7. The Hamiltonian

In this chapter, we consider a different approach to analytical mechanics, using the Hamiltonian instead of the Lagrangian. In Lagrangian mechanics, we generally obtained n second order differential equations corresponding to the n degrees of freedom in configuration space. In the Hamiltonian formalism, we will show that we can obtain $2n$ first order differential equations for the n degrees of freedom and the n generalized momenta. The generalized momenta are “promoted” to variables that describe the motion, and we speak of describing a system in “phase space” by specifying q_i and p_i, instead of specifying only the q_i in configuration space. The main difference is that the position in phase space completely specifies the past and future motions of the system, while in configuration space, one also needs to specify the velocities.

7.1 The Legendre Transform

The Hamiltonian formalism can be formally derived through the use of the Legendre transforms, which we first introduce here. Consider a function, $f(u_1, \dots, u_n)$, that depends on n variables, u_i. Now consider new variables, v_i, given by:

$$v_i \equiv \frac{\partial f}{\partial u_i} \tag{7.1}$$

The Legendre transformation takes the function $f(u_i)$ into a new function, $g(v_i)$, that only depends on the v_i. The transformation is given by:

$$g(v_i) = \sum_i u_i v_i - f \tag{7.2}$$

It is not immediately apparent that g does not depend on the u_i, but we can verify this by considering the variation of g:

$$\begin{aligned}\delta g &= \delta\left(\sum_i u_i v_i - f\right)\\ &= \sum_i (u_i\delta v_i + v_i \delta u_i) - \sum_i \frac{\partial f}{\partial u_i}\delta u_i\\ &= \sum_i u_i \delta v_i + \sum_i\left(v_i - \frac{\partial f}{\partial u_i}\right)\delta u_i\\ &= \sum_i u_i\delta v_i\end{aligned} \tag{7.3}$$

where in the last line, we used the definition of v_i, to set the last term equal to zero. Thus, the variation of g only depends on the variations of the v_i and not of the u_i. We can thus write $g = g(v_i)$ and the variation of g as:

$$\delta g = \sum_i \frac{\partial g}{\partial v_i}\delta v_i \tag{7.4}$$

and make the following identification:

$$u_i = \frac{\partial g}{\partial v_i} \tag{7.5}$$

The Legendre Transformation thus has a nice set of symmetries:

$$\begin{aligned} v_i &= \frac{\partial f}{\partial u_i} \\ u_i &= \frac{\partial g}{\partial v_i} \\ g &= \sum_i u_i v_i - f \\ f &= \sum_i u_i v_i - g \end{aligned} \tag{7.6}$$

Now consider the case when f depends on two sets of variables, u_i, and w_i, $f = f(u_i, w_i)$, and we again define the v_i as follows:

$$v_i \equiv \frac{\partial f(u_i, w_i)}{\partial u_i} \tag{7.7}$$

We thus call the u_i the “active” variables, and the w_i, the “passive” variables. Again, we define g:

$$g \equiv \sum_i u_i v_i - f(u_i, w_i) \tag{7.8}$$

The variation of g is given by:

$$\begin{aligned}\delta g &= \delta\left(\sum_i u_i v_i - f(u_i, w_i)\right)\\ &= \sum_i (u_i\delta v_i + v_i\delta u_i) - \sum_i\left(\frac{\partial f}{\partial u_i}\delta u_i + \frac{\partial f}{\partial w_i}\delta w_i\right)\\ &= \sum_i\left(u_i\delta v_i - \frac{\partial f}{\partial w_i}\delta w_i\right) + \sum_i\left(v_i - \frac{\partial f}{\partial u_i}\right)\delta u_i\\ &= \sum_i\left(u_i\delta v_i - \frac{\partial f}{\partial w_i}\delta w_i\right)\end{aligned} \tag{7.9}$$

which again is independent of the variation on the u_i. Since $g = g(v_i, w_i)$, we can also write the variation as:

$$\delta g_i = \sum_i\left(\frac{\partial g}{\partial v_i}\delta v_i + \frac{\partial g}{\partial w_i}\delta w_i\right) \tag{7.10}$$

and immediately identify:

$$\begin{aligned}u_i &= \frac{\partial g}{\partial v_i}\\ \frac{\partial g}{\partial w_i} &= -\frac{\partial f}{\partial w_i}\end{aligned} \tag{7.11}$$

Note that we used the case where there are the same number of w_i as there are u_i and v_i; it is easily seen that the results do not depend on this.

7.2 Legendre Transform of the Lagrangian

Consider now the Lagrangian, $L(q_i, \dot{q}_i, t)$, where we will consider the q_i as the passive variables, and the $\dot{q}_i$ as the active variables. We introduce a new set of variables, p_i, given by:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{7.12}$$

and we introduce a new function, $H(q_i, p_i, t)$, called the Hamiltonian, which is the Legendre transform of the Lagrangian:

$$H(q_i, p_i, t) \equiv \sum_i p_i\dot{q}_i - L \tag{7.13}$$

Again, consider the variation of the Hamiltonian:

$$\begin{aligned}\delta H &= \delta\left(\sum_i p_i\dot{q}_i - L\right)\\ &= \sum_i (p_i\delta\dot{q}_i + \dot{q}_i\delta p_i) - \sum_i\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right) - \frac{\partial L}{\partial t}\delta t\\ &= \sum_i\left(\dot{q}_i\delta p_i - \frac{\partial L}{\partial q_i}\delta q_i\right) - \frac{\partial L}{\partial t}\delta t\end{aligned} \tag{7.14}$$

Again, we can identify this with the variation of $H(q_i, p_i, t)$:

$$\delta H = \sum_i\left(\frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial q_i}\delta q_i\right) + \frac{\partial H}{\partial t}\delta t \tag{7.15}$$

to obtain:

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}\\ \frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i}\\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}\end{aligned} \tag{7.16}$$

Note that the Hamiltonian can and must always be written explicitly only in terms of the generalized momenta and the coordinates (that is, the velocities should always be eliminated in favour of the momenta).

7.3 The Canonical equations

Again, consider the relations for transforming back and forth between the Lagrangian and the Hamiltonian:

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}\\ p_i &= \frac{\partial L}{\partial \dot{q}_i}\end{aligned} \tag{7.17}$$

The second equation can be re-arranged using the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} &= \frac{\partial L}{\partial q_i}\\ \therefore \dot{p} &= \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}\end{aligned} \tag{7.18}$$

We can write the transformation equations as:

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}\\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}\end{aligned}\tag{7.19}$$

These are called the “canonical equations of Hamilton”. They are $2n$ first order differential equations that specify the location, p_i, q_i, of the system in phase space, and have the property that the total time derivatives are isolated on one side of the equation. Note that if a coordinate is cyclic in the Hamiltonian, conservation of the associated generalized momentum follows immediately. In order to solve these equations in configuration space, one can substitute the first equation into the second one and re-obtain the second-order equation for the coordinate as a function of time that one obtains from the Euler-Lagrange equations.

Consider now the total time derivative of the Hamiltonian, $H(p_i, q_i, t)$:

$$\begin{aligned}\frac{dH}{dt} &= \sum_i \left(\frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i\right) + \frac{\partial H}{\partial t}\\ &= \sum_i \left(\dot{q}_i\dot{p}_i - \dot{p}_i\dot{q}_i\right) + \frac{\partial H}{\partial t}\\ &= \frac{\partial H}{\partial t}\left(= -\frac{\partial L}{\partial t}\right)\end{aligned}\tag{7.20}$$

where we used the canonical equations. We see that if the Lagrangian (or Hamiltonian) does not explicitly depend on time, then the total time derivative of H is zero (i.e. H is a constant). Often, the Hamiltonian is equal to the total energy, in particular, if it does not depend on time explicitly and if the potential does not depend on velocity. This is the result that we obtained for the Jacobian integral. In the case that the Hamiltonian is the total energy, it can be written as:

$$H = T + V \tag{7.21}$$

One should however be careful and keep in mind that this form is not the general definition of the Hamiltonian, which is given by the Legendre transformation.

7.4 Canonical equations from Hamilton’s principle

Recall that we obtained the equations of motion by requiring that the action, S, is stationary under variations of the q_i:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt \tag{7.22}$$

The Euler-Lagrange equations of motion were obtained by requiring that the variations of the q_i were zero at the end points. We can write the variation of S in terms of the Hamiltonian:

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt = \delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(q_i, p_i, t) \right] dt \tag{7.23}$$

In the Lagrangian formalism, we found that variations of the action with respect to the q_i led to the equations of motion. The $\dot{q}_i$ did not vary independently from the q_i, since:

$$\delta \dot{q}_i = \delta \frac{dq_i}{dt} = \frac{d}{dt} \delta q_i \tag{7.24}$$

In the Hamiltonian formalism, we must treat the q_i and p_i on equal footing, that is, we must allow them to be varied independently. We know from the properties of the Legendre transformation that variations of the p_i do not affect the Lagrangian, and hence the variation of the action will not be affected by independently varying the p_i:

$$\begin{aligned} p_i &\to p_i + \delta p_i \\ q_i &\to q_i + \delta q_i \\ \delta S &= \int_{t_1}^{t_2} \left[\sum_i \left(p_i \delta \dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \right] dt \end{aligned} \tag{7.25}$$

We can write:

$$p_i \delta \dot{q}_i = p_i \frac{d}{dt} \delta q_i = \frac{d}{dt}(p_i \delta q_i) - \dot{p}_i \delta q_i \tag{7.26}$$

The first term, being a total time derivative, will not contribute to the variation of the action (since it just adds a constant), so it can be dropped:

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \left[\sum_i \left(-\dot{p}_i \delta q_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \right] dt \\ &= \int_{t_1}^{t_2} \left[\sum_i \left(-\dot{p}_i - \frac{\partial H}{\partial q_i} \right) \delta q_i + \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i \right] dt \end{aligned} \tag{7.27}$$

For the variation of S to be zero when q_i and p_i are varied independently, then the terms in front of the δq_i and δp_i must always be zero, which is precisely the canonical equations.

7.5 Symplectic notation

Symplectic notation is a way to handle the Hamiltonian formalism using matrices. It is often implemented in computerized algorithms, for example for solving for the motion of astrophysical bodies (think of calculating the trajectory of a probe on its way to Mars). Such computerized algorithms are called “symplectic integrators”.

One starts by defining a vector, $\vec{\eta}$, of dimension $2n$ where the first n coordinates are the generalized coordinates, q_i, and the next n coordinates are the generalized momenta, p_i.

$$\vec{\eta} \equiv \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \tag{7.28}$$

We define a second vector, $\frac{\partial \vec{H}}{\partial \eta}$:

$$\frac{\partial \vec{H}}{\partial \eta} \equiv \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix} \tag{7.29}$$

and a $2n \times 2n$ square matrix J:

$$J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{7.30}$$

where I is the $n \times n$ identity matrix. Hamilton’s equations in symplectic notation are thus written as:

$$\frac{d\vec{\eta}}{dt} = J \frac{\partial \vec{H}}{\partial \eta} \tag{7.31}$$

7.6 Phase space, the phase space fluid and Liouville's Theorem

Recall that in the Lagrangian formalism, one can describe a system by specifying the value of its generalized coordinates in configuration space. In order to know the future (or past) development of the system, it is also necessary to specify the velocities. For example, a canon ball's trajectory through (say, Cartesian) configuration space, depends on its velocity. In general, for a given starting position in configuration space, different initial velocities can lead to intersecting trajectories in configuration space (see Figure 7.1).

In the Hamiltonian formalism, the system is described by the generalized coordinates **and** by the generalized momenta. This can be viewed as a set of coordinates in "phase space". The trajectory of the system through phase space is completely specified by the equations of motion. If the Hamiltonian does not change with time, the paths of different systems through phase space cannot intersect (this would mean that two systems with different positions (in configuration space) and momentum, following the same equations of motion, could end up at the same position and momentum). One can make the analogy with fluid dynamics, where the trajectories through phase space, for a system following a given Hamiltonian, are similar to the flow lines for a fluid (see Figure 7.1). Each trajectory corresponds to a particular value of the Hamiltonian. This fluid is called the "phase space fluid".

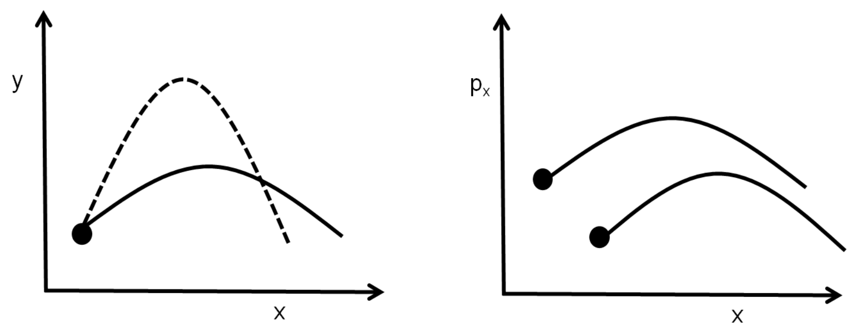

Figure 7.1 *Paths of a systems through configuration space (left) and phase space (right). In configuration space, the trajectories depend on the initial velocities and the equations of motion, so they can intersect. In phase space, the trajectories are completely determined by the equations of motion and the starting position in phase space;the trajectories for different starting points cannot intersect.*

EXAMPLE 7-1

Use the Hamiltonian formalism to describe the simple harmonic oscillator of mass m and spring constant k, and describe the motion in phase space.

The Lagrangian is given by:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The Hamiltonian is thus:

$$\begin{aligned}
H &= p\dot{x} - L \\
&= \frac{\partial L}{\partial \dot{x}}\dot{x} - L \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\
&= \frac{p^2}{2m} + \frac{1}{2}kx^2
\end{aligned}$$

Note that since L did not depend explicitly on time (and the kinetic energy is quadratic in the velocities, and the potential energy does not depend on velocity), H is equal to the total energy ($T + V$). Hamilton's canonical equations are thus:

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\
\dot{p} &= -\frac{\partial H}{\partial x} = -kx
\end{aligned}$$

Since H is constant ($= E$), the Hamiltonian describes an ellipse in phase space:

$$\frac{1}{2m}p^2 + \frac{k}{2}x^2 = E \tag{7.32}$$

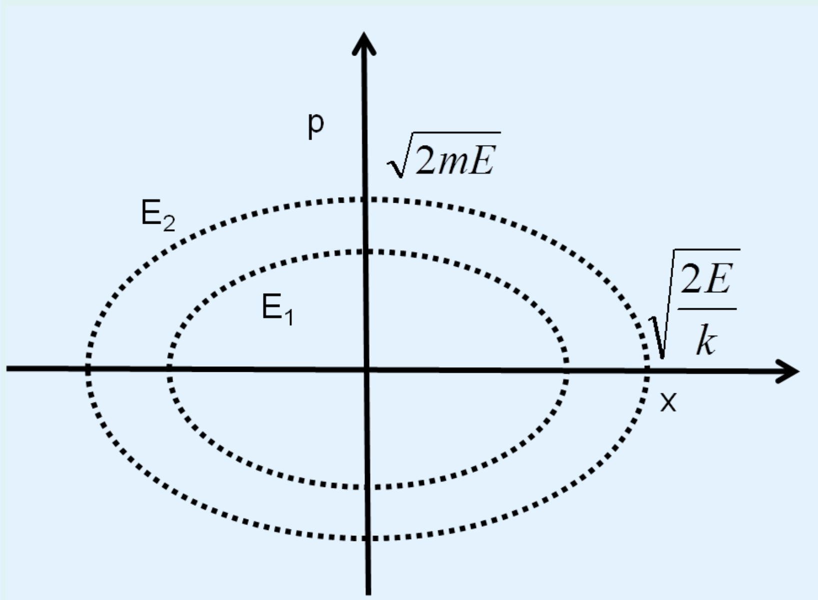

Figure 7.2 *Motion of the simple harmonic oscillator in phase space for two different values of energy.*

As the system evolves in time, it goes from a maximum value in x and zero momentum to a maximum value of momentum and zero value of x, and so on. Momentum and position in phase space can easily be thought of on an equal footing.

Liouville's theorem (although it was formulate by Gibbs and does not have much to do with Liouville) states that the phase space fluid acts like an incompressible fluid. That is, if you select a closed volume and let it evolve as the fluid moves in time, that total volume does not change with time. We know from fluid dynamics, that the divergence of the velocity field must be zero for an incompressible fluid. Indeed, consider the continuity equation for a fluid of density ρ and velocity field $\vec{v}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{7.33}$$

If the density is constant (the fluid incompressible), then the divergence of the velocity vector is zero.

Treating the generalized momenta and the generalized coordinates as “regular” coordinates of a particle in configuration space, the velocity of that particle is given by a vector of dimension $2n$:

$$\vec{v} = \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \\ \dot{p}_1 \\ \vdots \\ \dot{p}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial q_1} \\ \vdots \\ -\frac{\partial H}{\partial q_n} \end{pmatrix} \tag{7.34}$$

where we have also substituted Hamilton's canonical equations in the second equal sign.

If the divergence of the corresponding velocity field is zero:

$$\begin{aligned} \nabla \cdot \vec{v} &= 0 \\ &= \sum_i^n \left(\frac{\partial v_i}{\partial q_i} + \frac{\partial v_{n+i-1}}{\partial p_i} \right) \\ &= \sum_i^n \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) \\ &= \sum_i^n \left(\frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0 \end{aligned} \tag{7.35}$$

which is equal to zero, since the partial derivatives commute. The velocity field in field space does indeed behave like an incompressible fluid.

7.7 Poisson Brackets

The Poisson Bracket between two functions of the canonical variable, $U(q_i, p_i, t)$, and $V(q_i, p_i, t)$ is defined as:

$$\{U, V\} \equiv \sum_i^n \left(\frac{\partial U}{\partial q_i} \frac{\partial V}{\partial p_i} - \frac{\partial U}{\partial p_i} \frac{\partial V}{\partial q_i} \right) \tag{7.36}$$

The Poisson Bracket has several properties that are easily verified (in the following, capital letters denote functions of q_i and p_i, whereas k is a constant):

$$\begin{aligned}
\{U,V\} &= -\{V,U\} \\
\{U,U\} &= 0 \\
\{kU,V\} &= k\{U,V\} \\
\{A+B,V\} &= \{A,V\}+\{B,V\} \\
\{AB,V\} &= A\{B,V\}+B\{A,V\} \\
\{q_i,q_j\} = \{p_i,p_j\} &= 0 \\
\{q_i,p_j\} &= \delta_{ij} \\
\{q_i^n,p_j\} &= nq_i^{n-1}\delta_{ij} \\
\{U,p_i\} &= \frac{\partial U}{\partial q_i} \\
\{U,q_i\} &= -\frac{\partial U}{\partial p_i} \\
\{U,\{V,W\}\}+\{V,\{W,U\}\}+\{W,\{U,V\}\} &= 0
\end{aligned} \tag{7.37}$$

This last relation is called “Jacobi’s identity”. δ_{ij} is the Kronecker delta, and is equal to zero unless $i=j$. It seems a little strange to introduce the Poisson Brackets as a mathematical artefact, but they are related to the commutators that appear in Quantum Mechanics, so it is worthwhile to explore them further. Consider for example the Poisson Bracket of a canonical variable with the Hamiltonian:

$$\begin{aligned}
\{q_j,H\} &= \sum_i^n \left(\frac{\partial q_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\
&= \frac{\partial H}{\partial p_j} \\
\{p_j,H\} &= \sum_i^n \left(\frac{\partial p_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\
&= -\frac{\partial H}{\partial q_j}
\end{aligned}$$

where the terms $\frac{\partial q_j}{\partial q_i}$ and $\frac{\partial p_j}{\partial p_i}$ are zero unless $i=j$, and the terms $\frac{\partial q_j}{\partial p_i}$ and $\frac{\partial p_j}{\partial q_i}$ are zero. Identifying this with Hamilton’s canonical equations, we can re-write the canonical equations as:

$$\begin{aligned}
\dot{q}_i &= \{q_i,H\} \\
\dot{p}_i &= \{p_i,H\}
\end{aligned}$$

More generally, consider the total time derivative of a function, $F(q_i,p_i,t)$, of the canonical variables:

$$\begin{aligned}
\frac{dF}{dt} &= \sum_i \left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right) + \frac{\partial F}{\partial t} \\
&= \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial F}{\partial t} \\
&= \{F,H\} + \frac{\partial F}{\partial t}
\end{aligned} \tag{7.40}$$

We can see that the time derivative of a quantity is given by its Poisson Bracket with the Hamiltonian. If F does not depend explicitly on time, it is a constant of motion if its Poisson Bracket with the Hamiltonian is zero:

$$\{F(q_i,p_i),H\}=0\rightarrow F=\text{const.} \tag{7.41}$$

Poisson Brackets and symmetries

Recall that we showed in Chapter ? that there exists a conserved quantity, Q, for each axis of rotation about which the Lagrangian was invariant. For an infinitesimal rotation of angle $\delta\epsilon$ about the z-axis:

$$\begin{aligned}x'&=x+f_x\delta\epsilon=x-y\delta\epsilon\\ y&=y+f_y\delta\epsilon=y+x\delta\epsilon\\ z&=z+f_z\delta\epsilon=z\\ Q&=\sum_i f_ip_i=(xp_y-yp_x)=L_z\end{aligned}$$

and the conserved quantity was the z-component of angular momentum, L_z.

Now consider the Poisson Bracket:

$$\begin{aligned}\{x,L_z\}&=\{x,L_z\}\\ &=\{x,xp_y-yp_x\}\\ &=\{x,xp_y\}-\{x,yp_x\}\\ &=x\{x,p_y\}+p_y\{x,x\}-y\{x,p_x\}-p_x\{x,y\}\\ &=-y\end{aligned} \tag{7.43}$$

where we have made use of the properties from equations 7.37. One can easily show that:

$$\begin{aligned}\{x,L_z\}&=-y=f_x\\ \{y,L_z\}&=x=f_y\\ \{z,L_z\}&=0=f_z\end{aligned} \tag{7.44}$$

Thus, the Poisson Bracket of a coordinate with a component of angular momentum gives the coefficient (f_i) corresponding to the transformation of that coordinate with respect to infinitesimal rotations.

Now consider the Poisson Bracket with the components of momentum:

$$\begin{aligned}\{p_x,L_z\}&=-p_y=f_x\\ \{p_y,L_z\}&=p_x=f_y\\ \{p_z,L_z\}&=0=f_z\end{aligned} \tag{7.45}$$

which are easily demonstrated. The momentum vector transforms the same way as coordinates under a rotation, so the Poisson Brackets also correspond to the correct coefficients for rotations of the momentum vector. We find that the f_i corresponding to how a particular quantity is rotated about the z-axis are given by the Poisson Bracket of the quantity with the z-component of angular momentum. We say that angular momentum is the “generator” of rotations.

EXAMPLE 7-2

Show that a charged sphere rotating about some arbitrary axis precesses about an axis parallel to a uniform magnetic field. Assume the magnetic field is in the z-direction

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The Hamiltonian for the sphere is simply the total energy of the rotating sphere. The kinetic energy is a constant and will thus not influence the motion, and we can ignore it. The potential energy is related to the magnetic dipole of the sphere:

$$V = \vec{\mu} \cdot \vec{B}$$

The magnetic moment will be proportional to the angular momentum of the sphere, and will point in the direction of the angular momentum:

$$\begin{aligned} \vec{\mu} &\propto \vec{L} \\ \therefore H = V &\propto \vec{L} \cdot \vec{B} = \omega L_z \end{aligned}$$

where ω is a constant including the magnitude of the magnetic field, the charge and the radius of the sphere, and we used the fact that B points in the z-direction. Now consider how the components of angular momentum change with time (using Poisson Brackets):

$$\dot{L}_z = \{L_z, H\} = \{L_z, \omega L_z\} = 0 \tag{7.46}$$
$$\dot{L}_x = \{L_x, H\} = \{L_x, \omega L_z\} = -kL_y \tag{7.47}$$
$$\dot{L}_y = \{L_y, H\} = \{L_y, \omega L_z\} = kL_x \tag{7.48}$$

Thus the z-component of the angular momentum does not change. The equations for the x and y components correspond to the equations for a particle rotating around a circle with angular speed ω. Indeed, consider a particle rotating on a circle:

$$\begin{aligned} x &= \cos(\omega t) \\ y &= \sin(\omega t) \\ \dot{x} &= -\omega \sin(\omega t) = -\omega y \\ \dot{y} &= \omega \cos(\omega t) = \omega x \end{aligned}$$

Thus the tip of the angular momentum vector of the charged sphere rotates at an angular speed ω in a circle that is perpendicular to the z axis. In other words, the charged sphere precesses about the axis of the magnetic field.

Let's examine the relation between Poisson Brackets, symmetries and conserved quantities further. Recall that we showed that if the Lagrangian is invariant under translation in a direction, then momentum in that direction is conserved. The infinitesimal translation about the x-axis transformation equations and the associated conserved quantity are given by:

$$\begin{aligned}
&x' = x + f_x\delta\epsilon = x + \delta\epsilon \\
&y' = y + f_y\delta\epsilon = y \\
&z' = z + f_z\delta\epsilon = z \\
&Q = p_x
\end{aligned} \tag{7.49}$$

And we find that the Poisson Brackets are evaluated trivially:

$$\begin{aligned}
\{x, p_x\} = 1 = f_x \\
\{y, p_x\} = 0 = f_y \\
\{z, p_x\} = 0 = f_z
\end{aligned} \tag{7.50}$$

More generally, given a function, F, we have:

$$\{F, p_x\} = \frac{\partial F}{\partial x} \tag{7.51}$$

Under an infinitesimal translation a distance $\delta x = \delta\epsilon$ in the x-direction, the variation of F is:

$$\begin{aligned}
&F' = F + \frac{\partial F}{\partial x}\delta\epsilon = F + \{F, p_x\}\delta\epsilon \\
&\delta F = \{F, p_x\}\delta\epsilon
\end{aligned} \tag{7.52}$$

Thus, taking the Poisson Bracket of a conserved quantity and a coordinate, tells us how that coordinate transforms under the symmetry that corresponds to that conserved quantity.

Recall that if the Hamiltonian does not depend on time, then energy is conserved. Energy in this case is the Hamiltonian (so the statement is a little redundant). We can thus find how a quantity transforms under a translation in time by taking the Poisson Bracket with the Hamiltonian. Under a time translation, $\delta t = \delta\epsilon$:

$$F' = F + \{F, H\}\delta\epsilon \tag{7.53}$$

Now, consider a general function, $G(q_i, p_i)$ of the $2n$ coordinates. Let's define a transformation of the coordinates given by:

$$\begin{aligned}
q_i' = q_i + \{q_i, G\}\delta\epsilon = q_i + \frac{\partial G}{\partial p_i}\delta\epsilon \\
p_i' = p_i + \{p_i, G\}\delta\epsilon = p_i - \frac{\partial G}{\partial q_i}\delta\epsilon
\end{aligned} \tag{7.54}$$

We will call G the “generator” for this transformation. The Hamiltonian will transform just as any other function of q_i and p_i. If the Hamiltonian is invariant under the transformation generated by G:

$$\begin{aligned} H' &= H + \{H, G\}\delta\epsilon \\ &= H \\ \therefore \{H, G\} &= 0 \end{aligned} \tag{7.55}$$

That is, if the Poisson Bracket $\{H, G\} = 0$, then the Hamiltonian in invariant under the coordinate transformation generated by G. But this also means that $\{G, H\} = 0$, hence that G does not change with time (or is a constant). Here, the Poisson Brackets give us this insight into how a conserved quantity is related to the coordinates transformations under which the Hamiltonian is invariant.

7.8 Problems

Problem 7-1: Simple pendulum

The pendulum in Figure 7-1 is composed of a mass m attached to a mass-less rigid rod of the length l. The pendulum can swing in the xy-plane.

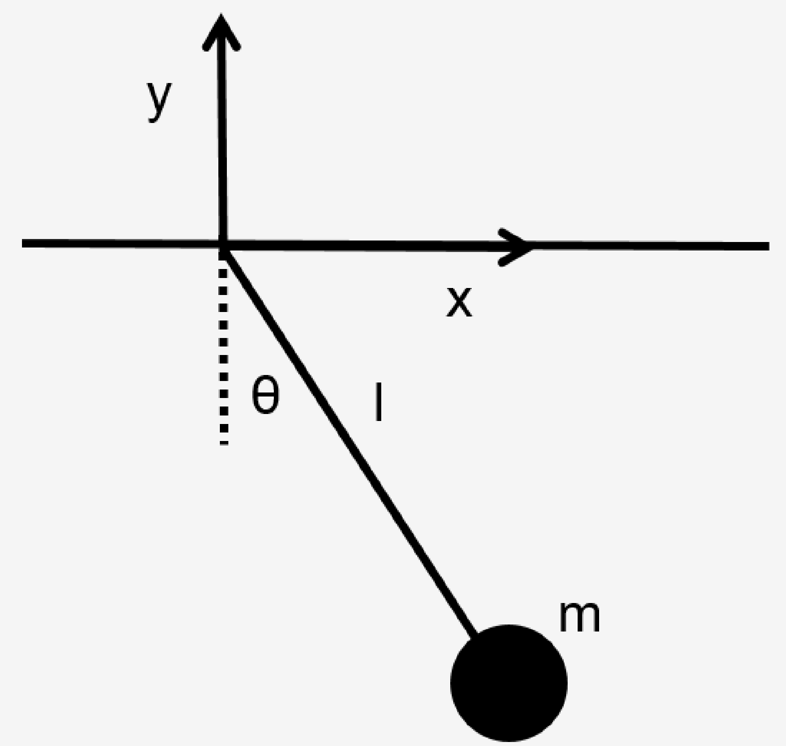

Figure 7.3 *The mass m is attached by mass-less rigid rod of length l and free to swing in the xy-plane under the action of gravity. (Problem 7-1)*

a)Choose suitable generalized coordinates and write the Hamiltonian for the system in terms of the generalized coordinates and their conjugate momenta
b)Use the canonical equations to obtain the equations of motion for the generalized coordinates and their conjugate momenta
c)Repeat parts a) and b) to obtain the equations of motion for the case where the rod has a mass M

Problem 7-2: Moving pendulum

The pendulum in Figure 7-2 is composed of a mass m attached to a mass-less rigid rod of the length l. The pendulum can swing in the xy-plane. The pivot point of the bar moves downwards at a fixed, known, speed v.

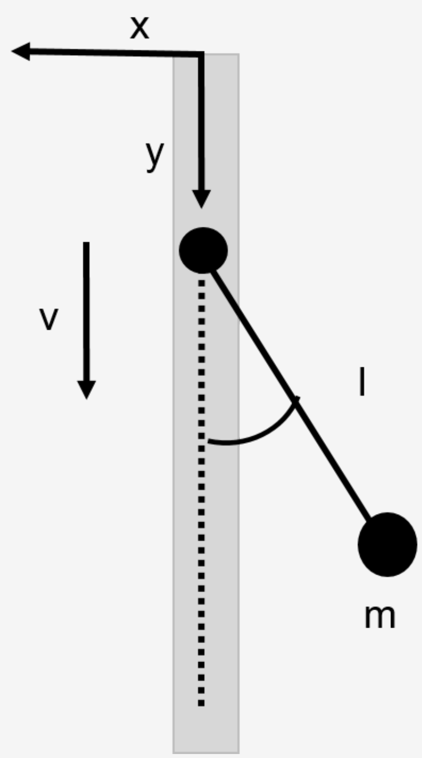

Figure 7.4 *The mass m is attached by mass-less rigid rod of length l and free to swing in the xy-plane under the action of gravity. The pivot point moves with a fixed, known velocity, v, and was at the origin at time $t = 0$. (Problem 7-2)*

a)Choose suitable generalized coordinates and write the Hamiltonian for the system in terms of the generalized coordinates and their conjugate momenta

b)Use the canonical equations to obtain the equations of motion for the generalized coordinates and their conjugate momenta

Problem 7-3: Two masses and two springs

Figure 7-3 shows two masses, m_1 and m_2, each connected to two springs with spring constants k_1 and k_2. Mass m_1 is constrained to slide without friction along the x-axis, whereas mass m_2 is constrained to move in the vertical direction, constrained by a massless frictionless vertical rod that is attached to m_1. Both springs have a resting length of l.

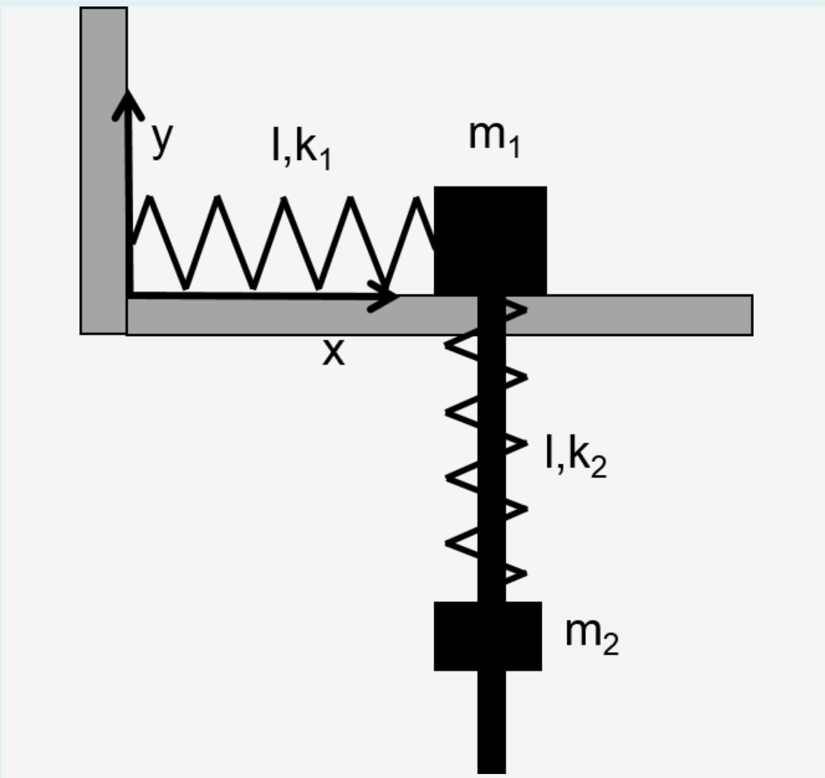

Figure 7.5 *Two masses and two springs, problem 7-3*

a)Choose suitable generalized coordinates and write the Hamiltonian for the system in terms of the generalized coordinates and their conjugate momenta
b)Use the canonical equations to obtain the equations of motion for the generalized coordinates and their conjugate momenta

Problem 7-4: Pendulum with a spring

Figure 7-4 shows a bead of mass, m, that can slide freely along a long mass-less rail which has one end fixed at the origin, forming a pendulum. The mass is connected to the pivot point at the origin by a mass-less spring of resting length, l, and spring constant k. The motion is constrained to be in the vertical plane (gravity pointing downwards in the figure).

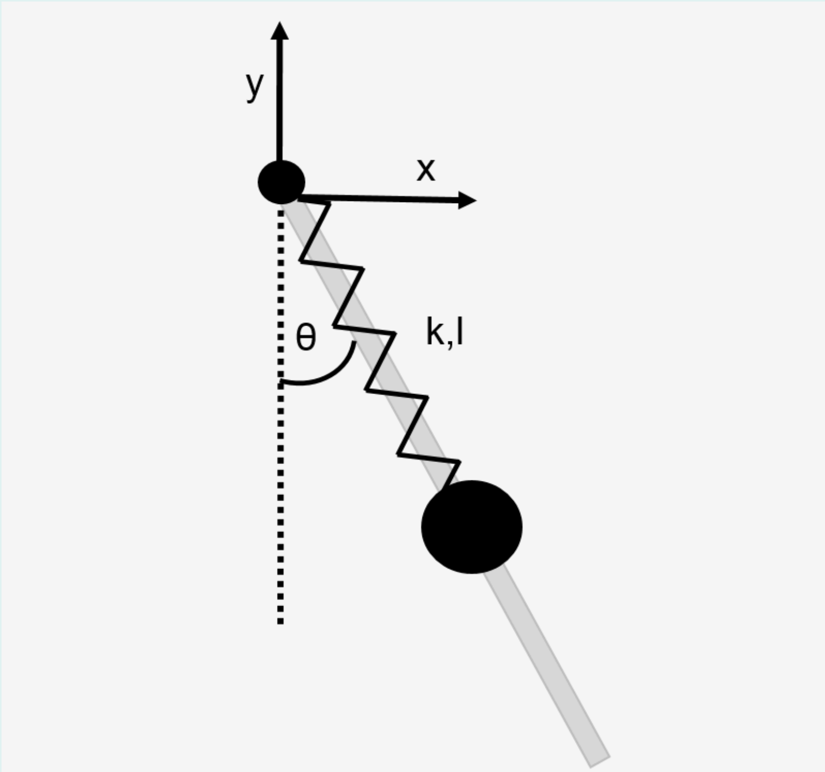

Figure 7.6 *Pendulum with a spring, problem 7-4*

a)Choose suitable generalized coordinates and write the Hamiltonian for the system in terms of the generalized coordinates and their conjugate momenta
b)Use the canonical equations to obtain the equations of motion for the generalized coordinates and their conjugate momenta

Problem 7-5: Phase space trajectory of a vertically thrown ball

a) Show that in phase space, the trajectory of a ball thrown vertically is a parabola. Assume that there is only one degree of freedom (in the vertical direction) and that gravity acts downwards with acceleration g.
b) Sketch the velocity lines of the corresponding phase space fluid (i.e. sketch the trajectory in phase space for different initial positions in phase space).

Problem 7-6: Harmonic oscillator trajectory in phase space

In phase space, the one-dimensional simple harmonic oscillator trajectory is an ellipse. The size of the ellipse depends on the initial conditions, or alternatively on the energy of the system. Show that that the time for a system to go around the ellipse in phase space is independent of energy.

Problem 7-7: Poisson Bracket Properties

Prove the following relations from equation 7.37:

$$
\begin{aligned}
&\text{a) } \{AB, V\} = A\{B, V\} + B\{A, V\} \\
&\text{b) } \{q_i, q_j\} = \{p_i, p_j\} = 0 \\
&\text{c) } \{q_i, p_j\} = \delta_{ij} \\
&\text{d) } \{q_i^n, p_j\} = n q_i^{n-1} \delta_{ij} \\
&\text{e) } \{U, p_i\} = \frac{\partial U}{\partial q_i} \\
&\text{f) } \{U, q_i\} = -\frac{\partial U}{\partial p_i} \\
&\text{g) } \{U, \{V, W\}\} + \{V, \{W, U\}\} + \{W, \{U, V\}\} = 0
\end{aligned}
$$

Problem 7-8: Angular momentum and rotation

a) Show that the Poisson Brackets between the cartesian coordinates and the components of angular momentum give the coefficients corresponding to the infinitesimal rotation about that axis of angular momentum.
b) Show that the Poisson Brackets of angular momentum components satisfy:

$$\{L_i, L_j\} = \epsilon_{ijk} L_k$$

where ϵ_{ijk} is the Levi-Cevita symbol ($+1$ for even permutations of the indices, -1 for odd permutations, and zero if two indices or more are equal).

Problem 7-9: Transformation of the Hamiltonian

Show that under the transformation:

$$\begin{aligned} q_i' &= q_i + \{q_i, G\}\delta\epsilon \\ p_i' &= p_i + \{p_i, G\}\delta\epsilon \end{aligned}$$

The Hamiltonian transforms to:

$$H' = H + \{H, G\}\delta\epsilon$$

Problem 7-10: Hamiltonian for a charged particle

Given the Lagrangian for a charged particle:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\phi + e\vec{A} \cdot \vec{v} \tag{7.56}$$

find equations for the generalized momenta and write the Hamiltonian for the system. Show that the canonical equations for Hamilton give the expected result.

Problem 7-11: Poisson's theorem

Show that if $F(q_i, p_i, t)$ and $G(q_i, p_i, t)$ are two constants of motion, then the Poisson Bracket:

$$\{F, G\}$$

is also a constant of motion.

Problem 7-12: Poisson brackets and conserved quantities

Use Poisson Brackets to show that, given the following Hamiltonian:

$$H = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2$$

the product of q_1q_2 is conserved.