna is the sum of

$$P_{rad} \triangleq \frac{1}{2}\left|\tilde{I}_A\right|^2 R_{rad} \tag{10.29}$$

representing power transferred to the radiating electromagnetic field, and

$$P_{loss} \triangleq \frac{1}{2}\left|\tilde{I}_A\right|^2 R_{loss} \tag{10.30}$$

representing power dissipated within the antenna.

The reactance X_A will play a role in determining $\tilde{I}_A$ given $\widetilde{V}_A$ (and vice versa), but does not by itself account for disposition of power. Again, this is exactly analogous to the role played by inductors and capacitors in a circuit.

The utility of this equivalent circuit formalism is that it allows us to treat the antenna in the same manner as any other component, and thereby facilitates analysis using conventional electric circuit theory and transmission line theory. For example: Given Z_A, we know how to specify the output impedance Z_S of the transmitter so as to minimize reflection from the antenna: We would choose $Z_S = Z_A$, since in this case the voltage reflection coefficient would be

$$\Gamma = \frac{Z_A - Z_S}{Z_A + Z_S} = 0 \tag{10.31}$$

Alternatively, we might specify Z_S so as to maximize power transfer to the antenna: We would choose $Z_S = Z_A^*$; i.e., conjugate matching.

In order to take full advantage of this formalism, we require values for R_{rad}, R_{loss}, and X_A. These quantities are considered below.

Radiation resistance. R_{rad} is referred to as *radiation resistance*. Equation 10.29 tells us that

$$R_{rad} = 2P_{rad}\left|\tilde{I}_A\right|^{-2} \tag{10.32}$$

This equation suggests the following procedure: We apply current $\tilde{I}_A$ to the antenna terminals, and then determine the total power P_{rad} radiated from the antenna in response. For an example of this procedure, see Section 10.2 (“Total Power Radiated by an Electrically-Short Dipole”). Given $\tilde{I}_A$ and P_{rad}, one may then use Equation 10.32 to determine R_{rad}.

Loss resistance. *Loss resistance* represents the dissipation of power within the antenna, which is usually attributable to loss intrinsic to materials comprising or surrounding the antenna. In many cases, antennas are made from good conductors – metals, in particular – so that R_{loss} is very low compared to R_{rad}. For such antennas, loss is often so low compared to R_{rad} that R_{loss} may be neglected. In the case of the electrically-short dipole, R_{loss} is typically very small but R_{rad} is also very small, so both must be considered. In many other cases, antennas contain materials with substantially greater loss than metal. For example, a microstrip patch antenna implemented on a printed circuit board typically has non-negligible R_{loss} because the dielectric material comprising the antenna exhibits significant loss.

Antenna reactance. The reactance term jX_A accounts for energy stored by the antenna. This may be due to reflections internal to the antenna, or due to energy associated with non-propagating electric and magnetic fields surrounding the antenna. The presence of significant reactance (i.e., $|X_A|$ comparable to or greater than $|R_A|$) complicates efforts to establish the desired impedance match to the source. For an example, see Section 10.4 ("Reactance of the Electrically-Short Dipole").

Radiation efficiency. When R_{loss} is non-negligible, it is useful to characterize antennas in terms of their *radiation efficiency* e_{rad}, defined as the fraction of power which is radiated compared to the total power delivered to the antenna; i.e.,

$$e_{rad} \triangleq \frac{P_{rad}}{P_A} \tag{10.33}$$

Using Equations 10.28–10.30, we see that this efficiency can be expressed as follows:

$$e_{rad} = \frac{R_{rad}}{R_{rad} + R_{loss}} \tag{10.34}$$

Once again, the equivalent circuit formalism proves useful.

Example

Example 10.4

Impedance of an antenna.

The total power radiated by an antenna is 60 mW when 20 mA (rms) is applied to the antenna terminals. The radiation efficiency of the antenna is known to be 70%. It is observed that voltage and current are in-phase at the antenna terminals. Determine (a) the radiation resistance, (b) the loss resistance, and (c) the impedance of the antenna.

Solution. From the problem statement, $P_{rad} = 60$ mW, $\left|\tilde{I}_A\right| = 20$ mA (rms), and $e_{rad} = 0.7$. Also, the fact that voltage and current are in-phase at the antenna terminals indicates that $X_A = 0$. From Equation 10.32, the radiation resistance is

$$R_{rad} \approx \frac{2 \cdot (60\ \mathrm{mW})}{\left|\sqrt{2} \cdot 20\ \mathrm{mA}\right|^2} = \underline{150\ \Omega} \tag{10.35}$$

Solving Equation 10.34 for the loss resistance, we find:

$$R_{loss} = \frac{1 - e_{rad}}{e_{rad}} R_{rad} \cong \underline{64.3\ \Omega} \tag{10.36}$$

Since $Z_A = R_{rad} + R_{loss} + jX_A$, we find $Z_A \cong 214.3 + j0\ \Omega$. This will be the ratio of voltage to current at the antenna terminals regardless of the source current.

1. Note that "delivered" power means power accepted by the antenna. We are not yet considering power reflected from the antenna due to impedance mismatch. ↩

10.6 Impedance of the Electrically-Short Dipole

In this section, we determine the impedance of the electrically-short dipole (ESD) antenna. The physical theory of operation for the ESD is introduced in Section 9.5, and additional relevant aspects of the ESD antenna are addressed in Sections 10.1–10.4. The concept of antenna impedance is addressed in Section 10.5. Review of those sections is suggested before attempting this section.

The impedance of any antenna may be expressed as

$$Z_A = R_{rad} + R_{loss} + jX_A \tag{10.37}$$

where the real-valued quantities R_{rad}, R_{loss}, and X_A represent the radiation resistance, loss resistance, and reactance of the antenna, respectively.

Radiation resistance. The radiation resistance of any antenna can be expressed as:

$$R_{rad} = 2P_{rad}\left|I_0\right|^{-2} \tag{10.38}$$

where $|I_0|$ is the magnitude of the current at the antenna terminals, and P_{rad} is the resulting total power radiated. For an ESD (Section 10.2):

$$P_{rad} \approx \eta\frac{\left|I_0\right|^2(\beta L)^2}{48\pi} \tag{10.39}$$

so

$$R_{rad} \approx \eta\frac{(\beta L)^2}{24\pi} \tag{10.40}$$

It is useful to have an alternative form of this expression in terms of wavelength λ. This is derived as follows. First, note:

$$\beta L = \frac{2\pi}{\lambda}L = 2\pi\frac{L}{\lambda} \tag{10.41}$$

where L/λ is the antenna length in units of wavelength. Substituting this expression into Equation 10.40:

$$\begin{aligned} R_{rad} &\approx \eta\left(2\pi\frac{L}{\lambda}\right)^2\frac{1}{24\pi} \\ &\approx \eta\frac{\pi}{6}\left(\frac{L}{\lambda}\right)^2 \end{aligned} \tag{10.42}$$

Assuming free-space conditions, $\eta \cong 376.7~\Omega$, which is $\approx 120\pi~\Omega$. Subsequently,

$$\boxed{R_{rad} \approx 20\pi^2\left(\frac{L}{\lambda}\right)^2} \tag{10.43}$$

This remarkably simple expression indicates that the radiation resistance of an ESD is very small (since $L \ll \lambda$ for an ESD), but increases as the square of the length.

At this point, a warning is in order. The radiation resistance of an “electrically-short dipole” is sometimes said to be $80\pi^2\,(L/\lambda)^2$; i.e., 4 times the right side of Equation 10.43. This higher value is not for a *physically-realizable* ESD, but rather for the *Hertzian dipole* (sometimes also referred to as an “ideal dipole”). The Hertzian dipole is an electrically-short dipole with a current distribution that has uniform magnitude over the length of the dipole.[1] The Hertzian dipole is quite difficult to realize in practice, and nearly all practical electrically-short dipoles exhibit a current distribution that is closer to that of the ESD. The ESD has a current distribution which is maximum in the center and goes approximately linearly to zero at the ends. The factor-of-4 difference in the radiation resistance of the Hertzian dipole relative to the (practical) ESD is a straightforward consequence of the difference in the current distributions.

Loss resistance. The loss resistance of the ESD is derived in Section 10.3 and is found to be

$$R_{loss} \approx \frac{L}{6a}\sqrt{\frac{\mu f}{\pi\sigma}} \tag{10.44}$$

where a, μ, and σ are the radius, permeability, and conductivity of the wire comprising the antenna, and f is frequency.

Reactance. The reactance of the ESD is addressed in Section 10.4. A suitable expression for this reactance is (see, e.g., Johnson (1993) in “Additional References” at the end of this section):

$$X_A \approx -\frac{120\ \Omega}{\pi L/\lambda}\left[\ln\left(\frac{L}{2a}\right) - 1\right] \tag{10.45}$$

where $a \ll L$ is assumed.

Example

Example 10.5

Impedance of an ESD.

A thin, straight dipole antenna operates at 30 MHz in free space. The length and radius of the dipole are 1 m and 1 mm respectively. The dipole is comprised of aluminum having conductivity $\approx 3.7 \times 10^7$ S/m and $\mu \approx \mu_0$. What is the impedance and radiation efficiency of this antenna?

Solution. The free-space wavelength at $f = 30$ MHz is $\lambda = c/f \cong 10$ m. Therefore, $L \cong 0.1\lambda$, and this is an ESD. Since this is an ESD, we may compute the radiation resistance using Equation 10.43, yielding $R_{rad} \approx 1.97\ \Omega$. The radius $a = 1$ mm and $\sigma \approx 3.7 \times 10^7$ S/m; thus, we find that the loss resistance $R_{loss} \approx 94.9\ \text{m}\Omega$ using Equation 10.44. Using Equation 10.45, the reactance is found to be $X_A \approx -1991.8\ \Omega$. The impedance of this antenna is therefore

$$\begin{aligned} Z_A &= R_{rad} + R_{loss} + jX_A \\ &\approx \underline{2.1 - j1991.8\ \Omega} \end{aligned} \tag{10.46}$$

and the radiation efficiency of this antenna is

$$e_{rad} = \frac{R_{rad}}{R_{rad} + R_{loss}} \approx \underline{95.4\%} \tag{10.47}$$

In the preceding example, the radiation efficiency is a respectable 95.4%. However, the real part of the impedance of the ESD is much less than the characteristic impedance of typical transmission lines (typically 10s to 100s of ohms). Another problem is that the reactance of the ESD is very large. Some form of impedance matching is required to efficiently transfer power from a transmitter or transmission line to this form of antenna. Failure to do so will result in reflection of a large fraction of the power incident on antenna terminals. A common solution is to insert series inductance to reduce – nominally, cancel – the negative reactance of the ESD. In the preceding example, about $10\ \mu$H would be needed. The remaining mismatch in the real-valued components of impedance is then relatively easy to mitigate using common “real-to-real” impedance matching techniques.

Additional Reading:

- R.C. Johnson (Ed.), *Antenna Systems Handbook* (Ch. 4), McGraw-Hill, 1993.

---

1. See Section 9.4 for additional information about the Hertzian dipole. ↩

10.7 Directivity and Gain

A transmitting antenna does not radiate power uniformly in all directions. Inevitably more power is radiated in some directions than others. *Directivity* quantifies this behavior. In this section, we introduce the concept of directivity and the related concepts of *maximum directivity* and *antenna gain*.

Consider an antenna located at the origin. The power radiated in a single direction (θ,ϕ) is formally zero. This is because a single direction corresponds to a solid angle of zero, which intercepts an area of zero at any given distance from the antenna. Since the power flowing through any surface having zero area is zero, the power flowing in a single direction is formally zero. Clearly we need a different metric of power in order to develop a sensible description of the spatial distribution of power flow.

The appropriate metric is *spatial power density*; that is, power per unit area, having SI base units of W/m^2. Therefore, directivity is defined in terms of spatial power density in a particular direction, as opposed to power in a particular direction. Specifically, directivity in the direction (θ,ϕ) is:

$$D(\theta,\phi) \triangleq \frac{S(\mathbf{r})}{S_{ave}(r)} \tag{10.48}$$

In this expression, $S(\mathbf{r})$ is the power density at (r,θ,ϕ); i.e., at a distance r in the direction (θ,ϕ). $S_{ave}(r)$ is the *average* power density at that distance; that is, $S(\mathbf{r})$ averaged over all possible directions at distance r. Since directivity is a ratio of power densities, it is unitless. Summarizing:

> Directivity is ratio of power density in a specified direction to the power density averaged over all directions at the same distance from the antenna.

Despite Equation 10.48, directivity does not depend on the distance from the antenna. To be specific, directivity is the same at every distance r. Even though the numerator and denominator of Equation 10.48 both vary with r, one finds that the distance dependence always cancels because power density and average power density are both proportional to r^{-2}. This is a key point: Directivity is a convenient way to characterize an antenna because it does not change with distance from the antenna.

In general, directivity is a function of direction. However, one is often not concerned about all directions, but rather only the directivity in the direction in which it is maximum. In fact it is quite common to use the term “directivity” informally to refer to the maximum directivity of an antenna. This is usually what is meant when the directivity is indicated to be a single number; in any event, the intended meaning of the term is usually clear from context.

Example

Example 10.6

Directivity of the electrically-short dipole.

An electrically-short dipole (ESD) consists of a straight wire having length $L \ll \lambda/2$. What is the directivity of the ESD?

Solution. The field radiated by an ESD is derived in Section 9.5. In that section, we find that the electric field intensity in the far field of a $\hat{\mathbf{z}}$-oriented ESD located at the origin is:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\eta \frac{I_0 \cdot \beta L}{8\pi} \left(\sin\theta\right) \frac{e^{-j\beta r}}{r} \tag{10.49}$$

where I_0 represents the magnitude and phase of the current applied to the terminals, η is the wave impedance of the medium, and $\beta = 2\pi/\lambda$. In Section 10.2, we find that the power density of this field is:

$$S(\mathbf{r}) \approx \eta \frac{|I_0|^2 (\beta L)^2}{128\pi^2} \left(\sin\theta\right)^2 \frac{1}{r^2} \tag{10.50}$$

and we subsequently find that the total power radiated is:

$$P_{rad} \approx \eta \frac{|I_0|^2 (\beta L)^2}{48\pi} \tag{10.51}$$

The average power density S_{ave} is simply the total power divided by the area of a sphere centered on the ESD. Let us place this sphere at distance r, with $r \gg L$ and $r \gg \lambda$ as required for the validity of Equations 10.49 and 10.50. Then:

$$S_{ave} = \frac{P_{rad}}{4\pi r^2} \approx \eta \frac{|I_0|^2 (\beta L)^2}{192\pi^2 r^2} \tag{10.52}$$

Finally the directivity is determined by applying the definition:

$$D(\theta,\phi) \triangleq \frac{S(\mathbf{r})}{S_{ave}(r)} \tag{10.53}$$

$$\approx \underline{1.5 \left(\sin\theta\right)^2} \tag{10.54}$$

The maximum directivity occurs in the $\theta = \pi/2$ plane. Therefore, the maximum directivity is 1.5, meaning the maximum power density is 1.5 times greater than the power density averaged over all directions.

Since directivity is a unitless ratio, it is common to express it in decibels. For example, the maximum directivity of the ESD in the preceding example is $10 \log_{10} 1.5 \cong 1.76$ dB. (Note "$10 \log_{10}$" here since directivity is the ratio of power-like quantities.)

Gain. The gain $G(\theta, \phi)$ of an antenna is its directivity modified to account for loss within the antenna. Specifically:

$$G(\theta, \phi) \triangleq \frac{S(\mathbf{r}) \text{ for actual antenna}}{S_{ave}(r) \text{ for identical but lossless antenna}} \tag{10.55}$$

In this equation, the numerator is the actual power density radiated by the antenna, which is less than the nominal power density due to losses within the antenna. The denominator is the average power density for an antenna which is identical, but lossless. Since the actual antenna radiates less power than an identical but lossless version of the same antenna, gain in any particular direction is always less than directivity in that direction. Therefore, an equivalent definition of antenna gain is

$$G(\theta, \phi) \triangleq e_{rad} D(\theta, \phi) \tag{10.56}$$

where e_{rad} is the radiation efficiency of the antenna (Section 10.5).

Gain is directivity times radiation efficiency; that is, directivity modified to account for loss within the antenna.

The receive case. To conclude this section, we make one additional point about directivity, which applies equally to gain. The preceding discussion has presumed an antenna which is radiating; i.e., transmitting. Directivity can also be defined for the receive case, in which it quantifies the effectiveness of the antenna in converting power in an incident wave to power in a load attached to the antenna. Receive directivity is formally introduced in Section 10.13 ("Effective Aperture"). When receive directivity is defined as specified in Section 10.13, it is equal to transmit directivity as defined in this section. Thus, it is commonly said that the directivity of an antenna is the same for receive and transmit.

Additional Reading:

- "Directivity (https://en.wikipedia.org/wiki/Directivity)" on Wikipedia.

10.8 Radiation Pattern

The *radiation pattern* of a transmitting antenna describes the magnitude and polarization of the field radiated by the antenna as a function of angle relative to the antenna. A pattern may also be defined for a receiving antenna, however, we defer discussion of the receive case to a later section.

The concept of radiation pattern is closely related to the concepts of directivity and gain (Section 10.7). The principal distinction is the explicit consideration of polarization. In many applications, the polarization of the field radiated by a transmit antenna is as important as the power density radiated by the antenna. For example, a radio communication link consists of an antenna which is transmitting separated by some distance from an antenna which is receiving. Directivity determines the power density delivered to the receive antenna, but the receive antenna must be co-polarized with the arriving wave in order to capture all of this power.

The radiation pattern concept is perhaps best explained by example. The simplest antenna encountered in common practice is the electrically-short dipole (ESD), which consists of a straight wire of length L that is much less than one-half of a wavelength. In Section 9.5, it is shown that the field radiated by an ESD which is located at the origin and aligned along the z-axis is:

$$\widetilde{\mathbf{E}}(\mathbf{r}) \approx \hat{\theta} j\eta \frac{I_0 \cdot \beta L}{8\pi} \ (\sin\theta) \ \frac{e^{-j\beta r}}{r} \tag{10.57}$$

where I_0 represents the magnitude and phase of the current applied to the terminals, η is the wave impedance of the medium, and $\beta = 2\pi/\lambda$ is the phase propagation constant of the medium. Note that the reference direction of the electric field is in the $\hat{\theta}$ direction; therefore, a receiver must be $\hat{\theta}$-polarized relative to the transmitter's coordinate system in order to capture all available power. A receiver which is not fully $\hat{\theta}$-polarized will capture less power. In the extreme, a receiver which is $\hat{\phi}$-polarized with respect to the transmitter's coordinate system will capture zero power. Thus, for the $\hat{\mathbf{z}}$-oriented ESD, we refer to the $\hat{\theta}$-polarization of the transmitted field as "co-polarized" or simply "co-pol," and the $\hat{\phi}$-polarization of the transmitted field as "cross-pol."

At this point, the reader may wonder what purpose is served by defining cross polarization, since the definition given above seems to suggest that cross-pol should always be zero. In common engineering practice, cross-pol is non-zero when co-pol is different from the *intended* or *nominal* polarization of the field radiated by the antenna. In the case of the ESD, we observe that the electric field is always $\hat{\theta}$-polarized, and therefore we consider that to be the nominal polarization. Since the actual polarization of the ESD in the example is precisely the same as the nominal polarization of the ESD, the cross-pol of the ideal ESD is zero. If, on the other hand, we arbitrarily define $\hat{\theta}$ to be the nominal polarization and apply this definition to a different antenna that does not produce a uniformly $\hat{\theta}$-polarized electric field, then we observe non-zero cross-pol. Cross-pol is similarly used to quantify effects due to errors in position or orientation, or due to undesired modification of the field due to materials (e.g., feed or mounting structures) near the transmit antenna. Summarizing:

> *Co-pol* is commonly defined to be the intended or nominal polarization for a particular application, which is not necessarily the actual polarization radiated by the antenna under consideration. *Cross-pol* measures polarization in the orthogonal plane; i.e., deviation from the presumed co-pol.

Returning to the ESD: Since $\widetilde{\mathbf{E}}(\mathbf{r})$ depends only on θ and not ϕ, the co-pol pattern is the same in any plane that contains the z axis. We refer to any such plane as the *E-plane*. In general:

> The *E-plane* is any plane in which the nominal or intended vector $\widetilde{\mathbf{E}}$ lies.

Since the ESD is $\hat{\theta}$-polarized, the E-plane pattern of the ESD is simply:

$$\left|\widetilde{\mathbf{E}}(\mathbf{r})\right| \approx \eta \frac{I_0 \cdot \beta L}{8\pi} \ (\sin\theta) \ \frac{1}{r} \tag{10.58}$$

This pattern is shown in Figure 10.5.

%

Figure 10.5. *E-plane co-pol pattern for the $\hat{z}$-oriented ESD. In the unnormalized pattern scaling, the dashed circle indicates the maximum value of Equation 10.58.*

Equation 10.58 is referred to as an *unnormalized* pattern. The associated normalized pattern is addressed later in this section.

Similarly, we define the *H-plane* as follows:

The *H-plane* is any plane in which the nominal or intended vector $\widetilde{\mathbf{H}}$ lies, and so is perpendicular to both the E-plane and the direction of propagation.

In the case of the ESD, the one and only H-plane is the $z = 0$ plane. The H-plane pattern is shown in Figure 10.6.

%

Figure 10.6. *H-plane co-pol pattern for the $\hat{z}$-oriented ESD. In the unnormalized pattern scaling, the radius of the circle is the maximum value of Equation 10.58.*

It is often useful to normalize the pattern, meaning that we scale the pattern so that its maximum magnitude corresponds to a convenient value. There are two common scalings. One common scaling sets the maximum value of the pattern equal to 1, and is therefore independent of source magnitude and distance. This is referred to as the *normalized pattern*. The normalized co-pol pattern can be defined as follows:

$$F(\theta,\phi) \triangleq \frac{\left|\hat{\mathbf{e}}\cdot\widetilde{\mathbf{E}}(\mathbf{r})\right|}{\left|\hat{\mathbf{e}}\cdot\widetilde{\mathbf{E}}(\mathbf{r})\right|_{max}} \tag{10.59}$$

where $\hat{\mathrm{e}}$ is the co-pol reference direction, and the denominator is the maximum value of the electric field at distance r.

A *normalized pattern* is scaled to a maximum magnitude of 1, using the definition expressed in Equation 10.59.

Note that the value of r is irrelevant, since numerator and denominator both scale with r in the same way. Thus, the normalized pattern, like directivity, does not change with distance from the antenna.

For the ESD, $\hat{\mathrm{e}} = \hat{\theta}$, so the normalized co-pol pattern is:

$$F(\theta,\phi) = \frac{\left|\hat{\theta}\cdot\widetilde{\mathbf{E}}(\mathbf{r})\right|}{\left|\hat{\theta}\cdot\widetilde{\mathbf{E}}(\mathbf{r})\right|_{max}} \quad \text{(ESD)} \tag{10.60}$$

which yields simply

$$F(\theta,\phi) \approx \sin\theta \quad \text{(ESD)} \tag{10.61}$$

Thus, the E-plane normalized co-pol pattern of the ESD is Figure 10.5 where the radius of the maximum value circle is equal to 1, which is 0 dB. Similarly, the H-plane normalized co-pol pattern of the ESD is Figure 10.6 where the radius of the circle is equal to 1 (0 dB).

The other common scaling for patterns sets the maximum value equal to maximum directivity. Directivity is proportional to power density, which is proportional to $\left|\widetilde{\mathbf{E}}(\mathbf{r})\right|^2$. Therefore, this "directivity normalized" pattern can be expressed as:

$$D_{max}\left|F(\theta,\phi)\right|^2 \tag{10.62}$$

where D_{max} is the directivity in whichever direction it is maximum. For the ESD considered previously, $D_{max} = 1.5$ (Section 10.7). Therefore, the co-pol pattern of the ESD using this particular scaling is

$$1.5\sin^2\theta \quad \text{(ESD)} \tag{10.63}$$

Thus, the E-plane co-pol pattern of the ESD using this scaling is similar to Figure 10.5, except that it is squared in magnitude and the radius of the maximum value circle is equal to 1.5, which is 1.76 dB. The H-plane co-pol pattern of the ESD, using this scaling, is Figure 10.6 where the radius of the circle is equal to 1.5 (1.76 dB).

Pattern lobes. Nearly all antennas in common use exhibit directivity that is greater than that of the ESD. The patterns of these antennas subsequently exhibit more complex structure. Figure 10.7 shows an example.

%

Figure 10.7. *Co-pol pattern typical of a highly directional antenna, such as a yagi.*

The region around the direction of maximum directivity is referred to as the *main lobe*. The main lobe is bounded on each side by a *null*, where the magnitude reaches a local minimum, perhaps zero. Many antennas also exhibit a lobe in the opposite direction, known as a *backlobe*. (Many other antennas exhibit a null in this direction.) Lobes between the main lobe and the backlobe are referred to as *sidelobes*. Other commonly-used pattern metrics include *first sidelobe level*, which is the ratio of the maximum magnitude of the sidelobe closest to the main lobe to that of the main lobe; and *front-to-back ratio*, which is the ratio of the maximum magnitude of the backlobe to that of the main lobe.

When the main lobe is narrow, it is common to characterize the pattern in terms of *half-power beamwidth* (HPBW). This is shown in Figure 10.8.

%

Figure 10.8. *Half-power beamwidth (HPBW).*

HPBW is the width of the main lobe measured between two points at which the directivity is one-half its maximum value. As a pattern metric, HPBW depends on the plane in which it is measured. In particular, HPBW may be different in the E- and H-planes. For example, the E-plane HPBW of the ESD is found to be 90°, whereas the H-plane HPBW is undefined since the pattern is constant in the H-plane.

Omnidirectional and isotropic antennas. The ESD is an example of an *omnidirectional* antenna. An omnidirectional antenna is an antenna whose pattern magnitude is nominally constant in a plane containing the maximum directivity. For the ESD, this plane is the H-plane, so the ESD is said to be omnidirectional in the H-plane. The term "omnidirectional" does *not* indicate constant pattern in all directions. (In fact, note that the ESD exhibits pattern nulls in two directions.)

An *omnidirectional* antenna, such as the ESD, exhibits constant and nominally maximum directivity in one plane.

An antenna whose pattern is uniform in all directions is said to be *isotropic*. No physically-realizable antenna is isotropic; in fact, the ESD is about as close as one can get. Nevertheless, the "isotropic antenna" concept is useful as a standard against which other antennas can be quantified. Since the pattern of an isotropic antenna is constant with direction, the power density radiated by such an antenna in any direction is equal to the power density averaged over all directions. Thus, the directivity of an isotropic antenna is exactly 1. The directivity of any other antenna may be expressed in units of "dBi," which is simply dB relative to that of an isotropic antenna. For example, the maximum directivity of the ESD is 1.5, which is $10\log_{10}(1.5/1) =$ 1.76 dBi. Summarizing:

An *isotropic* antenna exhibits constant directivity in every direction. Such an antenna is not physically-realizable, but is nevertheless useful as a baseline for describing other antennas.

Additional Reading:

- "Radiation pattern (https://en.wikipedia.org/wiki/Radiation_pattern)" on Wikipedia.

10.9 Equivalent Circuit Model for Reception

In this section, we begin to address antennas as devices that convert incident electromagnetic waves into potentials and currents in a circuit. It is convenient to represent this process in the form of a Thévenin equivalent circuit. The particular circuit addressed in this section is shown in Figure 10.9.

%

Figure 10.9. *Thévenin equivalent circuit for an antenna in the presence of an incident electromagnetic wave.*

The circuit consists of a voltage source $\widetilde{V}_{OC}$ and a series impedance Z_A. The source potential $\widetilde{V}_{OC}$ is the potential at the terminals of the antenna when there is no termination; i.e., when the antenna is open-circuited. The series impedance Z_A is the output impedance of the circuit, and so determines the magnitude and phase of the current at the terminals once a load is connected. Given $\widetilde{V}_{OC}$ and the current through the equivalent circuit, it is possible to determine the power delivered to the load. Thus, this model is quite useful, but only if we are able to determine $\widetilde{V}_{OC}$ and Z_A. This section provides an informal derivation of these quantities that is sufficient to productively address the subsequent important topics of effective aperture and impedance matching of receive antennas.[1]

Vector effective length. With no derivation required, we can deduce the following about $\widetilde{V}_{OC}$:

- $\widetilde{V}_{OC}$ must depend on the incident electric field intensity $\widetilde{\mathbf{E}}^i$. Presumably the relationship is linear, so $\widetilde{V}_{OC}$ is proportional to the magnitude of $\widetilde{\mathbf{E}}^i$.
- Since $\widetilde{\mathbf{E}}^i$ is a vector whereas $\widetilde{V}_{OC}$ is a scalar, there must be some vector $\mathbf{l}_e$ for which

$$\widetilde{V}_{OC} = \widetilde{\mathbf{E}}^i \cdot \mathbf{l}_e \tag{10.64}$$

- Since $\widetilde{\mathbf{E}}^i$ has SI base units of V/m and $\widetilde{V}_{OC}$ has SI base units of V, $\mathbf{l}_e$ must have SI base units of m; i.e., length.
- We expect that $\widetilde{V}_{OC}$ increases as the size of the antenna increases, so the magnitude of $\mathbf{l}_e$ likely increases with the size of the antenna.
- The direction of $\mathbf{l}_e$ must be related to the orientation of the incident electric field relative to that of the antenna, since this is clearly important yet we have not already accounted for this.

It may seem at this point that $\mathbf{l}_e$ is unambiguously determined, and we need merely to derive its value. However, this is not the case. There are in fact multiple unique definitions of $\mathbf{l}_e$ that will reduce the vector $\widetilde{\mathbf{E}}^i$ to the observed scalar $\widetilde{V}_{OC}$ via Equation 10.64. In this section, we shall employ the most commonly-used definition, in which $\mathbf{l}_e$ is referred to as *vector effective length*. Following this definition, the scalar part l_e of $\mathbf{l}_e = \hat{\mathbf{l}} l_e$ is commonly referred to as any of the following: *effective length* (the term used in this book), *effective height*, or *antenna factor*.

In this section, we shall merely define vector effective length, and defer a formal derivation to Section 10.11. In this definition, we arbitrarily set $\hat{\mathbf{l}}$, the real-valued unit vector indicating the direction of $\mathbf{l}_e$, equal to the direction in which the electric field transmitted from this antenna would be polarized in the far field. For example, consider a $\hat{\mathbf{z}}$-oriented electrically-short dipole (ESD) located at the origin. The electric field transmitted from this antenna would have only a $\hat{\theta}$ component, and no $\hat{\phi}$ component (and certainly no $\hat{\mathbf{r}}$ component). Thus, $\hat{\mathbf{l}} = \hat{\theta}$ in this case.

Applying this definition, $\widetilde{\mathbf{E}}^i \cdot \hat{\mathbf{l}}$ yields the scalar component of $\widetilde{\mathbf{E}}^i$ that is co-polarized with electric field radiated by the antenna when transmitting. Now l_e is uniquely defined to be the factor that converts this component into $\widetilde{V}_{OC}$. Summarizing:

The *vector effective length* $\mathbf{l}_e = \hat{\mathbf{l}} l_e$ is defined as follows: $\hat{\mathbf{l}}$ is the real-valued unit vector corresponding to the polarization of the electric field that would be transmitted from the antenna in the far field. Subsequently, the *effective length* l_e is

$$l_e \triangleq \frac{\widetilde{V}_{OC}}{\widetilde{\mathbf{E}}^i \cdot \hat{\mathbf{l}}} \tag{10.65}$$

where $\widetilde{V}_{OC}$ is the open-circuit potential induced at the antenna terminals in response to the incident electric field intensity $\widetilde{\mathbf{E}}^i$.

While this definition yields an unambiguous value for l_e, it is not yet clear what that value is. For most antennas, effective length is quite difficult to determine directly, and one must instead determine effective length indirectly from the transmit characteristics via reciprocity. This approach is relatively easy (although still quite a bit of effort) for thin dipoles, and is presented in Section 10.11.

To provide an example of how effective length works right away, consider the $\hat{\mathbf{z}}$-oriented ESD described earlier in this section. Let the length of this ESD be L. Let $\widetilde{\mathbf{E}}^i$ be a $\hat{\theta}$-polarized plane wave arriving at the ESD. The ESD is open-circuited, so the potential induced in its terminals is $\widetilde{V}_{OC}$. One observes the following:

- When $\widetilde{\mathbf{E}}^i$ arrives from anywhere in the $\theta = \pi/2$ plane (i.e., broadside to the ESD), $\widetilde{\mathbf{E}}^i$ points in the $-\hat{\mathbf{z}}$ direction, and we find that $l_e \approx L/2$. It should not be surprising that l_e is proportional to L; this expectation was noted earlier in this section.
- When $\widetilde{\mathbf{E}}^i$ arrives from the directions $\theta = 0$ or $\theta = \pi$ – i.e., along the axis of the ESD – $\widetilde{\mathbf{E}}^i$ is perpendicular to the axis of the ESD. In this case, we find that l_e equals zero.

Taken together, these findings suggest that l_e should contain a factor of $\sin\theta$. We conclude that the vector effective length for a $\hat{\mathbf{z}}$-directed ESD of length L is

$$\mathbf{l}_e \approx \hat{\theta}\frac{L}{2}\sin\theta \qquad \text{(ESD)} \tag{10.66}$$

Example

Example 10.7

Potential induced in an ESD.

A thin straight dipole of length 10 cm is located at the origin and aligned with the z-axis. A plane wave is incident on the dipole from the direction $(\theta = \pi/4, \phi = \pi/2)$. The frequency of the wave is 30 MHz. The magnitude of the incident electric field is $10\ \mu$V/m (rms). What is the magnitude of the induced open-circuit potential when the electric field is (a) $\hat{\theta}$-polarized and (b) $\hat{\phi}$-polarized?

Solution. The wavelength in this example is $c/f \cong 10$ m, so this dipole is electrically-short. Using Equation 10.66:

$$\begin{aligned} \mathbf{l}_e &\approx \hat{\theta}\frac{10\ \text{cm}}{2}\sin\frac{\pi}{4} \\ &\approx \hat{\theta}\,(3.54\ \text{cm}) \end{aligned}$$

Thus, the effective length $l_e = 3.54$ cm. When the electric field is $\hat{\theta}$-polarized, the magnitude of the induced open-circuit voltage is

$$\begin{aligned} \left|\widetilde{V}_{OC}\right| &= \left|\widetilde{\mathbf{E}}^i \cdot \mathbf{l}_e\right| \\ &\approx (10\ \mu\text{V/m})\,\hat{\theta}\cdot\hat{\theta}\,(3.54\ \text{cm}) \\ &\approx \underline{354\ \text{nV}\ \ \text{rms}} \quad \text{(a)} \end{aligned}$$

When the electric field is $\hat{\phi}$-polarized:

$$\begin{aligned} \left|\widetilde{V}_{OC}\right| &\approx (10\ \mu\text{V/m})\,\hat{\phi}\cdot\hat{\theta}\,(3.54\ \text{cm}) \\ &\approx \underline{0} \quad \text{(b)} \end{aligned}$$

This is because the polarization of the incident electric field is orthogonal to that of the ESD. In fact, the answer to part (b) is zero for *any* angle of incidence (θ, ϕ).

Output impedance. The output impedance Z_A is somewhat more difficult to determine without a formal derivation, which is presented in Section 10.12. For the purposes of this section, it suffices to jump directly to the result:

The output impedance Z_A of the equivalent circuit for an antenna in the receive case is equal to the input impedance of the same antenna in the *transmit* case.

This remarkable fact is a consequence of the reciprocity property of antenna systems, and greatly simplifies the analysis of receive antennas.

Now a demonstration of how the antenna equivalent circuit can be used to determine the power delivered by an antenna to an attached electrical circuit:

Example

Example 10.8

Power captured by an ESD.

Continuing with part (a) of Example 10.7: If this antenna is terminated into a conjugate-matched load, then what is the power delivered to that load? Assume the antenna is lossless.

Solution. First, we determine the impedance Z_A of the equivalent circuit of the antenna. This is equal to the input impedance of the antenna in transmission. Let R_A and X_A be the real and imaginary parts of this impedance; i.e., $Z_A = R_A + jX_A$. Further, R_A is the sum of the radiation resistance R_{rad} and the loss resistance. The loss resistance is zero because the antenna is lossless. Since this is an ESD:

$$R_{rad} \approx 20\pi^2 \left(\frac{L}{\lambda}\right)^2 \tag{10.67}$$

Therefore, $R_A = R_{rad} \approx 4.93$ mΩ. We do not need to calculate X_A, as will become apparent in the next step.

A conjugate-matched load has impedance Z_A^*, so the potential $\widetilde{V}_L$ across the load is

$$\widetilde{V}_L = \widetilde{V}_{OC} \frac{Z_A^*}{Z_A + Z_A^*} = \widetilde{V}_{OC} \frac{Z_A^*}{2R_A} \tag{10.68}$$

The current $\tilde{I}_L$ through the load is

$$\tilde{I}_L = \frac{\widetilde{V}_{OC}}{Z_A + Z_A^*} = \frac{\widetilde{V}_{OC}}{2R_A} \tag{10.69}$$

Taking $\widetilde{V}_{OC}$ as an RMS quantity, the power P_L delivered to the load is

$$P_L = \text{Re}\left\{V_L I_L^*\right\} = \frac{\left|\widetilde{V}_{OC}\right|^2}{4R_A} \tag{10.70}$$

In part (a) of Example 10.7, $\left|\widetilde{V}_{OC}\right|$ is found to be ≈ 354 nV rms, so $P_L \approx$ 6.33 pW.

Additional Reading:

- "Thévenin's Theorem (https://en.wikipedia.org/wiki/Th%C3%A9venin%27s_theorem)" on Wikipedia.
- W.L. Stutzman & G.A. Thiele, *Antenna Theory and Design*, 3rd Ed., Wiley, 2012. Sec. 4.2 ("Receiving Properties of Antennas").

---

1. Formal derivations of these quantities are provided in subsequent sections. The starting point is the section on reciprocity. ↩

10.10 Reciprocity

The term "reciprocity" refers to a class of theorems that relate the inputs and outputs of a linear system to those of an identical system in which the inputs and outputs are swapped. The importance of reciprocity in electromagnetics is that it simplifies problems that would otherwise be relatively difficult to solve. An example of this is the derivation of the receiving properties of antennas, which is addressed in other sections using results derived in this section.

As an initial and relatively gentle introduction, consider a well-known special case that emerges in basic circuit theory: The two-port device shown in Figure 10.10.

%

Figure 10.10. *Two-port device.*

The two-port is said to be reciprocal if the voltage v_2 appearing at port 2 due to a current applied at port 1 is the same as v_1 when the same current is applied instead at port 2. This is normally the case when the two-port consists exclusively of linear passive devices such as ideal resistors, capacitors, and inductors. The fundamental underlying requirement is linearity: That is, outputs must be proportional to inputs, and superposition must apply.

This result from basic circuit theory is actually a special case of a more general theorem of electromagnetics, which we shall now derive. Figure 10.11 shows a scenario in which a current distribution $\widetilde{\mathbf{J}}_1$ is completely contained within a volume $\mathcal{V}$.

%

Figure 10.11. *An electromagnetic system consisting of a current distribution $\widetilde{\mathbf{J}}_1$ radiating an electric field $\widetilde{\mathbf{E}}_1$ in the presence of linear matter.*

%

Figure 10.12. *The same electromagnetic system as shown in Figure 10.11 (including the same distribution of matter), but now with a different input, $\widetilde{\mathbf{J}}_2$, resulting in a different output, $\widetilde{\mathbf{E}}_2$.*

This current is expressed as a volume current density, having SI base units of A/m^2. Also, the current is expressed in phasor form, signaling that we are considering a single frequency. This current distribution gives rise to an electric field intensity $\widetilde{\mathbf{E}}_1$, having SI base units of V/m. The volume may consist of any combination of linear time-invariant matter; i.e., permittivity ϵ, permeability μ, and conductivity σ are constants that may vary arbitrarily with position but not with time.

Here's a key idea: We may interpret this scenario as a "two-port" system in which $\widetilde{\mathbf{J}}_1$ is the input, $\widetilde{\mathbf{E}}_1$ is the output, and the system's behavior is completely defined by Maxwell's equations in differential phasor form:

$$\nabla \times \widetilde{\mathbf{E}}_1 = -j\omega\mu\widetilde{\mathbf{H}}_1 \tag{10.71}$$

$$\nabla \times \widetilde{\mathbf{H}}_1 = \widetilde{\mathbf{J}}_1 + j\omega\epsilon\widetilde{\mathbf{E}}_1 \tag{10.72}$$

along with the appropriate electromagnetic boundary conditions. (For the purposes of our present analysis, $\widetilde{\mathbf{H}}_1$ is neither an input nor an output; it is merely a coupled quantity that appears in this particular formulation of the relationship between $\widetilde{\mathbf{E}}_1$ and $\widetilde{\mathbf{J}}_1$.)

Figure 10.12 shows a scenario in which a different current distribution $\widetilde{\mathbf{J}}_2$ is completely contained within the same volume $\mathcal{V}$ containing the same distribution of linear matter. This new current distribution gives rise to an electric field intensity $\widetilde{\mathbf{E}}_2$. The relationship between $\widetilde{\mathbf{E}}_2$ and $\widetilde{\mathbf{J}}_2$ is governed by the same equations:

$$\nabla \times \widetilde{\mathbf{E}}_2 = -j\omega\mu\widetilde{\mathbf{H}}_2 \tag{10.73}$$

$$\nabla \times \widetilde{\mathbf{H}}_2 = \widetilde{\mathbf{J}}_2 + j\omega\epsilon\widetilde{\mathbf{E}}_2 \tag{10.74}$$

along with the same electromagnetic boundary conditions. Thus, we may interpret this scenario as the *same* electromagnetic system depicted in Figure 10.11, except now with $\widetilde{\mathbf{J}}_2$ as the input and $\widetilde{\mathbf{E}}_2$ is the output.

The input current and output field in the second scenario are, in general, completely different from the input current and output field in the first scenario. However, both scenarios involve precisely the same electromagnetic system; i.e., the same governing equations and the same distribution of matter. This leads to the following question: Let's say you know nothing about the system, aside from the fact that it is linear and time-invariant. However, you are able to observe the first scenario; i.e., you know $\widetilde{\mathbf{E}}_1$ in response to $\widetilde{\mathbf{J}}_1$. Given this very limited look into the behavior of the system, what can you infer about $\widetilde{\mathbf{E}}_2$ given $\widetilde{\mathbf{J}}_2$, or vice-versa? At first glance, the answer might seem to be nothing, since you lack a description of the system. However, the answer turns out to be that a surprising bit more information is available. This information is provided by reciprocity. To show this, we must engage in some pure mathematics.

Derivation of the Lorentz reciprocity theorem. First, let us take the dot product of $\widetilde{\mathbf{H}}_2$ with each side of Equation 10.71:

$$\widetilde{\mathbf{H}}_2 \cdot \left(\nabla \times \widetilde{\mathbf{E}}_1\right) = -j\omega\mu\widetilde{\mathbf{H}}_1 \cdot \widetilde{\mathbf{H}}_2 \tag{10.75}$$

Similarly, let us take the dot product of $\widetilde{\mathbf{E}}_1$ with each side of Equation 10.74:

$$\widetilde{\mathbf{E}}_1 \cdot \left(\nabla \times \widetilde{\mathbf{H}}_2\right) = \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 + j\omega\epsilon\widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{E}}_2 \tag{10.76}$$

Next, let us subtract Equation 10.75 from Equation 10.76. The left side of the resulting equation is

$$\widetilde{\mathbf{E}}_1 \cdot \left(\nabla \times \widetilde{\mathbf{H}}_2\right) - \widetilde{\mathbf{H}}_2 \cdot \left(\nabla \times \widetilde{\mathbf{E}}_1\right) \tag{10.77}$$

This can be simplified using the vector identity (Appendix B.3):

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{10.78}$$

Yielding

$$\widetilde{\mathbf{E}}_1 \cdot \left(\nabla \times \widetilde{\mathbf{H}}_2\right) - \widetilde{\mathbf{H}}_2 \cdot \left(\nabla \times \widetilde{\mathbf{E}}_1\right) = \nabla \cdot \left(\widetilde{\mathbf{H}}_2 \times \widetilde{\mathbf{E}}_1\right) \tag{10.79}$$

So, by subtracting Equation 10.75 from Equation 10.76, we have found:

$$\nabla \cdot \left(\widetilde{\mathbf{H}}_2 \times \widetilde{\mathbf{E}}_1\right) = \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 + j\omega\epsilon \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{E}}_2 + j\omega\mu \widetilde{\mathbf{H}}_1 \cdot \widetilde{\mathbf{H}}_2 \tag{10.80}$$

Next, we repeat the process represented by Equations 10.75–10.80 above to generate a complementary equation to Equation 10.80. This time we take the dot product of $\widetilde{\mathbf{E}}_2$ with each side of Equation 10.72:

$$\widetilde{\mathbf{E}}_2 \cdot \left(\nabla \times \widetilde{\mathbf{H}}_1\right) = \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 + j\omega\epsilon \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{E}}_2 \tag{10.81}$$

Similarly, let us take the dot product of $\widetilde{\mathbf{H}}_1$ with each side of Equation 10.73:

$$\widetilde{\mathbf{H}}_1 \cdot \left(\nabla \times \widetilde{\mathbf{E}}_2\right) = -j\omega\mu \widetilde{\mathbf{H}}_1 \cdot \widetilde{\mathbf{H}}_2 \tag{10.82}$$

Next, let us subtract Equation 10.82 from Equation 10.81. Again using the vector identity, the left side of the resulting equation is

$$\widetilde{\mathbf{E}}_2 \cdot \left(\nabla \times \widetilde{\mathbf{H}}_1\right) - \widetilde{\mathbf{H}}_1 \cdot \left(\nabla \times \widetilde{\mathbf{E}}_2\right) = \nabla \cdot \left(\widetilde{\mathbf{H}}_1 \times \widetilde{\mathbf{E}}_2\right) \tag{10.83}$$

So we find:

$$\nabla \cdot \left(\widetilde{\mathbf{H}}_1 \times \widetilde{\mathbf{E}}_2\right) = \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 + j\omega\epsilon \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{E}}_2 + j\omega\mu \widetilde{\mathbf{H}}_1 \cdot \widetilde{\mathbf{H}}_2 \tag{10.84}$$

Finally, subtracting Equation 10.84 from Equation 10.80, we obtain:

$$\nabla \cdot \left(\widetilde{\mathbf{H}}_2 \times \widetilde{\mathbf{E}}_1 - \widetilde{\mathbf{H}}_1 \times \widetilde{\mathbf{E}}_2\right) = \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 - \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 \tag{10.85}$$

This equation is commonly known by the name of the theorem it represents: The *Lorentz reciprocity theorem*. The theorem is a very general statement about the relationship between fields and currents at each point in space. An associated form of the theorem applies to contiguous regions of space. To obtain this form, we simply integrate both sides of Equation 10.85 over the volume $\mathcal{V}$:

$$\int_{\mathcal{V}} \nabla \cdot \left(\widetilde{\mathbf{H}}_2 \times \widetilde{\mathbf{E}}_1 - \widetilde{\mathbf{H}}_1 \times \widetilde{\mathbf{E}}_2\right) dv$$
$$= \int_{\mathcal{V}} \left(\widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 - \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1\right) dv \tag{10.86}$$

We now take the additional step of using the divergence theorem (Appendix B.3) to transform the left side of the equation into a surface integral:

$$\oint_{\mathcal{S}} \left(\widetilde{\mathbf{H}}_2 \times \widetilde{\mathbf{E}}_1 - \widetilde{\mathbf{H}}_1 \times \widetilde{\mathbf{E}}_2\right) \cdot d\mathbf{s}$$
$$= \int_{\mathcal{V}} \left(\widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 - \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1\right) dv \tag{10.87}$$

where $\mathcal{S}$ is the closed mathematical surface which bounds $\mathcal{V}$. This is also the Lorentz reciprocity theorem, but now in integral form. This version of the theorem relates fields on the bounding surface to sources within the volume.

The integral form of the theorem has a particularly useful feature. Let us confine the sources to a finite region of space, while allowing $\mathcal{V}$ to grow infinitely large, expanding to include all space. In this situation, the closest distance between any point containing non-zero source current and $\mathcal{S}$ is infinite. Because field magnitude diminishes with distance from the source, the fields $(\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{H}}_1)$ and $(\widetilde{\mathbf{E}}_2, \widetilde{\mathbf{H}}_2)$ are all effectively zero on $\mathcal{S}$. In this case, the left side of Equation 10.87 is zero, and we find:

$$\boxed{\int_{\mathcal{V}} \left(\widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 - \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1\right) dv = 0} \tag{10.88}$$

for any volume $\mathcal{V}$ which contains *all* the current.

The *Lorentz reciprocity theorem* (Equation 10.88) describes a relationship between one distribution of current and the resulting fields, and a second distribution of current and resulting fields, when both scenarios take place in identical regions of space filled with identical distributions of linear matter.

Why do we refer to this relationship as reciprocity? Simply because the expression is identical when the subscripts “1” and “2” are swapped. In other words, the relationship does not recognize a distinction between “inputs” and “outputs;” there are only “ports.”

A useful special case pertains to scenarios in which the current distributions $\widetilde{\mathbf{J}}_1$ and $\widetilde{\mathbf{J}}_2$ are *spatially disjoint*. By “spatially disjoint,” we mean that there is no point in space at which both $\widetilde{\mathbf{J}}_1$ and $\widetilde{\mathbf{J}}_2$ are non-zero; in other words, these distributions do not overlap. (Note that the currents shown in Figures 10.11 and 10.12 are depicted as spatially disjoint.) To see what happens in this case, let us first rewrite Equation 10.88 as follows:

$$\int_{\mathcal{V}} \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 \; dv = \int_{\mathcal{V}} \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 \; dv \tag{10.89}$$

Let $\mathcal{V}_1$ be the contiguous volume over which $\widetilde{\mathbf{J}}_1$ is non-zero, and let $\mathcal{V}_2$ be the contiguous volume over which $\widetilde{\mathbf{J}}_2$ is non-zero. Then Equation 10.89 may be written as follows:

$$\int_{\mathcal{V}_2} \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 \; dv = \int_{\mathcal{V}_1} \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 \; dv \tag{10.90}$$

The utility of this form is that we have reduced the region of integration to just those regions where the current exists.

Reciprocity of two-ports consisting of antennas. Equation 10.90 allows us to establish the reciprocity of two-ports consisting of pairs of antennas. This is most easily demonstrated for pairs of thin dipole antennas, as shown in Figure 10.13.

%

Figure 10.13. *A two-port consisting of two dipole antennas.*

Here, port 1 is defined by the terminal quantities $(\widetilde{V}_1, \tilde{I}_1)$ of the antenna on the left, and port 2 is defined by the terminal quantities $(\widetilde{V}_2, \tilde{I}_2)$ of the antenna on the right. These quantities are defined with respect to a small gap of length Δl between the perfectly-conducting arms of the dipole. Either antenna may transmit or receive, so $(\widetilde{V}_1, \tilde{I}_1)$ depends on $(\widetilde{V}_2, \tilde{I}_2)$, and vice-versa.

The transmit and receive cases for port 1 are illustrated in Figure 10.14(a) and (b), respectively.

%

Figure 10.14. *The port 1 antenna (a) transmitting and (b) receiving.*

Note that $\widetilde{\mathbf{J}}_1$ is the current distribution on this antenna when transmitting (i.e., when driven by the impressed current $\tilde{I}_1^t$) and $\widetilde{\mathbf{E}}_2$ is the electric field incident on the antenna when the other antenna is transmitting. Let us select $\mathcal{V}_1$ to be the cylindrical volume defined by the exterior surface of the dipole, including the gap between the arms of the dipole. We are now ready to consider the right side of Equation 10.90:

$$\int_{\mathcal{V}_1} \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 \; dv \tag{10.91}$$

The electromagnetic boundary conditions applicable to the perfectly-conducting dipole arms allow this integral to be dramatically simplified. First, recall that all current associated with a perfect conductor must flow on the surface of the material, and therefore $\widetilde{\mathbf{J}}_1 = 0$ everywhere except on the surface. Therefore, the direction of $\widetilde{\mathbf{J}}_1$ is always tangent to the surface. The tangent component of $\widetilde{\mathbf{E}}_2$ is zero on the surface of a perfectly-conducting material, as required by the applicable boundary condition. Therefore, $\widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 = 0$ everywhere $\widetilde{\mathbf{J}}_1$ is non-zero.

There is only one location where the current is non-zero and yet there is no conductor: This location is the gap located precisely at the terminals. Thus, we find:

$$\int_{\mathcal{V}_1} \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 \ dv = \int_{gap} \widetilde{\mathbf{E}}_2 \cdot \tilde{I}_1^t \hat{\mathbf{l}}_1 \ dl \tag{10.92}$$

where the right side is a line integral crossing the gap that defines the terminals of the antenna. We assume the current $\tilde{I}_1^t$ is constant over the gap, and so may be factored out of the integral:

$$\int_{\mathcal{V}_1} \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 \ dv = \tilde{I}_1^t \int_{gap} \widetilde{\mathbf{E}}_2 \cdot \hat{\mathbf{l}}_1 \ dl \tag{10.93}$$

Recall that potential between two points in space is given by the integral of the electric field over any path between those two points. In particular, we may calculate the open-circuit potential $\widetilde{V}_1^r$ as follows:

$$\widetilde{V}_1^r = -\int_{gap} \widetilde{\mathbf{E}}_2 \cdot \hat{\mathbf{l}}_1 \ dl \tag{10.94}$$

Remarkably, we have found:

$$\int_{\mathcal{V}_1} \widetilde{\mathbf{E}}_2 \cdot \widetilde{\mathbf{J}}_1 \ dv = -\tilde{I}_1^t \widetilde{V}_1^r \tag{10.95}$$

Applying the exact same procedure for port 2 (or by simply exchanging subscripts), we find:

$$\int_{\mathcal{V}_2} \widetilde{\mathbf{E}}_1 \cdot \widetilde{\mathbf{J}}_2 \ dv = -\tilde{I}_2^t \widetilde{V}_2^r \tag{10.96}$$

Now substituting these results into Equation 10.90, we find:

$$\boxed{\tilde{I}_1^t \widetilde{V}_1^r = \tilde{I}_2^t \widetilde{V}_2^r} \tag{10.97}$$

At the beginning of this section, we stated that a two-port is reciprocal if $\widetilde{V}_2$ appearing at port 2 due to a current applied at port 1 is the same as $\widetilde{V}_1$ when the same current is applied instead at port 2. If the present two-dipole problem is reciprocal, then $\widetilde{V}_2^r$ due to $\tilde{I}_1^t$ should be the same as $\widetilde{V}_1^r$ when $\tilde{I}_2^t = \tilde{I}_1^t$. Is it? Let us set $\tilde{I}_1^t$ equal to some particular value I_0, then the resulting value of $\widetilde{V}_2^r$ will be some particular value V_0. If we subsequently set $\tilde{I}_2^t$ equal to I_0, then the resulting value of $\widetilde{V}_1^r$ will be, according to Equation 10.97:

$$\widetilde{V}_1^r = \frac{\tilde{I}_2^t \widetilde{V}_2^r}{\tilde{I}_1^t} = \frac{I_0 V_0}{I_0} = V_0 \tag{10.98}$$

Therefore, Equation 10.97 is simply a mathematical form of the familiar definition of reciprocity from basic circuit theory, and we have found that our system of antennas is reciprocal in precisely this same sense.

The above analysis presumed pairs of straight, perfectly conducting dipoles of arbitrary length. However, the result is readily generalized – in fact, is the same – for *any* pair of passive antennas in linear time-invariant media. Summarizing:

> The potential induced at the terminals of one antenna due to a current applied to a second antenna is equal to the potential induced in the second antenna by the same current applied to the first antenna (Equation 10.97).

Additional Reading:

- "Reciprocity (electromagnetism) (https://en.wikipedia.org/wiki/Reciprocity_(electromagnetism))" on Wikipedia.
- "Two-port network (https://en.wikipedia.org/wiki/Two-port_network)" on Wikipedia.

10.11 Potential Induced in a Dipole

An electromagnetic wave incident on an antenna will induce a potential at the terminals of the antenna. In this section, we shall derive this potential. To simplify the derivation, we shall consider the special case of a straight thin dipole of arbitrary length that is illuminated by a plane wave. However, certain aspects of the derivation will apply to antennas generally. In particular, the concepts of *effective length* (also known as *effective height*) and *vector effective length* emerge naturally from this derivation, so this section also serves as a stepping stone in the development of an equivalent circuit model for a receiving antenna. The derivation relies on the transmit properties of dipoles as well as the principle of reciprocity, so familiarity with those topics is recommended before reading this section.

The scenario of interest is shown in Figure 10.15. Here a thin $\hat{\mathbf{z}}$-aligned straight dipole is located at the origin. The total length of the dipole is L. The arms of the dipole are perfectly-conducting. The terminals consist of a small gap of length Δl between the arms. The incident plane wave is described in terms of its electric field intensity $\widetilde{\mathbf{E}}^i$. The question is: What is $\widetilde{V}_{OC}$, the potential at the terminals when the terminals are open-circuited?

%

Figure 10.15. *A potential is induced at the terminals of a thin straight dipole in response to an incident plane wave.*

There are multiple approaches to solve this problem. A direct attack is to invoke the principle that potential is equal to the integral of the electric field intensity over a path. In this case, the path begins at the "$-$" terminal and ends at the "$+$" terminal, crossing the gap that defines the antenna terminals. Thus:

$$\widetilde{V}_{OC} = -\int_{gap} \widetilde{\mathbf{E}}_{gap} \cdot d\mathbf{l} \tag{10.99}$$

where $\widetilde{\mathbf{E}}_{gap}$ is the electric field in the gap. The problem with this approach is that the value of $\widetilde{\mathbf{E}}_{gap}$ is not readily available. It is not simply $\widetilde{\mathbf{E}}^i$, because the antenna structure (in particular, the electromagnetic boundary conditions) modify the electric field in the vicinity of the antenna.[1]

Fortunately, we can bypass this obstacle using the principle of reciprocity. In a reciprocity-based strategy, we establish a relationship between two scenarios that take place within the same electromagnetic system. The first scenario is shown in Figure 10.16.

%

Figure 10.16. *The dipole of interest driven by current $\tilde{I}_1^t$ radiates electric field $\widetilde{\mathbf{E}}_1$, resulting in open-circuit potential $\widetilde{V}_2^r$ at the terminals of a Hertzian dipole in the far field.*

%

Figure 10.17. *The Hertzian dipole driven by current $\tilde{I}_2^t$ radiates electric field $\widetilde{\mathbf{E}}_2$, resulting in open-circuit potential $\widetilde{V}_1^r$ at the terminals of the dipole of interest.*

In this scenario, we have two dipoles. The first dipole is precisely the dipole of interest (Figure 10.15), *except* that a current $\tilde{I}_1^t$ is applied to the antenna terminals. This gives rise to a current distribution $\tilde{I}(z)$ (SI base units of A) along the dipole, and subsequently the dipole radiates the electric field (Section 9.6):

$$\begin{aligned}\widetilde{\mathbf{E}}_1(\mathbf{r}) \approx \hat{\theta} j \frac{\eta}{2} \frac{e^{-j\beta r}}{r}\ (\sin\theta) \\ \cdot \left[\frac{1}{\lambda}\int_{-L/2}^{+L/2} \tilde{I}(z')e^{+j\beta z'\cos\theta}dz'\right]\end{aligned} \tag{10.100}$$

The second antenna is a $\hat{\theta}$-aligned Hertzian dipole in the far field, which receives $\widetilde{\mathbf{E}}_1$. (For a refresher on the properties of Hertzian dipoles, see Section 9.4. A key point is that Hertzian dipoles are vanishingly small.) Specifically, we measure (conceptually, at least) the open-circuit potential $\widetilde{V}_2^r$ at the terminals of the Hertzian dipole. We select a Hertzian dipole for this purpose because – in contrast to essentially all other antennas – it is simple to determine the open circuit potential. As explained earlier:

$$\widetilde{V}_2^r = -\int_{gap} \widetilde{\mathbf{E}}_{gap} \cdot d\mathbf{l} \tag{10.101}$$

For the Hertzian dipole, $\widetilde{\mathbf{E}}_{gap}$ *is* simply the incident electric field, since there is negligible structure (in particular, a negligible amount of material) present to modify the electric field. Thus, we have simply:

$$\widetilde{V}_2^r = -\int_{gap} \widetilde{\mathbf{E}}_1 \cdot d\mathbf{l} \tag{10.102}$$

Since the Hertzian dipole is very short and very far away from the transmitting dipole, $\widetilde{\mathbf{E}}_1$ is essentially constant over the gap. Also recall that we required the Hertzian dipole to be aligned with $\widetilde{\mathbf{E}}_1$. Choosing to integrate in a straight line across the gap, Equation 10.102 reduces to:

$$\widetilde{V}_2^r = -\widetilde{\mathbf{E}}_1(\mathbf{r}_2)\cdot\hat{\theta}\Delta l \tag{10.103}$$

where Δl is the length of the gap and $\mathbf{r}_2$ is the location of the Hertzian dipole. Substituting the expression for $\widetilde{\mathbf{E}}_1$ from Equation 10.100, we obtain:

$$\begin{aligned}\widetilde{V}_2^r \approx & -j\frac{\eta}{2}\frac{e^{-j\beta r_2}}{r_2}\ (\sin\theta) \\ & \cdot\left[\frac{1}{\lambda}\int_{-L/2}^{+L/2}\tilde{I}(z')e^{+j\beta z'\cos\theta}dz'\right]\Delta l\end{aligned} \tag{10.104}$$

where $r_2 = |\mathbf{r}_2|$.

The second scenario is shown in Figure 10.17. This scenario is identical to the first scenario, with the exception that the Hertzian dipole transmits and the dipole of interest receives. The field radiated by the Hertzian dipole in response to applied current $\tilde{I}_2^t$, evaluated at the origin, is (Section 9.4):

$$\widetilde{\mathbf{E}}_2(\mathbf{r}=0) \approx \hat{\theta}j\eta\frac{\tilde{I}_2^t\cdot\beta\Delta l}{4\pi}\ (1)\ \frac{e^{-j\beta r_2}}{r_2} \tag{10.105}$$

The "$\sin\theta$" factor in the general expression is equal to 1 in this case, since, as shown in Figure 10.17, the origin is located broadside (i.e., at $\pi/2$ rad) relative to the axis of the Hertzian dipole. Also note that because the Hertzian dipole is presumed to be in the far field, $\mathbf{E}_2$ may be interpreted as a plane wave in the region of the receiving dipole of interest.

Now we ask: What is the induced potential $\widetilde{V}_1^r$ in the dipole of interest? Once again, Equation 10.99 is not much help, because the electric field in the gap is not known. However, we do know that $\widetilde{V}_1^r$ should be proportional to $\widetilde{\mathbf{E}}_2(\mathbf{r}=0)$, since this is presumed to be a linear system. Based on this much information alone, there must be some vector $\mathbf{l}_e = \hat{\mathbf{l}}l_e$ for which

$$\widetilde{V}_1^r = \widetilde{\mathbf{E}}_2(\mathbf{r}=0)\cdot\mathbf{l}_e \tag{10.106}$$

This does not uniquely define either the unit vector $\hat{\mathbf{l}}$ nor the scalar part l_e, since a change in the definition of the former can be compensated by a change in the definition of the latter and vice-versa. So at this point we invoke the standard definition of $\mathbf{l}_e$ as the *vector effective length*, introduced in Section 10.9. Thus, $\hat{\mathbf{l}}$ is defined to be the direction in which an electric field *transmitted* from the antenna would be polarized. In the present example, $\hat{\mathbf{l}} = -\hat{\theta}$, where the minus sign reflects the fact that positive terminal potential results in terminal current which flows in the $-\hat{\mathbf{z}}$ direction. Thus, Equation 10.106 becomes:

$$\widetilde{V}_1^r = -\widetilde{\mathbf{E}}_2(\mathbf{r}=0)\cdot\hat{\theta}l_e \tag{10.107}$$

We may go a bit further and substitute the expression for $\widetilde{\mathbf{E}}_2(\mathbf{r}=0)$ from Equation 10.105:

$$\widetilde{V}_1^r \approx -j\eta \, \frac{\widetilde{I}_2^t \cdot \beta \Delta l}{4\pi} \, \frac{e^{-j\beta r_2}}{r_2} l_e \tag{10.108}$$

Now we invoke reciprocity. As a two-port linear time-invariant system, it must be true that:

$$\widetilde{I}_1^t \widetilde{V}_1^r = \widetilde{I}_2^t \widetilde{V}_2^r \tag{10.109}$$

Thus:

$$\widetilde{V}_1^r = \frac{\widetilde{I}_2^t}{\widetilde{I}_1^t} \widetilde{V}_2^r \tag{10.110}$$

Substituting the expression for $\widetilde{V}_2^r$ from Equation 10.104:

$$\begin{aligned} \widetilde{V}_1^r \approx & -\frac{\widetilde{I}_2^t}{\widetilde{I}_1^t} \cdot j\frac{\eta}{2} \frac{e^{-j\beta r_2}}{r_2} \; (\sin\theta) \\ & \cdot \left[\frac{1}{\lambda} \int_{-L/2}^{+L/2} \widetilde{I}(z') e^{+j\beta z' \cos\theta} dz' \right] \Delta l \end{aligned} \tag{10.111}$$

Thus, reciprocity has provided a second expression for $\widetilde{V}_1^r$. We may solve for l_e by setting this expression equal to the expression from Equation 10.108, yielding

$$l_e \approx \frac{2\pi}{\beta \widetilde{I}_1^t} \left[\frac{1}{\lambda} \int_{-L/2}^{+L/2} \widetilde{I}(z') e^{+j\beta z' \cos\theta} dz' \right] \sin\theta \tag{10.112}$$

Noting that $\beta = 2\pi/\lambda$, this simplifies to:

$$l_e \approx \left[\frac{1}{\widetilde{I}_1^t} \int_{-L/2}^{+L/2} \widetilde{I}(z') e^{+j\beta z' \cos\theta} dz' \right] \sin\theta \tag{10.113}$$

Thus, you can calculate l_e using the following procedure:

1. Apply a current $\widetilde{I}_1^t$ to the dipole of interest.
2. Determine the resulting current distribution $\widetilde{I}(z)$ along the length of the dipole. (Note that precisely this is done for the electrically-short dipole in Section 9.5 and for the half-wave dipole in Section 9.7.)
3. Integrate $\widetilde{I}(z)$ over the length of the dipole as indicated in Equation 10.113. Then divide ("normalize") by $\widetilde{I}_1^t$ (which is simply $\widetilde{I}(0)$). Note that the result is independent of the excitation $\widetilde{I}_1^t$, as expected since this is a linear system.
4. Multiply by $\sin\theta$.

We have now determined that the open-circuit terminal potential $\widetilde{V}_{OC}$ in response to an incident electric field $\widetilde{\mathbf{E}}^i$ is

$$\boxed{\widetilde{V}_{OC} = \widetilde{\mathbf{E}}^i \cdot \mathbf{l}_e} \tag{10.114}$$

where $\mathbf{l}_e = \hat{\mathbf{l}} l_e$ is the vector effective length defined previously.

This result is remarkable. In plain English, we have found that:

The potential induced in a dipole is the co-polarized component of the incident electric field times a normalized integral of the *transmit* current distribution over the length of the dipole, times sine of the angle between the dipole axis and the direction of incidence.

In other words, the reciprocity property of linear systems allows this property of a receiving antenna to be determined relatively easily if the transmit characteristics of the antenna are known.

Example

Example 10.9

Effective length of a thin electrically-short dipole (ESD).

A explained in Section 9.5, the current distribution on a thin ESD is

$$\tilde{I}(z) \approx I_0 \left(1 - \frac{2}{L}\left|z\right|\right) \tag{10.115}$$

where L is the length of the dipole and I_0 is the terminal current. Applying Equation 10.113, we find:

$$l_e \approx \left[\frac{1}{I_0}\int_{-L/2}^{+L/2} I_0 \left(1 - \frac{2}{L}\left|z'\right|\right) e^{+j\beta z' \cos\theta} dz'\right] \cdot \sin\theta \tag{10.116}$$

Recall $\beta = 2\pi/\lambda$, so $\beta z' = 2\pi\left(z'/\lambda\right)$. Since this is an *electrically-short* dipole, $z' \ll \lambda$ over the entire integral, and subsequently we may assume $e^{+j\beta z' \cos\theta} \approx 1$ over the entire integral. Thus:

$$l_e \approx \left[\int_{-L/2}^{+L/2} \left(1 - \frac{2}{L}\left|z'\right|\right) dz'\right] \sin\theta \tag{10.117}$$

The integral is easily solved using standard methods, or simply recognize that the "area under the curve" in this case is simply one-half "base" (L) times "height" (1). Either way, we find

$$l_e \approx \frac{L}{2} \sin\theta \tag{10.118}$$

Example 10.7 (Section 10.9) demonstrates how Equation 10.114 with the vector effective length determined in the preceding example is used to obtain the induced potential.

1. Also, if this were true, then the antenna itself would not matter; only the relative spacing and orientation of the antenna terminals would matter! ↩

10.12 Equivalent Circuit Model for Reception, Redux

Section 10.9 provides an informal derivation of an equivalent circuit model for a receiving antenna. This model is shown in Figure 10.18.

%

Figure 10.18. *Thévenin equivalent circuit model for an antenna in the presence of an incident electric field* $\mathbf{E}^i$.

The derivation of this model was informal and incomplete because the open-circuit potential $\mathbf{E}^i \cdot \mathbf{l}_e$ and source impedance Z_A were not rigorously derived in that section. While the open-circuit potential was derived in Section 10.11 ("Potential Induced in a Dipole"), the source impedance has not yet been addressed. In this section, the source impedance is derived, which completes the formal derivation of the model. Before reading this section, a review of Section 10.10 ("Reciprocity") is recommended.

The starting point for a formal derivation is the two-port model shown in Figure 10.19.

%

Figure 10.19. *Two-port system.*

Inductiveload, *public domain (modified)*

If the two-port is passive, linear, and time-invariant, then the potential v_2 is a linear function of potentials and currents present at ports 1 and 2. Furthermore, v_1 must be proportional to i_1, and similarly v_2 must be proportional to i_2, so any pair of "inputs" consisting of either potentials or currents completely determines the two remaining potentials or currents. Thus, we may write:

$$v_2 = Z_{21} i_1 + Z_{22} i_2 \tag{10.119}$$

where Z_{11} and Z_{12} are, for the moment, simply constants of proportionality. However, note that Z_{11} and Z_{12} have SI base units of Ω, and so we refer to these quantities as impedances. Similarly we may write:

$$v_1 = Z_{11} i_1 + Z_{12} i_2 \tag{10.120}$$

We can develop expressions for Z_{12} and Z_{21} as follows. First, note that $v_1 = Z_{12} i_2$ when $i_1 = 0$. Therefore, we may define Z_{12} as follows:

$$Z_{12} \triangleq \left.\frac{v_1}{i_2}\right|_{i_1=0} \tag{10.121}$$

The simplest way to make $i_1 = 0$ (leaving i_2 as the sole "input") is to leave port 1 open-circuited. In previous sections, we invoked special notation for these circumstances. In particular, we defined $\tilde{I}_2^t$ as i_2 in phasor representation, with the superscript "t" ("transmitter") indicating that this is the sole "input"; and $\widetilde{V}_1^r$ as v_1 in phasor representation, with the superscript "r" ("receiver") signaling that port 1 is both open-circuited and the "output." Applying this notation, we note:

$$Z_{12} = \frac{\widetilde{V}_1^r}{\tilde{I}_2^t} \tag{10.122}$$

Similarly:

$$Z_{21} = \frac{\widetilde{V}_2^r}{\tilde{I}_1^t} \tag{10.123}$$

In Section 10.10 (“Reciprocity”), we established that a pair of antennas could be represented as a passive linear time-invariant two-port, with v_1 and i_1 representing the potential and current at the terminals of one antenna (“antenna 1”), and v_2 and i_2 representing the potential and current at the terminals of another antenna (“antenna 2”). Therefore for any pair of antennas, quantities Z_{11}, Z_{12}, Z_{22}, and Z_{21} can be identified that completely determine the relationship between the port potentials and currents.

We also established in Section 10.10 that:

$$\tilde{I}_1^t \widetilde{V}_1^r = \tilde{I}_2^t \widetilde{V}_2^r \tag{10.124}$$

Therefore,

$$\frac{\widetilde{V}_1^r}{\tilde{I}_2^t} = \frac{\widetilde{V}_2^r}{\tilde{I}_1^t} \tag{10.125}$$

Referring to Equations 10.122 and 10.123, we see that Equation 10.125 requires that:

$$Z_{12} = Z_{21} \tag{10.126}$$

This is a key point. Even though we derived this equality by open-circuiting ports one at a time, the equality must hold generally since Equations 10.119 and 10.120 must apply – with the same values of Z_{12} and Z_{21} – regardless of the particular values of the port potentials and currents.

We are now ready to determine the Thévenin equivalent circuit for a receiving antenna. Let port 1 correspond to the transmitting antenna; that is, i_1 is $\tilde{I}_1^t$. Let port 2 correspond to an open-circuited receiving antenna; thus, $i_2 = 0$ and v_2 is $\widetilde{V}_2^r$. Now applying Equation 10.119:

$$\begin{aligned} v_2 &= Z_{21} i_1 + Z_{22} i_2 \\ &= \left(\widetilde{V}_2^r / \tilde{I}_1^t\right) \tilde{I}_1^t + Z_{22} \cdot 0 \\ &= \widetilde{V}_2^r \end{aligned} \tag{10.127}$$

We previously determined $\widetilde{V}_2^r$ from electromagnetic considerations to be (Section 10.11):

$$\widetilde{V}_2^r = \widetilde{\mathbf{E}}^i \cdot \mathbf{l}_e \tag{10.128}$$

where $\widetilde{\mathbf{E}}^i$ is the field incident on the receiving antenna, and $\mathbf{l}_e$ is the vector effective length as defined in Section 10.11. Thus, the voltage source in the Thévenin equivalent circuit for the receive antenna is simply $\widetilde{\mathbf{E}}^i \cdot \mathbf{l}_e$, as shown in Figure 10.18.

The other component in the Thévenin equivalent circuit is the series impedance. From basic circuit theory, this impedance is the ratio of v_2 when port 2 is open-circuited (i.e., $\widetilde{V}_2^r$) to i_2 when port 2 is short-circuited. This value of i_2 can be obtained using Equation 10.119 with $v_2 = 0$:

$$0 = Z_{21}\tilde{I}_1^t + Z_{22}i_2 \tag{10.129}$$

Therefore:

$$i_2 = -\frac{Z_{21}}{Z_{22}}\tilde{I}_1^t \tag{10.130}$$

Now using Equation 10.123 to eliminate $\tilde{I}_1^t$, we obtain:

$$i_2 = -\frac{\widetilde{V}_2^r}{Z_{22}} \tag{10.131}$$

Note that the reference direction for i_2 as defined in Figure 10.19 is opposite the reference direction for short-circuit current. That is, given the polarity of v_2 shown in Figure 10.19, the reference direction of current flow through a passive load attached to this port is from "$+$" to "$-$" through the load. Therefore, the source impedance, calculated as the ratio of the open-circuit potential to the short circuit current, is:

$$\frac{\widetilde{V}_2^r}{+\widetilde{V}_2^r/Z_{22}} = Z_{22} \tag{10.132}$$

We have found that the series impedance Z_A in the Thévenin equivalent circuit is equal to Z_{22} in the two-port model.

To determine Z_{22}, let us apply a current $i_2 = \tilde{I}_2^t$ to port 2 (i.e., antenna 2). Equation 10.119 indicates that we should see:

$$v_2 = Z_{21}i_1 + Z_{22}\tilde{I}_2^t \tag{10.133}$$

Solving for Z_{22}:

$$Z_{22} = \frac{v_2}{\tilde{I}_2^t} - Z_{21}\frac{i_1}{\tilde{I}_2^t} \tag{10.134}$$

Note that the first term on the right is precisely the impedance of antenna 2 *in transmission*. The second term in Equation 10.134 describes a contribution to Z_{22} from antenna 1. However, our immediate interest is in the equivalent circuit for reception of an electric field $\widetilde{\mathbf{E}}^i$ in the absence of any other antenna. We can have it both ways by imagining that $\widetilde{\mathbf{E}}^i$ is generated by antenna 1, but also that antenna 1 is far enough away to make Z_{21} – the factor that determines the effect of antenna 1 on antenna 2 – negligible. Then we see from Equation 10.134 that Z_{22} is the impedance of antenna 2 when transmitting.

Summarizing:

The Thévenin equivalent circuit for an antenna in the presence of an incident electric field $\widetilde{\mathbf{E}}^i$ is shown in Figure 10.18. The series impedance Z_A in this model is equal to the impedance of the antenna in transmission.

Mutual coupling. This concludes the derivation, but raises a follow-up question: What if antenna 2 is present and *not* sufficiently far away that Z_{21} can be assumed to be negligible? In this case, we refer to antenna 1 and antenna 2 as being "coupled," and refer to the effect of the presence of antenna 1 on antenna 2 as *coupling*. Often, this issue is referred to as *mutual coupling*, since the coupling affects both antennas in a reciprocal fashion. It is rare for coupling to be significant between antennas on opposite ends of a radio link. This is apparent from common experience. For example, changes to a receive antenna are not normally seen to affect the electric field incident at other locations. However, coupling becomes important when the antenna system is a *dense array*; i.e., multiple antennas separated by distances less than a few wavelengths. It is common for coupling among the antennas in a dense array to be significant. Such arrays can be analyzed using a generalized version of the theory presented in this section.

Additional Reading:

- "Thévenin's Theorem (https://en.wikipedia.org/wiki/Th%C3%A9venin%27s_theorem)" on Wikipedia.

10.13 Effective Aperture

When working with systems involving receiving antennas, it is convenient to have a single parameter that relates incident power (as opposed to incident electric field) to the power delivered to the receiver electronics. This parameter is commonly known as the *effective aperture* or *antenna aperture*.

From elementary circuit theory, the power delivered to a load Z_L is maximized when the load is conjugate matched to the antenna impedance; i.e., when $Z_L = Z_A^*$ where Z_A is the impedance of the antenna. Thus, a convenient definition for effective aperture A_e employs the relationship:

$$P_{R,max} \triangleq S_{co}^i A_e \tag{10.135}$$

where S_{co}^i is the incident power density (SI base units of W/m^2) that is co-polarized with the antenna, and $P_{R,max}$ is the power delivered to a load that is conjugate matched to the antenna impedance. Summarizing:

Effective aperture (SI base units of m^2) is the ratio of power delivered by an antenna to a conjugate matched load, to the incident co-polarized power density.

As defined, a method for calculation of effective aperture is not obvious, except perhaps through direct measurement. In fact, there are at least three ways we can calculate effective aperture: (a) via effective length, (b) via thermodynamics, and (c) via reciprocity. Each of these methods yields some insight and are considered in turn.

Effective aperture via effective length. The potential and current at the load of the receive antenna can be determined using the equivalent circuit model shown in Figure 10.20, using the Thévenin equivalent circuit model for the antenna developed in Section 10.9.

%

Figure 10.20. *Equivalent circuit model for an antenna in the presence of an incident electric field $\widetilde{\mathbf{E}}^i$, terminated into a load impedance Z_L.*

In this model, the voltage source is determined by the incident electric field intensity $\widetilde{\mathbf{E}}^i$ and the vector effective length $\mathbf{l}_\mathrm{e} = \hat{\mathbf{l}}_e l_e$ of the antenna. We determine the associated effective aperture as follows. Consider a co-polarized sinusoidally-varying plane wave $\widetilde{\mathbf{E}}^i_{co}$ incident on the antenna. Further, let

$$\widetilde{\mathbf{E}}^i_{co} = E^i \hat{\mathbf{e}} \tag{10.136}$$

where $\hat{\mathrm{e}}$ is the reference direction of $\widetilde{\mathbf{E}}^i_{co}$. The co-polarized power density incident on the antenna is:

$$S^i_{co} = \frac{\left|E^i\right|^2}{2\eta} \tag{10.137}$$

where η is the wave impedance of the medium (e.g., $\cong 377\ \Omega$ for an antenna in free space). In response, the time-average power delivered to the load is

$$P_R = \frac{1}{2}\mathrm{Re}\left\{\widetilde{V}_L \tilde{I}^*_L\right\} \tag{10.138}$$

where $\widetilde{V}_L$ and $\tilde{I}_L$ are the potential and current phasors, respectively, at the load. We may determine $\widetilde{V}_L$ and $\tilde{I}_L$ from the equivalent circuit model of Figure 10.20 using basic circuit theory. First, note that the magnitude and phase of the voltage source is:

$$\widetilde{\mathbf{E}}^i_{co} \cdot \mathbf{l}_e = E^i l_e \left(\hat{\mathbf{e}} \cdot \hat{\mathbf{l}}_e\right) \tag{10.139}$$

and since we have defined $\hat{\mathrm{l}}_e$ to be equal to $\hat{\mathrm{e}}$,

$$\widetilde{\mathbf{E}}^i_{co} \cdot \mathbf{l}_e = E^i l_e \tag{10.140}$$

Next, note that the load impedance creates a voltage divider with the source (antenna) impedance such that

$$\widetilde{V}_L = \left(E^i l_e\right) \frac{Z_L}{Z_A + Z_L} \tag{10.141}$$

Similarly, $\tilde{I}_L$ is the voltage source output divided by the total series resistance:

$$\tilde{I}_L = \frac{E^i l_e}{Z_A + Z_L} \tag{10.142}$$

Substituting these expressions into Equation 10.138, we obtain:

$$P_R = \frac{1}{2} \left| E^i l_e \right|^2 \frac{R_L}{\left| Z_A + Z_L \right|^2} \tag{10.143}$$

Let us identify the real and imaginary components of Z_A and Z_L explicitly, as follows:

$$Z_A = R_A + jX_A \tag{10.144}$$

$$Z_L = R_L + jX_L \tag{10.145}$$

Further, let us assume that loss internal to the antenna is negligible, so $R_A \approx R_{rad}$. If Z_L is conjugate matched for maximum power transfer, then $Z_L = Z_A^* = R_{rad} - jX_A$. Therefore:

$$\begin{aligned} P_{R,max} &= \frac{1}{2} \left| E^i l_e \right|^2 \frac{R_{rad}}{\left| Z_A + Z_A^* \right|^2} \\ &= \frac{1}{2} \left| E^i l_e \right|^2 \frac{R_{rad}}{\left| 2R_{rad} \right|^2} \\ &= \frac{\left| E^i l_e \right|^2}{8R_{rad}} \end{aligned} \tag{10.146}$$

Now using Equations 10.135, 10.137, and 10.146, we find:

$$A_e = \frac{\left| E^i l_e \right|^2 / 8R_{rad}}{\left| E^i \right|^2 / 2\eta} \tag{10.147}$$

which reduces to:

$$\boxed{A_e = \frac{\eta \left| l_e \right|^2}{4R_{rad}}} \tag{10.148}$$

It is not surprising that effective aperture should be proportional to the square of the magnitude of effective length. However, we see that the medium and the radiation resistance of the antenna also play a role.

Effective length and radiation resistance are easy to calculate for wire antennas, so it is natural to use Equation 10.148 to calculate the effective apertures of wire antennas. For the electrically-short dipole (ESD) of length L, $l_e \approx (L/2)\sin\theta$ and $R_{rad} \approx 20\pi^2 \left(L/\lambda\right)^2$. Thus, we find the effective aperture assuming free space (i.e., $\eta = \eta_0$) is:

$$A_e \approx 0.119\lambda^2 \left|\sin\theta\right|^2 \quad \text{(lossless ESD)} \tag{10.149}$$

Remarkably, the effective aperture of the ESD does not depend on its length. In fact, it depends only on the frequency of operation.

Also worth noting is that *maximum* effective aperture (i.e., the effective aperture in the $\theta = \pi/2$ direction) is

$$\boxed{A_e \approx 0.119\lambda^2} \quad \text{(lossless ESD, max.)} \tag{10.150}$$

Dipoles which are not electrically-short exhibit radiation resistance which is proportional to L^p, where $p > 2$. Therefore, the effective aperture of non-electrically-short dipoles *does* increase with L, as expected. However, this increase is not dramatic unless L is significantly greater than $\lambda/2$. Here's an example:

Example

Example 10.10

Effective aperture of a half-wave dipole.

The electrically-thin half-wave dipole exhibits radiation resistance $\cong 73\ \Omega$ and effective length λ/π. Assuming the dipole is lossless and in free space, Equation 10.148 yields:

$$A_e \approx 0.131\lambda^2 \quad \text{(half-wave dipole, max.)} \tag{10.151}$$

Again, this is the effective aperture for a wave incident from broadside to the dipole.

Effective aperture via thermodynamics. A useful insight into the concept of effective aperture can be deduced by asking the following question: How much power is received by an antenna when waves of equal power density arrive from all directions simultaneously? This question is surprisingly easy to answer using basic principles of thermodynamics. Thermodynamics is a field of physics that addresses heat and its relation to radiation and electrical energy. No previous experience with thermodynamics is assumed in this derivation.

Consider the scenario depicted in Figure 10.21.

%

Figure 10.21. *Thermodynamic analysis of the power exchanged between an antenna and its load.*

Chetvorno, public domain (modified)

In this scenario, the antenna is completely enclosed in a chamber whose walls do not affect the behavior of the antenna and which have uniform temperature T. The load (still conjugate matched to the antenna) is completely enclosed by a separate but identical chamber, also at temperature T.

Consider the load chamber first. Heat causes random acceleration of constituent charge carriers in the load, giving rise to a randomly-varying current at the load terminals. (This is known as *Johnson-Nyquist noise*.) This current, flowing through the load, gives rise to a potential; therefore, the load – despite being a passive device – is actually a source of power. The power associated with this noise is:

$$P_{load} = kTB \tag{10.152}$$

where $k \cong 1.38 \times 10^{-23}$ J/K is *Boltzmann's constant* and B is the bandwidth within which P_{load} is measured.

Similarly, the antenna is a source of noise power P_{ant}. P_{ant} can also be interpreted as captured *thermal radiation* – that is, electromagnetic waves stimulated by the random acceleration of charged particles comprising the chamber walls. These waves radiate from the walls and travel to the antenna. The *Rayleigh-Jeans law* of thermodynamics tells us that this radiation should have power density (i.e., SI base units of W/m^2) equal to

$$\frac{2kT}{\lambda^2}\ B \tag{10.153}$$

per steradian of solid angle.[1] The total power accessible to the antenna is one-half this amount, since an antenna is sensitive to only one polarization at a time, whereas the thermal radiation is equally distributed among any two orthogonal polarizations. P_A is the remaining power, obtained by integrating over all 4π steradians. Therefore, the antenna captures total power equal to[2]

$$\begin{aligned} P_{ant} &= \oint A_e(\theta',\phi') \left(\frac{1}{2}\ \frac{2kT}{\lambda^2}\ B\right) \sin\theta' d\theta' d\phi' \\ &= \left(\frac{kT}{\lambda^2}\ B\right) \oint A_e(\theta',\phi') \sin\theta' d\theta' d\phi' \end{aligned} \tag{10.154}$$

Given the conditions of the experiment, and in particular since the antenna chamber and the load chamber are at the same temperature, the antenna and load are in *thermodynamic equilibrium*. A consequence of this equilibrium is that power P_{ant} captured by the antenna and delivered to the load must be equal to power P_{load} generated by the load and delivered to the antenna; i.e.,

$$P_{ant} = P_{load} \tag{10.155}$$

Combining Equations 10.152, 10.154, and 10.155, we obtain:

$$\left(\frac{kT}{\lambda^2}\ B\right)\oint A_e(\theta',\phi')\sin\theta' d\theta' d\phi' = kTB \tag{10.156}$$

which reduces to:

$$\oint A_e(\theta',\phi')\sin\theta' d\theta' d\phi' = \lambda^2 \tag{10.157}$$

Let $\langle A_e\rangle$ be the *mean* effective aperture of the antenna; i.e., A_e averaged over all possible directions. This average is simply Equation 10.157 divided by the 4π sr of solid angle comprising all possible directions. Thus:

$$\langle A_e\rangle = \frac{\lambda^2}{4\pi} \tag{10.158}$$

Recall that a *isotropic* antenna is one which has the same effective aperture in every direction. Such antennas do not exist in practice, but the concept of an isotropic antenna is quite useful as we shall soon see. Since the effective aperture of an isotropic antenna must be the same as its mean effective aperture, we find:

$$\boxed{A_e = \frac{\lambda^2}{4\pi} \approx 0.080\lambda^2 \quad \text{(isotropic antenna)}} \tag{10.159}$$

Note further that this must be the *minimum possible* value of the maximum effective aperture for *any* antenna.

Effective aperture via reciprocity. The fact that all antennas have maximum effective aperture greater than that of an isotropic antenna makes the isotropic antenna a logical benchmark against which to compare the effective aperture of antennas. We can characterize the effective aperture of any antenna as being some unitless real-valued constant D times the effective aperture of an isotropic antenna; i.e.:

$$\boxed{A_e \triangleq D\frac{\lambda^2}{4\pi} \quad \text{(any antenna)}} \tag{10.160}$$

where D must be greater than or equal to 1.

Astute readers might notice that D seems a lot like *directivity*. Directivity is defined in Section 10.7 as the factor by which a *transmitting* antenna increases the power density of its radiation over that of an isotropic antenna. Let us once again consider the ESD, for which we previously determined (via the effective length concept) that $A_e \cong 0.119\lambda^2$ in the direction in which it is maximum. Applying Equation 10.160 to the ESD, we find:

$$D \triangleq A_e\frac{4\pi}{\lambda^2} \cong 1.50 \quad \text{(ESD, max.)} \tag{10.161}$$

Sure enough, D is equal to the directivity of the ESD in the transmit case.

This result turns out to be generally true; that is:

The effective aperture of *any* antenna is given by Equation 10.160 where D is the directivity of the antenna when transmitting.

A derivation of this result for the general case is possible using analysis similar to the thermodynamics presented earlier, or using the reciprocity theorem developed in Section 10.10.

The fact that effective aperture is easily calculated from transmit directivity is an enormously useful tool in antenna engineering. Without this tool, determination of effective aperture is limited to direct measurement or calculation via effective length; and effective length is generally difficult to calculate for antennas which are not well-described as distributions of current along a line. It is usually far easier to calculate the directivity of an antenna when transmitting, and then use Equation 10.160 to obtain effective aperture for receive operation.

Note that this principle is itself an expression of reciprocity. That is, one may fairly say that "directivity" is not exclusively a transmit concept nor a receive concept, but rather is a single quantifiable characteristic of an antenna that applies in both the transmit and receive case. Recall that radiation patterns are used to quantify the way (transmit) directivity varies with direction. Suddenly, we have found that precisely the same patterns apply to the receive case! Summarizing:

As long as the conditions required for formal reciprocity are satisfied, the directivity of a receiving antenna (defined via Equation 10.160) is equal to directivity of the same antenna when transmitting, and patterns describing receive directivity are equal to those for transmit directivity.

Finally, we note that the equivalence of transmit and receive directivity can, again via Equation 10.160, be used to define an effective aperture for the transmit case. In other words, we can define the effective aperture of a transmitting antenna to be $\lambda^2/4\pi$ times the directivity of the antenna. There is no new physics at work here; we are simply taking advantage of the fact that the concepts of effective aperture and directivity describe essentially the same characteristic of an antenna, and that this characteristic is the same for both transmit and receive operation.

Additional Reading:

- "Antenna aperture (https://en.wikipedia.org/wiki/Antenna_aperture)" on Wikipedia.
- "Boltzmann constant (https://en.wikipedia.org/wiki/Boltzmann_constant)" on Wikipedia.
- "Rayleigh-Jeans law (https://en.wikipedia.org/wiki/Rayleigh%E2%80%93Jeans_law)" on Wikipedia.
- "Johnson-Nyquist noise (https://en.wikipedia.org/wiki/Johnson%E2%80%93Nyquist_noise)" on Wikipedia.
- "Thermodynamics (https://en.wikipedia.org/wiki/Thermodynamics)" on Wikipedia.

---

1. Full disclosure: This is an approximation to the exact expression, but is very accurate at radio frequencies. See "Additional Reading" at the end of this section for additional details. ↩
2. Recall $\sin\theta' \, d\theta' \, d\phi'$ is the differential element of solid angle. ↩

10.14 Friis Transmission Equation

A common task in radio systems applications is to determine the power delivered to a receiver due to a distant transmitter. The scenario is shown in Figure 10.22:

Figure 10.22. *Radio link accounting for antenna gains and spreading of the transmitted wave.*

A transmitter delivers power P_T to an antenna which has gain G_T in the direction of the receiver. The receiver's antenna has gain G_R. As always, antenna gain is equal to directivity times radiation efficiency, so G_T and G_R account for losses internal to the antenna, but not losses due to impedance mismatch.

A simple expression for P_R can be derived as follows. First, let us assume "free space conditions"; that is, let us assume that the intervening terrain exhibits negligible absorption, reflection, or other scattering of the transmitted signal. In this case, the spatial power density at range R from the transmitter which radiates this power through a *lossless and isotropic* antenna would be:

$$\frac{P_T}{4\pi R^2} \tag{10.162}$$

that is, total transmitted power divided by the area of a sphere of radius R through which all the power must flow. The *actual* power density S^i is this amount times the gain of the transmit antenna, i.e.:

$$S^i = \frac{P_T}{4\pi R^2} G_T \tag{10.163}$$

The maximum received power is the incident co-polarized power density times the effective aperture A_e of the receive antenna:

$$\begin{aligned} P_{R,max} &= A_e S^i_{co} \\ &= A_e \frac{P_T}{4\pi R^2} G_T \end{aligned} \tag{10.164}$$

This assumes that the receive antenna is co-polarized with the incident electric field, and that the receiver is conjugate-matched to the antenna. The effective aperture can also be expressed in terms of the gain G_R of the receive antenna:

$$A_e = \frac{\lambda^2}{4\pi} G_R \tag{10.165}$$

Thus, Equation 10.164 may be written in the following form:

$$\boxed{P_{R,max} = P_T G_T \left(\frac{\lambda}{4\pi R}\right)^2 G_R} \tag{10.166}$$

This is the *Friis transmission equation*. Summarizing:

The *Friis transmission equation* (Equation 10.166) gives the power delivered to a conjugate-matched receiver in response to a distant transmitter, assuming co-polarized antennas and free space conditions.

The factor $(\lambda/4\pi R)^2$ appearing in the Friis transmission equation is referred to as *free space path gain*. More often this is expressed as the reciprocal quantity:

$$\boxed{L_p \triangleq \left(\frac{\lambda}{4\pi R}\right)^{-2}} \tag{10.167}$$

which is known as *free space path loss*. Thus, Equation 10.166 may be expressed as follows:

$$P_{R,max} = P_T G_T L_p^{-1} G_R \tag{10.168}$$

The utility of the concept of path loss is that it may also be determined for conditions which are different from free space. The Friis transmission equation still applies; one simply uses the appropriate (and probably significantly different) value of L_p.

A common misconception is that path loss is equal to the reduction in power density due to spreading along the path between antennas, and therefore this "spreading loss" increases with frequency. In fact, the reduction in power density due to spreading between any two distances $R_1 < R_2$ is:

$$\frac{P_T/4\pi R_1^2}{P_T/4\pi R_2^2} = \left(\frac{R_1}{R_2}\right)^2 \tag{10.169}$$

which is clearly independent of frequency. The path loss L_p, in contrast, depends only on the total distance R and does depend on frequency. The dependence on frequency reflects the dependence of the effective aperture on wavelength. Thus, path loss is not loss in the traditional sense, but rather accounts for a combination of spreading and the λ^2 dependence of effective aperture that is common to all receiving antennas.

Finally, note that Equation 10.168 is merely the simplest form of the Friis transmission equation. Commonly encountered alternative forms include forms in which G_T and/or G_R are instead represented by the associated effective apertures, and forms in which the effects of antenna impedance mismatch and/or cross-polarization are taken into account.

Example

Example 10.11

6 GHz point-to-point link.

Terrestrial telecommunications systems commonly aggregate large numbers of individual communications links into a single high-bandwidth link. This is often implemented as a radio link between dish-type antennas having gain of about 27 dBi (that's dB relative to a lossless isotropic antenna) mounted on very tall towers and operating at frequencies around 6 GHz. Assuming the minimum acceptable receive power is -120 dBm (that's -120 dB relative to 1 mW; i.e., 10^{-15} W) and the required range is 30 km, what is the minimum acceptable transmit power?

Solution. From the problem statement:

$$G_T = G_R = 10^{27/10} \cong 501 \tag{10.170}$$

$$\lambda = \frac{c}{f} \cong \frac{3 \times 10^8 \text{ m/s}}{6 \times 10^9 \text{ Hz}} \cong 5.00 \text{ cm} \tag{10.171}$$

$R = 30$ km, and $P_R \geq 10^{-15}$ W. We assume that the height and high directivity of the antennas yield conditions sufficiently close to free space. We further assume conjugate-matching at the receiver, and that the antennas are co-polarized. Under these conditions, $P_R = P_{R,max}$ and Equation 10.166 applies. We find:

$$\begin{aligned} P_T &\geq \frac{P_{R,max}}{G_T\,(\lambda/4\pi R)^2\,G_R} \\ &\cong 2.26 \times 10^{-7} \text{ W} \\ &\cong 2.26 \times 10^{-4} \text{ mW} \\ &\cong \underline{-36.5 \text{ dBm}} \end{aligned} \tag{10.172}$$

Additional Reading:

- "Free-space path loss (https://en.wikipedia.org/wiki/Free-space_path_loss)" on Wikipedia.
- "Friis transmission equation (https://en.wikipedia.org/wiki/Friis_transmission_equation)" on Wikipedia.

End Matter

Image Credits

- **Fig.** 10.1: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Standing_Wave_Creation.svg (https://commons.wikimedia.org/wiki/File:Standing_Wave_Creation.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.** 10.2: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Electrically-Short_Dipole.svg (https://commons.wikimedia.org/wiki/File:Electrically-Short_Dipole.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.** 10.3: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Zero-Length_Dipole_Open_Circuit.svg (https://commons.wikimedia.org/wiki/File:Zero-Length_Dipole_Open_Circuit.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.** 10.4: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Transmitting_Antenna_Circuit.svg (https://commons.wikimedia.org/wiki/File:Transmitting_Antenna_Circuit.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.** 10.5: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Magnitude_of_the_Radiated_Field.svg (https://commons.wikimedia.org/wiki/File:Magnitude_of_the_Radiated_Field.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.** 10.6: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:FHplaneMag.svg (https://commons.wikimedia.org/wiki/File:FHplaneMag.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.** 10.7: © T. Truckle, https://en.wikipedia.org/wiki/File:Sidelobes_en.svg (https://en.wikipedia.org/wiki/File:Sidelobes_en.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)). Modified.
- **Fig.** 10.8: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Half-Power_Beamwidth.svg (https://commons.wikimedia.org/wiki/File:Half-Power_Beamwidth.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.** 10.9: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Antenna_Equivalent_Circuit.svg (https://commons.wikimedia.org/wiki/File:Antenna_Equivalent_Circuit.svg), CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)). Modified.

- **Fig.**
10.10: Inductiveload, https://commons.wikimedia.org/wiki/File:Two_Port_Circuit.svg (https://commons.wikimedia.org/wiki/File:Two_Port_Circuit.svg),
public domain. Modified.
- **Fig.**
10.11: © Sevenchw (C. Wang),
https://commons.wikimedia.org/wiki/File:An_electromagnetic_system_consisting_of_a_current_distribution (https://commons.wikimedia.org/wiki/File:An_electromagnetic_system_consisting_of_a_current_distribution_radiating_an_electric_field.svg) _radiating_an_electric_field.svg,
CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.**
10.12: © Sevenchw (C. Wang),
https://commons.wikimedia.org/wiki/File:An_electromagnetic_system_consisting_of_a_different_current (https://commons.wikimedia.org/wiki/File:An_electromagnetic_system_consisting_of_a_different_current_distribution_radiating_an_electric_field.svg) _distribution_radiating_an_electric_field.svg,
CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.**
10.13: © Sevenchw (C. Wang), https://commons.wikimedia.org/wiki/File:A_two-port_consisting_of_two_dipole_antennas.svg (https://commons.wikimedia.org/wiki/File:A_two-port_consisting_of_two_dipole_antennas.svg),
CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.**
10.14: © Sevenchw (C. Wang),
https://commons.wikimedia.org/wiki/File:The_port_1_antenna_a_transmitting_and_b_receiving.svg (https://commons.wikimedia.org/wiki/File:The_port_1_antenna_a_transmitting_and_b_receiving.svg),
CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.**
10.15: © Sevenchw (C. Wang),
https://commons.wikimedia.org/wiki/File:Thin_straight_dipole_respond_to_incident_plane_wave.svg (https://commons.wikimedia.org/wiki/File:Thin_straight_dipole_respond_to_incident_plane_wave.svg),
CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.**
10.16: © Sevenchw (C. Wang),
https://commons.wikimedia.org/wiki/File:Dipole_of_interest_driven_by_current.svg (https://commons.wikimedia.org/wiki/File:Dipole_of_interest_driven_by_current.svg),
CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.**
10.17: © Sevenchw (C. Wang),
https://commons.wikimedia.org/wiki/File:Hertzian_dipole_drive_by_current.svg (https://commons.wikimedia.org/wiki/File:Hertzian_dipole_drive_by_current.svg),
CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).

- **Fig.**
  10.18: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Antenna_Equivalent_Circuit.svg (https://commons.wikimedia.org/wiki/File:Antenna_Equivalent_Circuit.svg),
  CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
- **Fig.**
  10.19: Inductiveload, https://commons.wikimedia.org/wiki/File:Two_Port_Circuit.svg (https://commons.wikimedia.org/wiki/File:Two_Port_Circuit.svg),
  public domain. Modified.
- **Fig.**
  10.20: © Offaperry (S. Lally), https://commons.wikimedia.org/wiki/File:Antenna_Equivalent_Circuit.svg (https://commons.wikimedia.org/wiki/File:Antenna_Equivalent_Circuit.svg),
  CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0/ (https://creativecommons.org/licenses/by-sa/4.0/)).
  Modified.
- **Fig.**
  10.21: Chetvorno, https://en.wikipedia.org/wiki/File:Antenna_and_resistor_in_cavity.svg (https://en.wikipedia.org/wiki/File:Antenna_and_resistor_in_cavity.svg),
  public domain. Modified.