

B Mathematical Formulas

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B.1 Trigonometry

$$e^{j\theta} = \cos \theta + j \sin \theta \quad (\text{B.1})$$

$$\cos \theta = \frac{1}{2} (e^{j\theta} + e^{-j\theta}) \quad (\text{B.2})$$

$$\sin \theta = \frac{1}{j2} (e^{j\theta} - e^{-j\theta}) \quad (\text{B.3})$$

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \quad (\text{B.4})$$

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta \quad (\text{B.5})$$

$$\sin (a \pm b) = \sin a \cos b \pm \cos a \sin b \quad (\text{B.6})$$

$$\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b \quad (\text{B.7})$$

Hyperbolic trigonometric functions:

$$\sinh \theta = \frac{1}{2} (e^{+\theta} - e^{-\theta}) \quad (\text{B.8})$$

$$\cosh \theta = \frac{1}{2} (e^{+\theta} + e^{-\theta}) \quad (\text{B.9})$$

B.2 Vector Operators

This section contains a summary of vector operators expressed in each of the three major coordinate systems:

- Cartesian (x, y, z)
- cylindrical (ρ, ϕ, z)
- spherical (r, θ, ϕ)

Associated basis vectors are identified using a caret (^) over the symbol. The vector operand $\mathbf{A}$ is expressed in terms of components in the basis directions as follows:

- Cartesian: $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$
- cylindrical: $\mathbf{A} = \hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{\mathbf{z}}A_z$
- spherical: $\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi$

Gradient

Gradient in Cartesian coordinates:

$$\nabla f = \hat{\mathbf{x}} \frac{\partial f}{\partial x} + \hat{\mathbf{y}} \frac{\partial f}{\partial y} + \hat{\mathbf{z}} \frac{\partial f}{\partial z} \quad (\text{B.10})$$

Gradient in cylindrical coordinates:

$$\nabla f = \hat{\rho} \frac{\partial f}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial f}{\partial z} \quad (\text{B.11})$$

Gradient in spherical coordinates:

$$\nabla f = \hat{\mathbf{r}} \frac{\partial f}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \quad (\text{B.12})$$

Divergence

Divergence in Cartesian coordinates:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (\text{B.13})$$

Divergence in cylindrical coordinates:

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad (\text{B.14})$$

Divergence in spherical coordinates:

$$\begin{aligned}\nabla \cdot \mathbf{A} = & \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) \\ & + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) \\ & + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}\end{aligned}\quad (\text{B.15})$$

Curl

Curl in Cartesian coordinates:

$$\begin{aligned}\nabla \times \mathbf{A} = & \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \\ & + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\ & + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)\end{aligned}\quad (\text{B.16})$$

Curl in cylindrical coordinates:

$$\begin{aligned}\nabla \times \mathbf{A} = & \hat{\rho} \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \\ & + \hat{\phi} \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \\ & + \hat{\mathbf{z}} \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right]\end{aligned}\quad (\text{B.17})$$

Curl in spherical coordinates:

$$\begin{aligned}\nabla \times \mathbf{A} = & \hat{\mathbf{r}} \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] \\ & + \hat{\theta} \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \\ & + \hat{\phi} \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right]\end{aligned}\quad (\text{B.18})$$

Laplacian

Laplacian in Cartesian coordinates:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\quad (\text{B.19})$$

Laplacian in cylindrical coordinates:

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \quad (\text{B.20})$$

Laplacian in spherical coordinates:

$$\begin{aligned} \nabla^2 f = & \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) \\ & + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial \theta} \sin \theta \right) \\ & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \end{aligned} \quad (\text{B.21})$$

B.3 Vector Identities

Algebraic Identities

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \quad (\text{B.22})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \quad (\text{B.23})$$

Identities Involving Differential Operators

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (\text{B.24})$$

$$\nabla \times (\nabla f) = 0 \quad (\text{B.25})$$

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) + (\nabla f) \times \mathbf{A} \quad (\text{B.26})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (\text{B.27})$$

$$\nabla \cdot (\nabla f) = \nabla^2 f \quad (\text{B.28})$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (\text{B.29})$$

$$\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \quad (\text{B.30})$$

Divergence Theorem: Given a closed surface $\mathcal{S}$ enclosing a contiguous volume $\mathcal{V}$,

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{A}) dv = \oint_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s} \quad (\text{B.31})$$

where the surface normal $d\mathbf{s}$ is pointing out of the volume.

Stokes' Theorem: Given a closed curve $\mathcal{C}$ bounding a contiguous surface $\mathcal{S}$,

$$\int_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l} \quad (\text{B.32})$$

where the direction of the surface normal $d\mathbf{s}$ is related to the direction of integration along $\mathcal{C}$ by the “right hand rule.”